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Existence of solutions for quasilinear degenerate elliptic equations *

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Abstract

In this paper , we study the existence of solutions for quasilinear de -

generate$^{\text{elliptic}}_{\text{isaLeray}-\text{Lions}}$equations$^{\text{ofthe}}_{\text{operatorfrom}}$$W_0^{\text{form}_{1,p}}(^{A}_{,\Omega}{}^{(u)}_{w)} +^{g(x\ ,}_{\text{toits dual.}}{}^{u,\nabla u)}_{\text{On}} =$the^{h,}nonlinear$^{\text{where}A}$

term $g(x, s, \xi)$, we assume growth conditions on ξ, not on s, and a sign condition on s.

1 Introduction

Let Ω be a bounded open subset of $\mathbb{R}^N, p$ be a real number with $1 < p < \infty$, and $w = \{w_i(x)\}0 \leq i \leq N$ be a vector of weight functions on Ω; i . e . each $w_i(x)$ is a measurable a . e . strictly positive function on Ω, satisfying some integrability conditions (see section 2) . Let $X = W_0^{1,p}(\Omega, w)$ be the weighted Sobolev space associated with the vector w. Assume :

(A 0) The norm

$$||| u ||| X = (\sum_{N}^{i=1} \int_\Omega | \partial_\partial^{u(x)}{}_{x_i} |^p w_i(x)dx)1/p$$

is equivalent to the usual norm on X; see (2 . 2) below . (A 1) There exists a weight function $\sigma(x)$ on Ω and a parameter $q, 1 < q < \infty$, such that the Hardy inequality ,

$$(\int_\Omega | u(x) |^q \sigma dx)1/q \leq c(\sum_{N}^{i=1} \int_\Omega | \partial_\partial^{u(x)}{}_{x_i} |^p w_i(x)dx)1/p$$

holds for every $u \in X$ with a constant $c > 0$ independent of u. Moreover , the imbedding $X \to L^q(\Omega, \sigma)$ is compact .

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Let A be the nonlinear operator from X into the dual X^* defined as

$$Au = -\mathrm{div}(a(x,u,\nabla u)), \tag{1.1}$$

where $a(x,s,\xi)=\{a_i(x,s,\xi)\}, 1\le i\le N:\Omega\times\mathbb{R}\times\mathbb{R}^N\to\mathbb{R}^N$ is a Carath $\acute{e}$ odory vector - valued function . (A 2) We assume that

$$\mid a_i(x,s,\xi)\mid\le c_1 1i_w{}^{/p}(x)[k(x)+\sigma^{1/p'}\mid s\mid p^{q\prime}+\sum_{j=1}^{N} 1j_w{}^{/p'}(x)\mid\xi_j\mid^{p-1}],$$

for$_{k(x)}$a.e$\in._{L^p}x\prime^{\in}_{(\Omega)}1^p_{\Omega(,}all^{(s\ \ ,\ \xi)}_{+p^{1\prime}=\ \ 1}$) $\in$ $\mathbb{R}$and $\times^{\mathbb{R}^N}_{\text{some constant}}{}^{,}$ $^{\text{all}i}$ $=1^{;}_{1_c}$..>., 0.$^{N,}_{\text{Here}}$ $^{\text{some}}\sigma$and$^{\text{function}}q$are

as in (A 1) . (A 3) For a . e .$x\in\Omega$, all $(s,\xi)\in\mathbb{R}\times\mathbb{R}^N$ and some constant $c_0>0$, we assume

that

$$a(x,s,\xi).\xi\ge c_0\sum_{i=1}^{N} w_i(x)\mid\xi i\mid^{p}.$$

Recently , Drabek , Kufner and Mustonen [5] proved that under the hypotheses

$(^{\text{A0}--\text{A3})}_{\text{with}}$ $_{\text{the}}$andcertainequationmonotonicity$_{Au=h,h\in X*}$ $^{\text{conditions},}$ $_{\text{has@least}}$ $^{\text{the}}$Dirichlet$^{\text{problem}}_{\text{1solution}}uinW_0^{\text{associated}1,p}(\Omega,w)$.

See also [1] , where A is of the form – div $(a(x,u,\nabla u))+a_0(x,u,\nabla u)$.

