MILD SOLUTIONS FOR SEMILINEAR FRACTIONAL DIFFERENTIAL EQUATIONS

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ABSTRACT . This paper concerns the existence of mild solutions for fractional semilinear differential equation with non local conditions in the $\alpha-$ norm . We prove existence and uniqueness , assuming that the linear part generates an analytic compact bounded semigroup , and the nonlinear part is a Lipschitz continuous function with respect to the fractional power norm of the linear part .

1 . INTRODUCTION

During the past decades , fractional differential equations have attracted many authors (see for instance [1 6 , 1 7 , 1 8 , 1 9 , 2 1 , 22 , 24 , 26 , 27] and references therein) .

This , mostly because it efficiently describes many phenomena arising in Engineer - ing , Physics , Economy , and Science .

Our aim in this paper is to discuss the existence and the uniqueness of the mild solution for fractional semilinear differential equation with nonlocal conditions :

$$D^q x(t) = -Ax(t)^{+}_{x(\ 0)} f_{+}(t_g^{;x(t);}{}_{(x)=} Bx_{x_0}(t)), \quad t \in [0, T], \tag{1.1}$$

where $T > 0, 0 < q < 1, -A$ generates an analytic compact semigroup $(S(t))t \geq 0$ of uniformly bounded linear operators on a Banach space $\mathbb{X}$. The term $Bx(t)$ which may be interpreted as a control on the system is defined by :

$$Bx(t) := \int_0^t K(t, s)x(s)ds,$$

where $K \in C(D, \mathbb{R}^+)$, the set of all positive function continuous on $D := \{(t, s) \in$

$$\mathbb{R}^2 : 0 \leq s \leq t \leq T\} \text{and}$$

$$B^* = \sup_{\in_t [0,T]} \int_0^t K(t, s)ds < \infty, \tag{1.2}$$

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f and g are continuous
.   The derivative D^q is understood here in the Riemann -
Liouville sense . The non lo cal condition

$$g(x) = \sum_{k=1}^{p} c_k x(t_k),$$

where c_k, k $=$ $1, 2, ...p,$ are given constants and
0 $<$ t_1 $<$ t_2 $<$ $\cdots$ $<$ t_p $\leq$
T.   Let us recall that such lo cal conditions were first used by Deng in   [4] .   In his
paper , Deng indicated that using the nonlocal condition $x(0) + g(x) = x_0$ to describe
for instance ,   the diffusion phenomenon of a small amount of gas in a
transparent tube can give better result than using the usual lo cal Cauchy Problem
$x(0)$ $=$ x_0.   Let ' s observe also that since Deng ' s paper ,   such problem has
also attracted several authors including Aizicovici , Byszewski , Ezzinbi , Fan , Liu ,
Liang , Lin , Xiao , Hern á ndez , Lee , etc . . (see for instance [1 , 2 , 3 , 4 , 1 3 , 1 1
, 6 , 1 2 , 22 , 1 9] and the references therein) .

However , among the previous research on nonlocal cauchy problems , few are con
- cerned with mild solutions of fractional semilinear differential equations . Recently ,
in [1 4] , the authors prove the existence and uniqueness of a mild solution for the
semilinear initial value problem of non - integer order when the linear part generates
a strongly continuous semigroup .   In [20] , we considered the fractional semilinear
differential equation with nonlocal conditions

$$D^q x(t) = Ax(t) + t^n f^{(t,}_{x}{}^{(0)}_{(0)} x_{=}(t)^{;Bx(t)),}_{x}{}^{}_{0+}{}^{}g(x)   t \in [0, T],   n \in \mathbb{Z}^+ \tag{1.3}$$

where T is a positive real , $0 < q < 1$, A is the generator of a C_0- semigroup $(S(t))t \geq 0$
on a Banach space $\mathbb{X}$, $Bx(t) := \int_0^t K(t,s)x(s)ds$, $K \in C(D, \mathbb{R}^+)$ with $D := \{(t,s) \in$

$$\mathbb{R}^2 : 0 \leq s \leq t \leq T\} \text{and} B^* = \sup_{\in_t [0,T]} \int_0^t K(t,s)ds < \infty, f : \mathbb{R} \times \mathbb{X} \times \mathbb{X} \to \mathbb{X} \text{is}$$

a nonlinear function , $g : \mathbb{C}([0,T], \mathbb{X}) \to D(A)$ is continuous and $0 < q < 1$.   The
derivative D^q is understood here in the Riemann - Liouville sense .

