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EXISTENCE AND UNIQUENESS OF SOLUTIONS FOR THE FRACTIONAL INTEGRO - DIFFERENTIAL EQUATIONS IN BANACH SPACES

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ABSTRACT . In this article , we established the existence and uniqueness of so - lutions for fractional integro - differential equations with nonlocal conditions in Banach spaces . Krasnoselskii - Krein - type conditions are used for obtaining the main result .

1 . INTRODUCTION

In this article , we are interesting in the existence and uniqueness of solutions for the Cauchy problem with a Caputo fractional derivative and nonlocal conditions :

$$D^q x(t) = f(t, x(t), [\theta x](t)), \quad q \in (0,1) t \in I := [0,1], \tag{1.1}$$

$$x(0) + g(x) = x_0, \tag{1.2}$$

where $q \in (0,1), f : I \times X \times X \to X, g : C(I, X) \to X, \theta : X \to X$ defined as

$$[\theta x](t) = \int_0^t k(t, s, x(s)) ds,$$

and $k : \Delta \times X \to X, \Delta = \{(t,s) : 0 \le s \le t \le 1\}$. Here , $(X, \| \cdot \|)$ is a Banach space and $C = C(I, X)$ denotes the Banach space of all bounded continuous functions from I into X equipped with the norm $\| \cdot \| C$.

The study of fractional differential equations and inclusions is linked to the wide applications of fractional calculus in physics , continuum mechanics , signal processing , and electromagnetics . The theory of fractional differential equations has seen considerable development , see for example the monographs of Kilbas et al . [5] and Lakshmikantham et al . [9] .

Recently , existence and uniqueness criteria for the various fractional (integro -) differential equations were considered by Ahmad and Nieto [1] , Bhaskar [4] , Lak - shmikantham and Leela et al [7 , 8] . For more information in this fields , see [2 , 3]
and the references therein .

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As indicated in many previous articles , the nonlocal condition $x(0) + g(x) = x_0$ generalizes the Cauchy condition $x(0) = x_0$, and can be applied in physics with better cases than the Cauchy condition . The term $g(x)$ denotes the nonlocal effects , which describe the diffusion phenomenon of the a small amount in a transparent tube , with the general form $g(x) = \sum_{i=1}^{p} c_i x(t_i)$. Also , the problem (1 . 1) - (1 . 2) includes many classical formulations . For example , $g(x) = x_0 - x(T)$ becomes a periodic boundary problem , $g(x) = x_0 + x(T)$ becomes an antiperiodic boundary problem , while $g(x) = 0$ becomes a Cauchy problem .

In [2] , the authors presented some existence and uniqueness results for the prob - lem (1 . 1) - (1 . 2) , when $f(t, x(t), [\theta x](t)) = p(t, x(t)) + \int_0^t k(t, s, x(s))ds$. In [3] , the authors presented some existence and uniqueness results for the problem (1 . 1) - (1 . 2) , when $f(t, x(t), [\theta x](t)) = \int_0^t k(t, s, x(s))ds$. The aim of this paper is to present some existence results for the problem (1 . 1) - (1 . 2) for some Krasnoselskii - Krein - type con - ditions . Our methods are based on the equivalence of norms and a fixed point theorem .

2 . Main results

For the next theorem , we sue the following assumptions :

(F 1) f is continuous and there exist constants $\alpha, \beta \in (0, 1], L_1, L_2 > 0$ such that

$$\text{for} t \in I \text{and} x_i, yi \in X,$$

$$\| f(t, x_1, y1) - f(t, x_2, y2) \| \le L_1 \| x_1 - y1 \| \alpha + L_2 \| x_2 - y2 \|^{\beta};$$

(F 2) k is continuous and there exist $\beta 1 \in (0, 1], h \in L^1(I)$ such that

$$\| k(t, s, x) - k(t, s, y) \| \le h(s) \| x - y \|^{\beta 1}, \quad (t, s) \in \Delta, x, y \in X;$$