The purpose in this paper , is to prove the same result for the general non - linear elliptic equation

$$Au+g(x,u,\nabla u)=h, h\in X^*$$

where g is a nonlinear lower - order term having natural growth (of order p) with respect to $\mid\nabla u\mid$. Regarding $\mid u\mid$, we do not assume any growth restrictions . However , we assume the “ sign condition ”

$$g(x,s,\xi).s\ge 0.$$

More precisely , we prove in theorem 3 . 1 an existence result for the problem

$u\in W_0^{1,p}(\Omega,)^{Au}_{w}$, $+^{g(x\ \ ,}_{g(x\ \ ;}u,u,\nabla^{\nabla u)}_{u)\ \in}=L^{h\,\text{in}}_{1\,(\Omega),}\mathcal{D}'^{(\Omega\ ),}_{g(x,}{}_{u,}\nabla u)u\in L^1(\Omega)$. (

However$^{\text{Itturnsout}}_{,}$for$^{\text{that}}$aforasolutiongeneral$v\in W^{1,p0}_{u}$of$(\Omega^{\text{this}}_{,}w)$, system$^{,}_{g(x,v,\nabla\ \ v)}$ $^{\text{the}}$term$^{g(\ \ \ x,\ \ \ u,}_{\text{canbevery}}\nabla u)$isinsingular$^{L^1(\Omega)}_{(}$

example[3]where$w\equiv$

Let us point out that more work in this direction can be found in [7] where the authors have studied the existence of bounded solutions for the degenerate elliptic equation

$$Au-c_0\mid u\mid^{p-2}u=h(x,u,\nabla u),$$

with some more general degeneracy , under some additional assumptions on h and $a(x, s, \xi)$. When $w \equiv 1$ (the non weighted case) existence results for the problem (1 . 2) have been shown in [3] .

The present paper is organized as follows : In section 2 , we give some prelim - inaries and we prove some technical lemmas concerning convergence in weighted Sobolev spaces . In section 3 , we state our general result which will be proved in section 4 . Section 5 is devoted to an example which illustrates our abstract hypotheses . Note that , in the proof of our main result , many ideas have been adapted from Bensoussan et al . [3] .

2 Preliminaries

Weighted Sobolev spaces . Let Ω be a bounded open subset of $\mathbb{R}^N (N \geq 1)$, let $1 < p < \infty$, and let $w = \{w_i(x)\}, 0 \leq i \leq N$ be a vector of weight functions ; i . e . every component $w_i(x)$ is a measurable function which is strictly positive a . e . in Ω. Further , we suppose in all our considerations that for $0 \leq i \leq N$,

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$$w_i \in L^1_{\text{loc}}(\Omega) \quad \text{and} \quad w_i^{-p-1} \quad \in L^1_{\text{loc}}(\Omega). \tag{2.1}$$

We define the weighted space with weight γ on Ω as

$$L^p(\Omega, \gamma) = \{u = u(x) : \quad u\gamma^{1/p} \in L^p(\Omega)\}.$$

In this space , we define the norm

$$\| u \|_{p,\gamma} = (\int_\Omega | u(x) |^p \gamma(x) dx) 1/p.$$

We denote by $W^{1,p}(\Omega, w)$ the space of all real - valued functions $u \in L^p(\Omega, w_0)$ such that the derivatives in the sense of distributions satisfy

$$\partial^{\partial u}{}_{x_i} \in L^p(\Omega, w_i) \text{foralli} = 1, ..., N.$$

This set of functions forms a Banach space under the norm

$$\| u \| 1, p, w = (\int_\Omega | u(x) |^p w_0(x) dx + \sum_N^{i=1} \int_\Omega | \partial_\partial^{u(x)}{}_{x_i} |^p w_i(x) dx) 1/p. \tag{2.2}$$

To deal with the Dirichlet problem , we use the space

$$X = W_0^{1,p}(\Omega, w)$$

$C_0^{\text{defined}\infty}(\Omega)$asisdense$^{\text{theclosure}}inW^{1,p0}{}_{\text{of}}(^{C\infty 0(\Omega)}_{\Omega,\ w)}and^{\text{with}}$(respect$^{\text{to}}_{X,\|.\|1,p,w)}is^{\text{the}}$anorm(2.reflexive2$_{\text{Banach}}$). Note$^{\text{that,}}_{\text{space}}$.