We used the Krasnoselkii and the contraction mapping principle to show the exis-
tence and uniqueness of a mild solution for a fractional semilinear differential equation
with non lo cal conditions .

In this paper , motivated by [8 , 1 3] , we investigate the existence and the unique
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ness of a mild solution for the fractional semilinear differential equation (1 . 1) , as -
suming that f is defined on $[0, T] \times \mathbb{X}_\alpha \times \mathbb{X}_\alpha$ where $\mathbb{X}_\alpha = D(A^\alpha)$, for some $0 < \alpha < 1$,
the domain of the fractional power of A.

The rest of this paper is organized as follows . In section 2 we give some known
preliminary results on the fractional powers of the generator of an analytic compact
semigroup .   In Section 3 , we study the existence and the uniqueness of the mild
solution for the fractional semilinear differential equation (1 . 1) .

2 .   PRELIMINARIES

For the rest of this article , we set $I = [0, T]$.   We denote by $\mathbb{X}$ a Banach space
with norm $\| \cdot \|$ and $-A : D(A) \to \mathbb{X}$ is the infinitesimal generator of a compact
analytic semigroup of uniformly bounded linear operators $(S(t))t \geq 0$.   This means
that there exists $M > 1$ such that

$$\| \, S(t) \, \| \leq M \qquad\qquad\qquad (2.1)$$

We assume without loss of generality that $0 \in \rho(A)$.   This allows us to define the fractional power A^α for $0 < \alpha < 1$, as a closed linear operator on its domain $D(A^\alpha)$ with inverse $A^{-\alpha}$(see [8]) . We have the following basic properties A^α **Theorem**

2 . 1 ([23 , pp .69 − 75]).   (1) $\mathbb{X}_\alpha = D(A\alpha)$ *is a Banach space with the*

$$norm \quad \| x \| \alpha := \| A^\alpha x \| \ for x \in D(A^\alpha).$$

$$(2) \quad S(t) : \mathbb{X} \to \mathbb{X}_\alpha \ for each t > 0.$$

$$(3) \quad A^\alpha S(t) x = S(t) A^\alpha x \ for \ each \quad x \in D(A^\alpha) \ and \ \ t \geq 0.$$

(4)   *For every* $t > 0, A\alpha S(t)$ *is bounded on* $\mathbb{X}$ *and th ere exist* $M_\alpha > 0$ *and* $\delta > 0$ *such that*

$$\| A^\alpha S(t) \| \quad \leq \frac{M_\alpha}{t\alpha} e^{-\delta t} \tag{2.2}$$

(5) $A − \alpha$ *is a bounded linear operator in* $\mathbb{X}$ *with* $D(A\alpha) = \mathrm{Im}\ (A − \alpha).$

$$(6) \quad If 0 < \alpha \leq \beta, then D(A^\beta) \to D(A\alpha).$$

Remark 2 . 2 .   Observe as in [1 3] that by Theorem 2 . 1 (ii) and (ii i) , the restriction $S_\alpha(t)$ of $S(t)$ to $\mathbb{X}_\alpha$ is exactly the part of $S(t)$ in $\mathbb{X}_\alpha$. Let $x \in \mathbb{X}_\alpha$. Since

$$\| S(t) x \|_\alpha = \quad \| A^\alpha S(t) x \| \quad = \| S(t) A^\alpha x \| \leq \quad \| S(t) \| \quad \| A^\alpha x \| \quad = \| S(t) \| \| x \|_\alpha,$$