(G) g is bounded , continuous , and there exists a constant $b \in (0, 1)$ such that

$$\| g(u) - g(v) \| \le b \| u - v \| .$$

Theorem 2 . 1 . *Under Assumptions* (F 1) , (F 2) , (G) , *Pro b lem* (1 . 1) - (1 . 2) *has a unique s o lution .*

For special cases of f, we obtain the following corollaries . **Corollary 2 . 2 .** *Let* $f(t, x(t), [\theta x](t)) = p(t, x(t)) + \int_0^t k(t, s, x(s))ds$. *Assume* (F 2) , (G) *and that* p *is continuous and there exist constants* $\beta \in (0, 1], L > 0$ *such that*

$$\| p(t, x) - p(t, y) \| \le L \| x - y \|^{\beta} \quad t \in I, x, y \in X.$$

Then (1 . 1) - (1 . 2) *has a unique so lution .*

Corollary 2 . 3 . *Assume* (F 1) , (G) *and that* $k(t, s, x(s)) = \gamma(t, s)x(s)$ *and* $\gamma \in C(\Delta)$. *Then* (1 . 1) - (1 . 2) *has a unique s o lutio n .*

For the next theorem , we use the assumptions :

(F 1 ') f is continuous and there exist constants $p1, p2 \in [0, q), L_1, L_2, C > 0$ such that

$$\| f(t, x, y) \| \le \frac{L_1}{tp1} \| x \| + \frac{L_2}{tp2} \| y \| + C, \quad t \in I, x, y \in X;$$

(F 2 ') k is continuous and there exist $h \in L^1(I), K > 0$ such that

$$\| k(t,s,x) \| \leq h(s) \| x \| + K, \quad (t,s) \in \Delta, x,y \in X.$$

Theorem 2 . 4 . *Assume* (F 1 ') , F (2 ') , (G) *. Then* (1 . 1) - (1 . 2) *has at least one s o lution .* We remark that Theorem 2 . 1 extends [2 , Theorem 2 . 1] and [3 , Theorem 2 . 1] .

3 . Proof of Theorem 2 . 1

The following lemma , due to Krasnoselskii , plays an important role in the proof of the existence part of Theorem 2 . 1 . **Lemma 3 . 1** ([6]) **.** *Let* M *be a clos ed convex and nonempty subset of a Banach space* X. *Let* A, B *be two operators such that* $(1) Ax + By \in M$ *whenever* $x, y \in M$; (2) A *is compact and continuous* ; $(3) B$ *is a contraction mapping .* *Then there exists*

$$z \in M such that z = Az + Bz.$$

Proof of Theorem 2 . 1 . First , we transform the Cauchy problem (1 . 1) - (1 . 2) into fixed point problem with $F : C(I, X) \to C(I, X)$ defined by

$$Fx(t) = x_0 - g(x) + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, x(s), [\theta x](s)) ds. \quad (3.1)$$

Let$F = A + B$, with

$$Ax(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, x(s), [\theta x](s)) ds; \quad (3.2)$$

$$Bx(t) = x_0 - g(x). \quad (3.3)$$

Define the norm $\| \cdot \| k$ in $C(I, X)$, for $u \in C(I, X)$ and for some $k \in \mathbb{N}$, by

$$\| u \| k = \max\{e^{-kt} \| u(t) \| : t \in I\}.$$

Note that the norms $\| \cdot \| C$ and $\| \cdot \| k$ are equivalent .
We prove Theorem 2 . 1 in the following two steps . **Step 1 : Existence .** Let $P = \sup_{x \in X} \| g(x) \|, M_0 = \sup_{t \in I} \| \int_0^t k(t, s, 0) ds \|, M_1 = \sup_{t \in I} \| f(t, 0, 0) \|$ and $Q = \| x_0 \| + P + \frac{M_1}{\Gamma(q+1)} + 3$. Choose a $k_1 \in N$ such that

$$\frac{1}{k_1^q} (L_1 Q^\alpha + L_2 (\| h \| L^1 Q^{\beta 1} + M_0)^\beta) < 3.$$