We recall that the dual space of the weighted Sobolev spaces $W_0^{1,p}(\Omega, w)$ is equivalent to $W^{-1,p'}(\Omega, w^*)$, where $w^* = \{w^*i = 1i_w{}^{-p'}\}i = 0, ..., N$, and p' is the conjugate of p i . e .$p' = pp_-1$. For more details , we refer the reader to [6] .

Definition . Let X be a reflexive Banach space . An operator B from X to the dual X^* satisfies property (M) if for any sequence $(u_n) \subset X$ satisfying $u_n \rightharpoonup u$ in X weakly , $B(u_n) \rightharpoonup \chi$ in X^* weakly and lim $\sup_{n\to\infty} \langle Bu_n, u_n\rangle \leq \langle \chi, u\rangle$ then

$$1\text{has}\chi = B(u).$$

Now we state the following assumption . (H 1) The expression

$$||| \ u \ ||| \ X = (\sum_{N}^{i=1} \int_\Omega \mid \partial_{\partial}^{u(x)}{}_{x_i} \mid^p w_i(x)dx)^{1/p} \tag{2.3}$$

is a norm defined on X and is equivalent to the norm (2 . 2) . Note that $(X, ||| \ . \ ||| \ X)$ is a uniformly convex (and thus reflexive) Banach space .
There exist a weight function σ on Ω and a parameter $q, 1 < q < \infty$, such that

$$\sigma^{1q'_-} \in L^1(\Omega), \tag{2.4}$$

with $q' = qq_{-}1$ and such that the Hardy inequality ,

$$(\int_\Omega \mid u(x) \mid^q \sigma dx)1/q \leq c(\sum_{N}^{i=1} \int_\Omega \mid \partial_{\partial}^{u(x)}{}_{x_i} \mid^p w_i(x)dx)1/p, \tag{2.5}$$

holds for every $u \in X$ with a constant $c > 0$ independent of u. Moreover , the imbedding

$$X \to L^q(\Omega, \sigma), \tag{2.6}$$

determined by the inequality (2 . 5) is compact .

Now we state and prove the following technical lemmas which are needed later .

Lemma 2 . 1 *Let* $g \in L^r(\Omega, \gamma)$ *and le t* $gn \in L^r(\Omega, \gamma)$, *with* $\parallel gn \parallel_{r,\gamma} \leq c, 1 < r < \infty$. *If* $gn(x) \to g(x)$ *a . e . in* Ω, *then* $gn \rightharpoonup g$ *in* $L^r(\Omega, \gamma)$, *where* $\rightharpoonup$ *denotes weak convergence and* γ *is a weight function on* Ω.

Proof . Since $gn\gamma^{1/r}$ is bounded in $L^r(\Omega)$ and $gn(x)\gamma^{1/r}(x) \to g(x)\gamma^{1/r}(x)$, a . e . in Ω, then by [1 1 , Lemma 3 . 2] , we have

$$gn\gamma^{1/r} \rightharpoonup g\gamma^{1/r} \quad \text{in} L^r(\Omega).$$

Moreover for all $\phi \in L^{r'}(\Omega, \gamma^{1-r'})$, we have $\phi\gamma^{1-_r} \in L^{r'}(\Omega)$. Then $\int_\Omega gn\phi dx \to \int_\Omega g\phi dx$, i . e $.gn \rightharpoonup g$ in $L^r(\Omega, \gamma)$.

Lemma$_{chitzian,}2.2 with Assume_{F(0)=0}{}^{that}_{.}(^{H1)}_{Letu\in} W_0^{holds_{1,p}.}{}_{(\Omega,}{}^{Let}w).F_{Then}: \quad \mathbb{R}^{\to}_{F\ (u)}\mathbb{R}_{\in} W_0^{beuniformly_{1,p}}(\Omega, w). \quad Lips^{-}_{M}$

o ver , if the s e t D *of dis continuity points of* F' *is finite , then*

$$\partial(F\partial^\circ_{x}{}_i u) = \begin{cases} F'(u)\partial^{\partial u}{}_{x_i} \quad a.e. \quad in\{x \in \Omega : u(x) element - negationslash D\} \\ 0 \quad a.e. \quad in\{x \in \Omega : u(x) \in D\}. \end{cases}$$