and as t decreases to 0

$$\| S(t) x − x \| \alpha = \| A^\alpha S(t) x − A^\alpha x \| \quad = \| S(t) A^\alpha x − A^\alpha x \| \quad \to 0,$$

for all $x \in \mathbb{X}_\alpha$, it follows that $(S(t)) t \geq 0$ is a family of strongly continuous semigroup on $\mathbb{X}_\alpha$ and $\| S_\alpha(t) \| \leq \| S(t) \|$ for all $t \geq 0$.
We have the the following result from [1 3] . **Lemma** 2.3. $(S_\alpha(t)) t \geq 0$ *is an immediately compact s emigroup in* $\mathbb{X}_\alpha$, *and h ence it*
is immediately norm - continuous .
Definition 2 . 4 ([1 4]) **.**   A continuous function $x : I \to \mathbb{X}$ satisfying the equation

$$x(t) = S(t)(x_0 − g(x)) + \frac{1}{\Gamma(q)} \int_0^t (t − s)^{q-1} S(t − s)(f(s, x(s), Bx(s)) ds \tag{2.3}$$

for $t \in [0, T]$ is called a mild solution of the equation (1 . 1)
    In the sequel , we will also use $\| f \|_p$ to denote the L^p norm of f whenever $f \in L^p(0, T)$ for some p with $1 \leq p < \infty$.   We will set $\alpha \in (0, 1)$ and we will denote by $\mathcal{C}_\alpha$, the Banach space $C([0, T], \mathbb{X}_\alpha)$ endowed with the supnorm given by

$$\| x \| \infty := \sup \| x \|_\alpha, \quad for x \in \mathcal{C}.$$
$$t \in I$$

3 .   Main Results

We assume the following conditions : (H 1)   The function $f : I \times \mathbb{X}_\alpha \times \mathbb{X}_\alpha \to \mathbb{X}$ is continuous , and there exists a positive
function $\mu \in L^1_{\mathrm{loc}}(I, \mathbb{R}^+)$ such that

$$\| f(t, x, y) \| \leq \mu(t), \tag{3.1}$$

(H 2) $g \in C(\mathcal{C}_\alpha, \mathbb{X}_\alpha)$ is completely continuous and there exist $\lambda, \gamma > 0$ such that

$$\| g(x) \|_\alpha \leq \lambda \| x \|_\infty + \gamma.$$

Theorem 3 . 1 . *Suppose that assumptions* (H 1) , (H 2) *hold .*   *If* $x_0 \in \mathbb{X}_\alpha$ *and*

$$M\lambda < \frac{1}{2} \tag{3.2}$$

then (1 . 1) *has a mild s o lution on* $[0,T]$. *Proof .*   We define the function $F : \mathcal{C}_\alpha \to \mathcal{C}_\alpha$ by

$$(Fx)(t) = S(t)(x_0 - g(x)) + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} S(t-s) f(s, x(s), Bx(s)) ds,$$

and we choose r such that

$$r \geq 2(\frac{M_\alpha T^{q-\alpha}}{(1-\alpha)\Gamma(q)} \parallel \mu \parallel L^1_{\text{loc}}(I, \mathbb{R}_+) + M(\parallel x_0 \parallel_\alpha + \gamma)).$$

Let $B_r = \{x \in \mathcal{C}_\alpha : \parallel x \parallel_\infty \leq r\}$. Then we proceed in three steps . **Step 1 .**   We show that $FB_r \subset B_r$. Let $x \in B_r$. Then for $t \in I$, we have

$$\parallel (Fx)(t) \parallel_\alpha$$

$$\leq \quad \parallel S(t)(x_0 - g(x)) \parallel_\alpha + \frac{1}{\Gamma(q)} \parallel \int_0^t (t-s)^{q-1} S(t-s) f(s, x(s), Bx(s)) ds \parallel_\alpha$$