Setting $B_Q = \{u \in C(I, X) : \| u \| k_1 \leq Q\}$. For $u \in B_Q$, noting the assumption (F 2) , we have

$$\begin{aligned} \| [\theta u](t) \| &\leq \int_0^t \| k(t, r, u(r)) - k(t, r, 0) + k(t, r, 0) \| dr \\ &\leq \| h \| L^1 \sup_{r \in [0,t]} \| x(r) \|^{\beta 1} + M_0 \\ &\leq \| h \| L^{1 e^{k_1 t}} Q^{\beta 1} + M_0. \end{aligned}$$

Thus

$$\| \theta u \| k_1 \leq \| h \| L^1 Q^{\beta 1} + M_0.$$

By assumption (F 1) , for $u \in B_Q$, we obtain

$$\begin{aligned} \| Fu(t) \| \leq \| x_0 \| + P &+ \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \| f(s, u(s), [\theta u](s)) - f(s, u(s), 0) \| ds \\ &+ \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \| f(s, u(s), 0) - f(s, 0, 0) \| ds \\ &+ \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \| f(s, 0, 0) \| ds \end{aligned}$$

$$\leq \ \| x_0 \| \ +P+\frac{L_2}{\Gamma(q)}\int_0^t (t-s)^{q-1} \| [\theta u](s) \|^{\beta} ds$$
$$+\frac{L_1}{\Gamma(q)}\int_0^t (t-s)^{q-1} \| u(s) \|^{\alpha} ds+\frac{M_1}{\Gamma(q+1)}$$
$$\leq \ \| x_0 \| \ +P+\frac{M_1}{\Gamma(q+1)}+\frac{L_2}{\Gamma(q)}\int_0^t (t-s)^{q-1}e^{\beta k_1 s}ds \| \theta u \|_{k_1}^{\beta}$$
$$+\frac{L_1}{\Gamma(q)}\int_0^t (t-s)^{q-1}e^{\alpha k_1 s}ds \| u \|_{k_1}^{\alpha}$$
$$\leq \| x_0 \| +P+\frac{M_1}{\Gamma(q+1)}+\frac{L_1}{\Gamma(q)}\int_0^t (t-s)^{q-1}e^{k_1 s}ds \| u \|_{k_1}^{\alpha}$$
$$+\frac{L_2}{\Gamma(q)}\int_0^t (t-s)^{q-1}e^{k_1 s}ds(\| h \| L^1 Q^{\beta 1}+M_0)^{\beta}$$
$$\leq \| x_0 \| +P+\frac{M_1}{\Gamma(q+1)}+e^{k_1 t}[\frac{L_1}{k_1^q}Q^{\alpha}+\frac{L_2}{k_1^q}(\| h \| L^1 Q^{\beta 1}+M_0)^{\beta}].$$

Thus

$$\| Fu \| k_1 \leq \ \| x_0 \| \ +P+\frac{M_1}{\Gamma(q+1)}+\frac{L_1}{k_1^q}Q^{\alpha}+\frac{L_2}{k_1^q}(\| h \| L^1 Q^{\beta 1}+M_0)^{\beta}<Q.$$
$$\text{This implies} F(B_Q) \subset B_Q.$$

On the other hand , for $u \in B_Q$ and $t_1, t_2 \in J(t_1 < t_2)$, we deduce that

$$\| Au(t_2)-Au(t_1) \|$$
$$=\frac{1}{\Gamma(q)} \| \int_0^{t_2}(t_2-s)^{q-1}f(s,u(s),[\theta u](s))ds-\int_0^{t_1}(t_1-s)^{q-1}f(s,u(s),[\theta u](s))ds \|$$
$$\leq \frac{M}{\Gamma(q+1)}[2(t_2-t_1)^q+(t_1)^q-(t_2)^q]$$
$$\leq \frac{2M}{\Gamma(q+1)}(t_2-t_1)^{q},$$

where $M = \sup \{\| f(t,x,y) \| \ : \ (t,x,y) \ \in I\times B_Q\times\theta(B_Q)\}$. This means $A(B_Q)$ is equicontinuous set . By Ascoli - Arzela theorem , we easily deduce that $A(B_Q)$ is relatively compact set . It follows from the continuousness of f that A is complete continuous .