Remark . The previous lemma is a generalization of the corresponding in [8 , pp . 1 5 1 - 1 52] , where $w \equiv 1$ and $F \in C^1(\mathbb{R})$ and $F' \in L^\infty(\mathbb{R})$, and of the corresponding in [2] , where $w_0 \equiv w_1 \equiv \cdots \equiv w_N$ is some weight function ,

functions$^{F\in C^1(\mathbb{R})}_{\text{in}}$ $^{\text{and}}W_0^{F_{p,1}\,\prime}{}_{(\Omega,}^{\in} L^{\infty\,(\mathbb{R})}_{w)}{}^{\text{.}}_{\text{can}}{}_{\text{be}}$ $^{\text{Also}}$notethattruncated. the previous lemma implies that

Proof of Lemma 2 . 2 First , note that the proof of the second part of Lemma

thecase$F \in C^1(\mathbb{R})^{W_0^{\text{dense}_{1,p}}}_{\text{2.2isidenticaltothe}}$ $(\Omega^{\text{in}}_{,} W^{0^{1,\,p(\Omega,}}_{w).\text{ Passing}}$ $^{w),}$ corresponding$_{\text{and}F'\in L^\infty}$ $^{\text{in non}}_{(\ \mathbb{R})}$ $.\text{Let}$weighted$^{\text{case}}_{u\in W_0^1}$ $^{\text{in}}_{,p(\Omega}$ $^{[8].}_{,w\ )}$. to there$^{\text{existsasequence}}_{\text{asubsequence}}$ $,\text{wecan}\,u_n\text{assume}\in$

Then

$F(u_n) \to F(u)$ a . e . in Ω. (2.7) On the other hand , from the relation $\mid F(u_n) \mid^p w_0 \leq \| F' \|_\infty \mid u_n \mid^p w_0$ and

$$\mid \partial F(u_n\partial^{)}_{x_i} \mid^p w_i =\mid F'(u_n)\partial^{u_n}_{\partial}{}_{x_i} \mid^p w_i \leq M \mid \partial^{u_n}_{\partial}{}_{x_i} \mid^p w_i,$$

we deduce that the function $F(u_n)$ remains bounded in $W_0^{1,p}(\Omega, w)$. Thus , going to a further subsequence , we obtain

$$F(u_n) \rightharpoonup v\text{in}W_0^{1,p}(\Omega, w). \tag{2.8}$$

Thanks to (2 . 7) , (2 . 8) and (2 . 6) we conclude that

$$v = F(u) \in W_0^{1,p}(\Omega, w).$$

We now turn our attention to the general case . Taking convolutions with

mollifiersbytheρnfirstin$^{\mathbb{R}}_{\text{case}}$$^{,}_{\text{we}}$ $^{\text{we}}$have$^{\text{have}}$ $F_n F_n(u) = \in W_0^{F_1^*,p\rho n,}{}_{(\Omega,}$ $F^n_{w)}$. $\in C^1(\mathbb{R})$Since$_{F_n}$and$_{\to F}F'n_{\text{uniformly}} \in L^\infty(\mathbb{R})_{\text{in}}$.

compact$^{;}_{\text{bounded}}$wein$W_0^{\text{have}_{1,p}}(\Omega^{Fn(u)}_{,\ w),}$ $\to$ then$^{F(u)}_{\text{for}}$aa.e. inΩ. Onsubsequence$^{\text{the}}_{F_n(u)}$other$^{\text{hand}}_{\rightharpoonup \bar{v}\text{in}}$$W^{1,p0}_{,}$(

a . e . in Ω(due to (2 . 6)) , then

$$\bar{v} = F(u) \in W_0^{1,p}(\Omega, w).$$

The following lemmas follow from the previous lemma .