$$\leq \parallel S(t) \parallel \parallel x_0 - g(x) \parallel \alpha + \frac{1}{\Gamma(q)} \int_0^t \parallel (t-s)^{q-1} A^\alpha S(t-s) f(s, x(s), Bx(s)) \parallel ds$$

$$\leq \parallel S(t) \parallel (\parallel x_0 \parallel_\alpha + \lambda \parallel x \parallel_\infty + \gamma) + \frac{T^{q-1}}{\Gamma(q)} \int_0^t \parallel A^\alpha S(t-s) \parallel \parallel f(s, x(s), Bx(s)) \parallel ds,$$

which according to (2 . 1) , (2 . 2) , (3 . 1) and (3 . 2) gives

$$\parallel (Fx)(t) \parallel_\alpha$$

$$\leq \parallel S(t) \parallel (\parallel x_0 \parallel_\alpha + \lambda \parallel x \parallel_\infty + \gamma) + \frac{T^{q-1}}{\Gamma(q)} \int_0^t M_\alpha (t-s)^{-\alpha} e^{-\delta(t-s)} \mu(s) ds$$

$$\leq \quad \parallel S(t) \parallel (\parallel x_0 \parallel \alpha + \lambda \parallel x \parallel \infty + \gamma) + \frac{M_\alpha T^{q-1}}{\Gamma(q)} \int_0^t (t-s)^{-\alpha} \mu(s) ds$$

$$\leq M(\parallel x_0 \parallel_\alpha + \lambda \parallel x \parallel_\infty + \gamma) + \frac{M_\alpha T^{q-\alpha}}{(1-\alpha)\Gamma(q)} \parallel \mu \parallel L^1_{\text{loc}}(I, \mathbb{R}_+) \leq r$$

for $t \in I$. Hence , we deduce $\parallel Fx \parallel_\infty \leq r$. **Step 2 .**   We prove that F is continuous .   Let (x_n) be a sequence of B_r such that

$$x_n \to x \text{in} B_r. \text{Then}$$
$$f(s, x_n(s), Bx_n(s)) \to f(s, x(s), Bx(s)), \quad n \to \infty$$

because the function f is continuous on $I \times \mathbb{X}_\alpha \times \mathbb{X}_\alpha$. Now , for $t \in I$, we have

$$\parallel Fx_n - Fx \parallel_\alpha$$
$$\leq \quad \parallel S(t)(g(x_n) - g(x)) \parallel_\alpha$$

$$+ \parallel \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} S(t-s)(f(s, x_n(s), Bx_n(s)) - f(s, x(s), Bx(s))) ds \parallel \alpha'$$

which in view of (2 . 1) and (2 . 2) gives

$$\| Fx_n - Fx \|_\alpha$$
$$\leq \quad \| S(t) \| \quad \| g(x_n) - g(x) \|_\alpha$$

$$+\frac{T^{q-1}}{\Gamma(q)}\int_0^t \parallel A^\alpha S(t-s) \parallel \;\; \parallel f(s,x_n(s),Bx_n(s)) - f(s,x(s),Bx(s)) \parallel ds$$

$$\leq M \parallel g(x_n) - g(x) \parallel_\alpha$$

$$+\frac{M_\alpha T^{q-1}}{\Gamma(q)}\int_0^t (t-s)^{-\alpha} \parallel f(s,x_n(s),Bx_n(s)) - f(s,x(s),Bx(s)) \parallel ds$$

$$\leq M \parallel g(x_n) - g(x) \parallel_\alpha$$

$$+\frac{M_\alpha T^{q-1}}{\Gamma(q)}\int_0^t (t-s)^{-\alpha} \parallel f(s,x_n(s),Bx_n(s)) - f(s,x(s),Bx(s)) \parallel ds$$

for $t \in I$. Therefore , using on the one hand the fact that

$$\parallel f(s,x_n(s),Bx_n(s)) - f(s,x(s),Bx(s)) \parallel \;\; \leq 2\mu(s) \quad \text{for} s \in I,$$

and for each $t \in I$ since f satisfies (H 1) and on the other hand the fact that the function $s \mapsto 2\mu(s)(t-s)^{-\alpha}$ is integrable on I, by means of the Lebesgue Dominated Convergence Theorem one proves that

$$\int_0^t (t-s)^{-\alpha} \parallel f(s,x_n(s),Bx_n(s)) - f(s,x(s),Bx(s)) \parallel ds \to 0.$$