By Assumption (G) , it is easy to see that B is contraction mapping . Following the Lemma 3 . 1 (Krasnoselskii ' s fixed point theorem) , we conclude that F has a fixed point in B_Q. Thus there exists a solution of Cauchy problem (1 . 1) - (1 . 2) . **Step 2 : Uniqueness .** Let $\varphi(t)$ and $\psi(t)$ be two solutions of Cauchy problem

$$\| \varphi(t)-\psi(t) \| . \qquad (1.1)-(1.2), \text{and set} m(t) =$$

First , we prove that $m(0) = 0$. Indeed , by the definition of operator B and assumption (G), we see that B is contraction on $C(I,X)$. Thus there exists a unique

$y(t)$ such that $By(t) = x_0 + g(y)$. On the other hand , noting that $\varphi(0) = x_0 + g(\varphi)$ and $\psi(0) = x_0 + g(\psi)$, we obtain $\varphi(0) = \psi(0)$.

Next , we prove $m(t) \equiv 0$ for $t \in I$ by contraction . If $m(t) \neq 0$ for some $t \in I$. Setting $t_* = \min \{t \in I : m(t) \neq 0\}$, then $m(t) \equiv 0$ for $t \in [0, t_*]$. Thus $m(t) \equiv 0$ for $t \in I$ if and only if $t_* = 1$. If $t_* < 1$, then we can choose positive numbers ε_0 and

$$k_2 \in N \text{suchthat}$$
$$\frac{e^{k_2\varepsilon_0}}{k_2^q}(L_1 m_{\varepsilon_0}^{\alpha-1} + L_2 \parallel h \parallel \beta_{L1}\varepsilon_m^{\beta\beta 1-1}{}_0) < 1,$$
$$\text{where} m_{\varepsilon_0} = \max\{\parallel \varphi(t) - \psi(t) \parallel : t \in [t_*, t_* + \varepsilon_0]\}.$$

Redefine the norm $\parallel \cdot \quad \parallel k_2$ on the interval $[t_*, t_* + \varepsilon_0]$ by

$$\parallel u \parallel k_2 = \sup\{e^{-k_2(t-t_*)} \parallel u(t) \parallel : t \in [t_*, t_* + \varepsilon_0]\},$$

then the norms $\parallel \cdot \quad \parallel k_2$ and $\parallel \quad \cdot \parallel C$ are equivalent on $[t_*, t_* + \varepsilon_0]$. Since $\varphi(0) = \psi(0)$, we claim that $g(\varphi) = g(\psi)$. Thus there exists $t_1 \in [t_*, t_* + \varepsilon_0]$ such that

$$\begin{aligned}
0 < m_{\varepsilon_0} &= \parallel \varphi(t_1) - \psi(t_1) \parallel \\
&= \parallel F\varphi(t_1) - F\psi(t_1) \parallel \\
&\leq \frac{1}{\Gamma(q)} \int_{t_*}^{t_1} (t_1 - s)^{q-1} \parallel f(s, \varphi(s), [\theta\varphi](s)) - f(s, \psi(s), [\theta\psi](s)) \parallel ds \\
&\leq \frac{L_1}{\Gamma(q)} \int_{t_*}^{t_1} (t_1 - s)^{q-1} \parallel \varphi(s) - \psi(s) \parallel^\alpha ds \\
&+ \frac{L_2}{\Gamma(q)} \int_{t_*}^{t_1} (t_1 - s)^{q-1} \parallel [\theta\varphi](s) - [\theta\psi](s) \parallel^\beta ds \\
&\leq \frac{1}{\Gamma(q)} \int_{t_*}^{t_1} (t_1 - s)^{q-1} [L_1 m^\alpha(s) + L_2 \parallel h \parallel_{L^1}^\beta \sup_{\in_r [0,s]} m^{\beta\beta 1}(r)] ds \\
&\leq \frac{L_1}{\Gamma(q)} \int_{t_*}^{t_1} (t_1 - s)^{q-1} e^{\alpha k_2(s-t_*)} ds \parallel \varphi - \psi \parallel_{k_2}^\alpha \\
&+ \frac{L_2 \parallel h \parallel \beta_{L1}}{\Gamma(q)} \int_{t_*}^{t_1} (t_1 - s)^{q-1} e^{\beta\beta 1^k 2(s-t_*)} ds \parallel \varphi - \psi \parallel_{k_2}^{\beta\beta 1} \\
&\leq \frac{e^{k_2\varepsilon_0}}{k_2^q} (L_1 m_{\varepsilon_0}^\alpha + L_2 \parallel h \parallel_{L^1}^\beta \varepsilon_m^{\beta\beta 1}{}_0) < m_{\varepsilon_0}.
\end{aligned}$$