Hence , since $g(x_n) \to g(x)$ as $n \to \infty$ because g is completely continuous on $\mathcal{C}_\alpha$, it can easily been shown that

$$\lim_{n\to\infty} \parallel Fx_n - Fx \parallel_\infty = 0, \quad \text{as} n \to \infty.$$

In other words F is continuous .

Step   3 .   We show that F is compact .   To this end ,   we use the Ascoli - Arzela ' s theorem . We first prove that $\{(Fx)(t) : x \in B_r\}$ is relatively compact in $\mathbb{X}_\alpha$, for all $t \in I$. Obviously , $\{(Fx)(0) : x \in B_r\}$ is compact . Let $t \in (0,T]$. For each $h \in (0,t)$ and $x \in B_r$, we define the operator F_h by

$$(F_h x)(t) = S(t)(x_0 - g(x)) + \frac{1}{\Gamma(q)}\int_0^{t-h}(t-s)^{q-1}S(t-s)f(s,x(s),Bx(s))ds$$

$$= S(t)(x_0 - g(x)) + \frac{S(h)}{\Gamma(q)}\int_0^{t-h}(t-s)^{q-1}S(t-h-s)f(s,x(s),Bx(s))ds.$$

Then the sets $\{(F_h x)(t) : x \in B_r\}$ are relatively compact in $\mathbb{X}_\alpha$ since by Lemma 2 . 3 , the operators $S_\alpha(t), t \geq 0$ are compact on X_α. Moreover , using (H 1) and (2 . 2) , we have

$$\parallel (Fx)(t) - (F_h x)(t) \parallel_\alpha \leq \frac{1}{\Gamma(q)}\int_{t-h}^t (t-s)^{q-1} \parallel S(t-s)f(s,x(s),Bx(s)) \parallel_\alpha ds$$

$$\leq \frac{T^{q-1}}{\Gamma(q)}\int_{t-h}^t \parallel A^\alpha S(t-s) \parallel \parallel f(s,x(s),Bx(s)) \parallel ds$$

$$\leq \frac{T^{q-1}M_\alpha}{\Gamma(q)} \parallel \mu \parallel L^1_{\text{loc}}(I,\mathbb{R}_+)\int_{t-h}^t (t-s)^{-\alpha}ds$$

$$\leq \frac{T^{q-1}M_\alpha \parallel \mu \parallel L^1_{\text{loc}}(I,\mathbb{R}_+)}{(1-\alpha)\Gamma(q)}h^{1-\alpha}$$

Therefore , we deduce that $\{(Fx)(t) : x \in B_r\}$ is relatively compact in $\mathbb{X}_\alpha$ for all $t \in (0, T]$ and since it is compact at $t = 0$ we have the relatively compactness in $\mathbb{X}_\alpha$ for all $t \in I$. Now , let us prove that $F(B_r)$ is equicontinuous . By the compactness

can prove that the functions $Fx, x \in B_r$ are equicontinuous a
$t = 0$. For $0 < t_2 < t_1 \leq T$, we have

$$\| (Fx)(t_1) - (Fx)(t_2) \|_\alpha$$
$$\leq \| (S(t_1) - S(t_2))(x_0 - g(x)) \|_\alpha$$
$$+ \frac{1}{\Gamma(q)} \| \int_0^{t_2} (t_1 - s)^{q-1}(S(t_1 - s) - S(t_2 - s))f(s, x(s), Bx(s))ds \|_\alpha$$
$$+ \frac{1}{\Gamma(q)} \| \int_0^{t_2} ((t_1 - s)^{q-1} - (t_2 - s)^{q-1})S(t_2 - s)f(s, x(s), Bx(s))ds \|_\alpha$$
$$+ \frac{1}{\Gamma(q)} \| \int_{t_2}^{t_1} (t_1 - s)^{q-1}S(t_1 - s)f(s, x(s), Bx(s))ds \|_\alpha$$
$$\leq I_1 + I_2 + I_3 + I_4$$

Where

$$I_1 = \| (S(t_1) - S(t_2))(x_0 - g(x)) \|_\alpha$$
$$I_2 = \frac{1}{\Gamma(q)} \| \int_0^{t_2} (t_1 - s)^{q-1}(S(t_1 - s) - S(t_2 - s))f(s, x(s), Bx(s))ds \|_\alpha$$
$$I_3 = \frac{1}{\Gamma(q)} \| \int_0^{t_2} ((t_1 - s)^{q-1} - (t_2 - s)^{q-1})S(t_2 - s)f(s, x(s), Bx(s))ds \|_\alpha$$
$$I_4 = \frac{1}{\Gamma(q)} \| \int_{t_2}^{t_1} (t_1 - s)^{q-1}S(t_1 - s)f(s, x(s), Bx(s))ds \|_\alpha$$

Actually , I_1, I_2, I_3 and I_4 tend to 0 independently of $x \in B_r$ when $t_2 \to t_1$. Indeed , let
$x \in B_r$ and $G = \sup_{x \in \mathcal{C}_\alpha} \| g(x) \|_\alpha$. We have

$$I_1 = \| (S(t_1) - S(t_2))(x_0 - g(x)) \|_\alpha$$
$$\leq \| S_\alpha(t_1) - S_\alpha(t_2) \| \alpha \| x_0 - g(x) \|_\alpha$$
$$\leq \quad \| S_\alpha(t_1) - S_\alpha(t_2) \|_\alpha (\| x_0 \|_\alpha + G)$$

from which we deduce that $\lim_{t_2 \to t_1} I_1 = 0$ since by Lemma 2 . 3 the function $t \mapsto$
$\| S_\alpha(t) \|_\alpha$ is continuous for $t \in (0, T]$.

$$I_2$$
$$\leq \frac{1}{\Gamma(q)} \int_0^{t_2} \quad \| (t_1 - s)^{q-1}(S(t_1 - s) - S(t_2 - s))f(s, x(s), Bx(s)) \|_\alpha \, ds$$
$$\leq \frac{T^{q-1}}{\Gamma(q)} \int_0^{t_2} \| [S(\frac{t_1 - t_2}{2} + \frac{t_1 - s}{2}) - S(\frac{t_2 - s}{2}] A^\alpha S(\frac{t_2 - s}{2}) f(s, x(s), Bx(s)) \| ds$$
$$\leq \frac{T^{q-1-\alpha}}{\Gamma(q)} \| \mu \| L^1_{\text{loc}}(I, \mathbb{R}_+) \int_0^{t_2} \| S(\frac{t_1 - t_2}{2} + \frac{t_1 - s}{2}) - S(\frac{t_2 - s}{2}) \| ds.$$

Therefore , the continuity of the function $t \mapsto \quad \| S(t) \| \quad$ for $t \in \quad (0, T) \quad$ allows us
to conclude that $\lim_{t_2 \to t_1} I_2 = 0$.

$$I_3 \leq \frac{1}{\Gamma(q)} \int_0^{t_2} \| \quad ((t_2 - s)^{q-1} - (t_1 - s)^{q-1})S(t_2 - s)f(s, x(s), Bx(s)) \|_\alpha \, ds$$
$$\leq \frac{1}{\Gamma(q)} \int_0^{t_2} |(t_2 - s)^{q-1} - (t_1 - s)^{q-1}| \quad \| A^\alpha S(t_2 - s) \| \| f(s, x(s), Bx(s)) \| \, ds$$

$$\leq \frac{1}{\Gamma(q)} \int_0^{t_2} |(t_2-s)^{q-1} - (t_1-s)^{q-1}|(t_2-s)^{-\alpha}\mu(s)ds$$