This is impossible . Thus $t_* = 1$ and we conclude that $\varphi(t) \equiv \psi(t)$ for $t \in [0, 1]$. The proof is complete . □

4 . Proof of Theorem 2 . 4

Define an operator $H : C(I, R^+) \to C(I, R^+)$ by

$$Hx(t) = \frac{1}{\Gamma(q)} \int_0^t (t - s)^{q-1} (as^{-p1} + bs^{-p}2) \sup_{\in_r [0,s]} x(r) ds,$$

where $p1, p2 \in [0, q)$ are constants and $a = L_1, b = L_2 \parallel h \parallel L^1$. **Lemma 4 . 1 .** *There exist an increasing function* $b \in C(I, R^+)$ *and a* $\delta \in (0, 1)$ *such*

$$that Hb(t) \leq \delta b(t).$$

Proof . We choose a positive number $\eta \in I$ such that

$$\frac{a\eta^{q-p1} B(q, 1 - p1)}{\Gamma(q)} + \frac{b\eta^{q-p2} B(q, 1 - p2)}{\Gamma(q)} + a\eta^{q-p1} + b\eta^{q-p2} < 1,$$

 where $B(\cdot,\cdot)$ is the Beta function $B(x,y)=\int_0^1(1-s)^{x-1}s^{y-1}ds$. Let

$$\delta=\frac{a\eta^{q-p1}B(q,1-p1)}{\Gamma(q)}+\frac{b\eta^{q-p2}B(q,1-p2)}{\Gamma(q)}+a\eta^{q-p1}+b\eta^{q-p2}$$

and define an increasing function $b:I\to\mathbb{R}$ by

$$b(t)=\begin{cases}1, & \text{if} t\in[0,\eta],\\ e^{(t-\eta)/\eta}, & \text{if} t\in(\eta,1].\end{cases}$$

We claim that $Hb(t)\le\delta b(t)$ for $t\in[0,1]$. For $t\in[0,\eta]$, recalling that $B(x,y)=$

$$\int_0^1(1-s)^{x-1}s^{y-1}ds, \text{wehave}$$

$$t$$

$$Hb(t)=\frac{1}{\Gamma(q)}\int_0(t-s)^{q-1}(as^{-p1}+bs^{-p}2)ds$$

$$=\frac{a}{\Gamma(q)}t^{q-p1}\int_0^1(1-z)^{q-1}z^{1-p1-1}dz+\frac{b}{\Gamma(q)}t^{q-p2}\int_0^1(1-z)^{q-1}z^{1-p2-1}dz$$

$$=\frac{aB(q,1-p1)}{\Gamma(q)}t^{q-p1}+\frac{bB(q,1-p2)}{\Gamma(q)}t^{q-p2}$$

$$\le\frac{aB(q,1-p1)}{\Gamma(q)}\eta^{q-p1}+\frac{bB(q,1-p2)}{\Gamma(q)}\eta^{q-p2}<\delta b(t).$$

For $t\in(\eta,1]$, we have

$$Hb(t)=\frac{1}{\Gamma(q)}\int_0^t(t-s)^{q-1}(as^{-p1}+bs^{-p}2)b(s)ds$$

$$=\frac{1}{\Gamma(q)}\int_0^\eta(t-s)^{q-1}(as^{-p1}+bs^{-p}2)ds$$

$$t$$

$$+\frac{1}{\Gamma(q)}\int_\eta(t-s)^{q-1}(as^{-p1}+bs^{-p}2)e\frac{s-\eta}{\eta}{}_{ds}$$

$$\le\frac{1}{\Gamma(q)}\int_0^\eta(\eta-s)^{q-1}(as^{-p1}+bs^{-p}2)ds$$

$$+\frac{1}{\Gamma(q)}\int_\eta^t(t-s)^{q-1}(as^{-p1}+bs^{-p}2)e\frac{s-\eta}{\eta}{}_{ds}$$

$$\le\frac{a\eta^{q-p1}B(q,1-p1)}{\Gamma(q)}+\frac{b\eta^{q-p2}B(q,1-p2)}{\Gamma(q)}$$

$$t$$

$$+\frac{1}{\Gamma(q)}\int_\eta(t-s)^{q-1}(as^{-p1}+bs^{-p}2)e^{-t_-^{-s}{}_\eta}dse\frac{t-\eta}{\eta}$$

$$\le[\frac{a\eta^{q-p1}B(q,1-p1)}{\Gamma(q)}+\frac{b\eta^{q-p2}B(q,1-p2)}{\Gamma(q)}+a\eta^{q-p1}+b\eta^{q-p2}]e\frac{t-\eta}{\eta}$$

$$=\delta b(t).$$

The proof is complete . □ *Proof of Theorem 2 . 4 .* As in the proof of Theorem 2 . 1 , we prove the operator F
admits a fixed point . Define the norm $\|\cdot\quad\|b$ in $C(I,X)$, for $u\in C(I,X)$, by

$$\| u \| b = \max\{\frac{1}{b(t)} \| u(t) \|: t \in I\}.$$

 Then the norms $\| \cdot \| C$ and $\| \cdot \| b$ are equivalent . Let $P = \sup_{x \in X} \| g(x) \|$,

$$Q = \frac{1}{1-\delta}(\| x_0 \| + P + \frac{C}{\Gamma(q+1)} + \frac{L_2 K B(q, 1-p2)}{\Gamma(q)}),$$

and $B_Q = \{u \in C(I, X) : \| u \| b \leq Q\}$. For $u \in B_Q$, noting the assumption (F 2 ') , we have

$$\| [\theta u](t) \| \leq \int_0^t \| k(t, r, u(r)) \| dr \leq \| h \| L^1 \sup_{r \in [0,t]} \| x(r) \| + K.$$

By the assumption (F 1 ') and Lemma 4 . 1 , for $u \in B_Q$, we obtain

$$\| Fu(t) \| \leq \| x_0 \| + P + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \| f(s, u(s), [\theta u](s)) \| ds$$

$$\leq \| x_0 \| + P + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} (L_1 s^{-p1} \| u(s) \| + L_2 s^{-p2} \| \theta u(s) \| + C) ds$$

$$\leq \| x_0 \| + P + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} (L_1 s^{-p1} + L_2 \| h \| L^1 s^{-p2}) r \sup_{\in [0,s]} \| u(s) \| ds$$

$$t$$

$$+ \frac{1}{\Gamma(q)} \int_0 (t-s)^{q-1} (L_2 K s^{-p} 2 + C) ds$$

$$\leq \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} (L_1 s^{-p1} + L_2 \| h \| L^{1 s^{-p2}}) b(s) ds \| u \| b$$

$$+ \| x_0 \| + P + \frac{C}{\Gamma(q+1)} + \frac{L_2 K B(q, 1-p2)}{\Gamma(q)}$$

$$\leq \delta b(t) \| u \| b + \| x_0 \| + P + \frac{C}{\Gamma(q+1)} + \frac{L_2 K B(q, 1-p2)}{\Gamma(q)}.$$

Thus

$$\| Fu \| b \leq \delta Q + \| x_0 \| + P + \frac{C}{\Gamma(q+1)} + \frac{L_2 K B(q, 1-p2)}{\Gamma(q)} = Q.$$

Thisimplies$F(B_Q) \subset B_Q$.

Similar arguments as in the proof of Theorem 2 . 1 show that A is completely continuous and B is contraction mapping . Thus , by Lemma 3 . 1 , we conclude that F has a fixed point in B_Q. Thus there exists a solution of Cauchy problem (1 . 1) -
(1 . 2) . The proof is complete . □

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