$$\leq \frac{T^{-\alpha}}{\Gamma(q)} \parallel \mu \parallel L^1_{\text{loc}}(I, \mathbb{R}_+) \int_0^{t_2} |(t_2-s)^{q-1} - (t_1-s)^{q-1}|ds$$

$$\leq \frac{T^{-\alpha}}{q\Gamma(q)} \parallel \mu \parallel L^1_{\text{loc}}(I, \mathbb{R}_+) \mid t_1 - t_2 \mid^q.$$

Hence $\lim_{t_2 \to t_1} I_3 = 0.$

$$I_4 \leq \frac{1}{\Gamma(q)} \int_{t_2}^{t_1} \parallel (t_1-s)^{q-1}S(t_1-s)f(s,x(s),Bx(s)) \parallel_\alpha ds$$

$$\leq \frac{T^{q-1}}{\Gamma(q)} \int_{t_2}^{t_1} \parallel A^\alpha S(t_1-s) \parallel \parallel f(s,x(s),Bx(s)) \parallel ds$$

$$\leq \frac{M_\alpha T^{q-1}}{\Gamma(q)} \int_{t_2}^{t_1} (t_1-s)^{-\alpha}\mu(s)ds$$

$$\leq \frac{M_\alpha T^{q-1}}{(1-\alpha)\Gamma(q)} \parallel \mu \parallel L^1_{\text{loc}}(I, \mathbb{R}_+) \mid t_1 - t_2 \mid^{1-\alpha}.$$

Since $1 - \alpha > 0$, we deduce that $\lim_{t_2 \to t_1} I_4 = 0$.

In summary , we have proven that $F(B_r)$ is relatively compact , for $t \in I, \{Fx : x \in B_r\}$ is a family of equicontinuous functions . Hence by the Arzela - Ascoli Theo - rem , F is compact . By Schauder fixed point theorem F has a fixed point $x \in B_r$. Consequently , (1 . 1) has a mild solution . $\square$

Now we make the following assumptions .

(H 1 ') $f : I \times \mathbb{X}_\alpha \times \mathbb{X}_\alpha \to \mathbb{X}$ is continuous and there exist functions $\mu 1, \mu 2 \in$

$$L^1_{\text{loc}}(I, \mathbb{R}^+) \text{suchthat}$$
$$\parallel f(t,x,u) - f(t,y,v) \parallel \leq \mu 1(t) \parallel x-y \parallel \alpha + \mu 2(t) \parallel u-v \parallel_\alpha,$$
$$\text{forall} t \in I, x,y,u,v \in \mathbb{X}_\alpha.$$

(H 2 ') $g : \mathcal{C}_\alpha \to \mathbb{X}_\alpha$ is continuous and there exists a constant b such that

$$\parallel g(x) - g(y) \parallel \alpha \leq b \parallel x-y \parallel_\infty, \quad \text{forall} x, y \in \mathcal{C}_\alpha.$$

(H 3) The function $\Omega_{\alpha,q} : I \to \mathbb{R}_+, 0 < \alpha, q < 1$ defined by

$$\Omega_{\alpha,q} = Mb + \frac{T^{q-1}M_\alpha t^{1-\alpha}}{(1-\alpha)\Gamma(q)}(\parallel \mu 1 \parallel L^1_{\text{loc}}(I, \mathbb{R}_+) + B^* \parallel \mu 2 \parallel L^1_{\text{loc}}(I, \mathbb{R}_+))$$

satisfies $0 < \Omega_{\alpha,q} \leq \tau < 1$, for all $t \in I$. **Theorem 3 . 2 .** Assume that (H 1 ') , (H 2 ') , (H 3) hold . If $x_0 \in \mathbb{X}_\alpha$ then (1 . 1) has a unique mild s o lutio n $x \in \mathcal{C}_\alpha$.
Proof . Define the function $F : \mathcal{C}_\alpha \to \mathcal{C}_\alpha$ by

$$(Fx)(t) = S(t)(x_0 - g(x)) + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1}S(t-s)f(s,x(s),Bx(s))ds.$$

Note that F is well defined on $\mathcal{C}_\alpha$. Now take $t \in I$ and $x, y \in \mathcal{C}_\alpha$. We have

$$\parallel (Fx)(t) - F(y)(t) \parallel_\alpha$$
$$\leq \parallel S(t)(g(x) - g(y)) \parallel_\alpha$$

$$+\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} \parallel S(t-s)(f(s,x(s),Bx(s)) - f(s,y(s),By(s))) \parallel_\alpha ds$$

$$\leq \quad \parallel S(t) \parallel \quad \parallel g(x) - g(y) \parallel_\alpha$$

$$+\frac{T^{q-1}}{\Gamma(q)}\int_0^t \parallel A^\alpha S(t-s) \parallel \quad \parallel f(s,x(s),Bx(s)) - f(s,y(s),By(s)) \parallel ds$$

which according to (2 . 1) , (2 . 2) , (H 1 ') , (H 2 ') and (1 . 2) gives

$$\parallel (Fx)(t) - F(y)(t) \parallel_\alpha$$

$$\leq Mb \parallel x - y \parallel_\infty + \frac{T^{q-1} M_\alpha}{\Gamma(q)}\int_0^t (t-s)^{-\alpha}\mu 1(s) \parallel x(s) - y(s) \parallel_\alpha ds$$

$$+\frac{T^{q-1} M_\alpha}{\Gamma(q)}\int_0^t (t-s)^{-\alpha}\mu 2(s) \parallel Bx(s) - By(s)) \parallel_\alpha ds$$

$$\leq Mb \parallel x - y \parallel_\infty$$

$$+\frac{T^{q-1} M_\alpha}{\Gamma(q)} \parallel \mu 1 \parallel L^1_{\text{loc}}(I,\mathbb{R}_+)(\int_0^t (t-s)^{-\alpha}ds) \parallel x(s) - y(s) \parallel_\infty$$

$$+\frac{T^{q-1} M_\alpha}{\Gamma(q)}\int_0^t (t-s)^{-\alpha}\mu 2(s)[\int_0^s K(s,\sigma) \parallel A^\alpha(x(\sigma) - y(\sigma)) \parallel d\sigma]ds$$

$$\leq (Mb + \frac{T^{q-1} M_\alpha t^{1-\alpha}}{(1-\alpha)\Gamma(q)} \parallel \mu 1 \parallel L^1_{\text{loc}}(I,\mathbb{R}_+)) \parallel x - y \parallel_\infty$$

$$+\frac{T^{q-1} M_\alpha B^* t^{1-\alpha}}{(1-\alpha)\Gamma(q)} \parallel \mu 2 \parallel L^1_{\text{loc}}(I,\mathbb{R}_+) \parallel x - y \parallel_\infty$$

$$\leq [Mb + \frac{T^{q-1} M_\alpha t^{1-\alpha}}{(1-\alpha)\Gamma(q)}(\parallel \mu 1 \parallel L^1_{\text{loc}}(I,\mathbb{R}_+) + B^* \parallel \mu 2 \parallel L^1_{\text{loc}}(I,\mathbb{R}_+))] \parallel x - y \parallel_\infty$$

$$\leq \Omega_{\alpha,q}(t) \parallel x - y \parallel_\infty .$$

So we get

$$\parallel (Fx)(t) - F(y)(t) \parallel \infty \leq \Omega_{\alpha,q}(t) \parallel x - y \parallel_\infty .$$

Therefore , assumption (H 3) allows us to conclude in view of the contraction map - ping principe that , F has a unique fixed point in $\mathcal{C}_\alpha$, and

$$x(t) = S(t)(x_0 - g(x)) + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} S(t-s)f(s,x(s),Bx(s))ds$$

which is the mild solution of (1.1). $\square$

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