$\mathbb{R}^{\textbf{Lemma}_+}$*, be2the*$^{.3}_{\textit{usual}}$ $^{\textit{Assume}}$*that*$^{(H}_{\textit{truncation}\,\textit{then}}$ $^{1)\ \ \textit{holds.}}_{T_k(u)}$ *Let*$^{u}_{\in W_0^1,p(\Omega\ ,}$ $^{\in W_0 1,p(\Omega,\ \ w),\textit{and}}_{w).\textit{Moreover},}$ $le_{we}tT_{khave}(u), k \in$

$$T_k(u) \to u \textit{ s tro ngly in } W_0^{1,p}(\Omega, w).$$

Lemma$_{andu^-=}2^{.4}_{\max(}{}^{Assume}{}_{-u,0)}that_{liein}(1^{0}_{H_W 1),p(\Omega,\ w).}{}^{holds.}{}^{Let u\in W_0^{1,p}(\ \Omega,}_{\ \ Moreover,\ \ we}w), thenhave$

$$\partial^{(u^+)}_{\partial}{}_{x_i} = \{ \quad \partial x_i^{0,}{}_{\partial u},$$

$$\partial^{(u^-)}_{\partial}{}_{x_i} = \{ \quad 0_-, \partial^{\partial u}_{x_i}$$

Lemma$_{such^{that}_{andu-n\rightharpoonup}}W_0^{1,p}(\Omega,w)^{,p(\Omega,w).}_{2_{u_{-}{}^{u}_{n}1^0\rightharpoonup uweaklyweakly_{inW}in}}{}^{.5}\quad _{Assumethat(H1)holds.\cdot}Let^{(u_n\quad)}_{Then,}{}_{n_u^+}be_{\rightharpoonup}a_{u^+}sequenceweaklyof_i$

Proof . Since $u_n \rightharpoonup u$ in $W_0^{1,p}(\Omega, w)$ and by (2 . 8) we have for a subsequence $u_n \to u$ in $L^q(\Omega, \sigma)$ and a . e . in Ω. On the other hand ,

$$\||\ u_n\ \||\ pX = \sum_{N}^{i=1} \int_\Omega \mid \partial^{u_n}_{\partial}{}_{x_i} \mid^p w_i \geq \sum_{N}^{i=1} \int_{\{u_n \geq 0\}} \mid \partial^{u_n}_{\partial}{}_{x_i} \mid^p w_i$$

$$= \sum_{N}^{i=1} \int_\Omega \mid \partial^{n_u^+}_{\partial}{}_{x_i} \mid^p w_i = \||\ n_u^+\ \||^p_X \ .$$

$\text{Then}(n_u^+\text{Similarly}^{)}_{,}\text{is}_{\text{we}} - \quad -^{u_n\rightharpoonup u}_{\text{bounded}^{\text{in}}_{\text{provethat}}}W_0^{1,p}(\Omega,w)\text{in}^{\text{hence by}}_{W}{}^{\quad (2.6),}_{\ 0^{1,p(\Omega\ ,w)}}.n_u^+ \rightharpoonup u^+\text{in}W_0^{1,p}(\Omega,w).$

3 Main result

Let A be the nonlinear operator from $W_0^{1,p}(\Omega, w)$ into the dual $W^{-1p'_,}(\Omega, w^*)$ defined as

$$Au = -\text{div}(a(x, u, \nabla u)),$$

where $a : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carath $\acute{e}$ odory vector - function satisfying the following assumptions :

$$\text{(H2)} \quad \text{For} i = 1, ..., N,$$

$$\mid a_i(x, s, \xi) \mid \leq \beta 1 i_w{}^{/p}(x)[k(x) + \sigma^{1/p'} \mid s \mid p^{q_{\prime}} + \sum_{j=1}^{N} 1 j_w{}^{/p'}(x) \mid \xi_j \mid^{p-1}], \quad (3.1)$$

$$[a(x, s, \xi) - a(x, s, \eta)](\xi - \eta) > 0 \quad \text{forall} \xi \neq \eta \in \mathbb{R}^N, \quad (3.2)$$

$$a(x, s, \xi).\xi \geq \alpha \sum_{i=1}^{N} w_i \mid \xi i \mid \ {}^{p}_{,} \quad (3.3)$$

where $k(x)$ is a positive function in $L^{p'}(\Omega)$ and α, β are positive constants .

(H 3) $g(x,s,\xi)$ is a Carath $\acute{e}$ odory function satisfying

$$g(x,s,\xi)s \geq 0, \tag{3.4}$$

$$\mid g(x,s,\xi) \mid \leq b(\mid s \mid)(\sum_{i=1}^{N} w_i \mid \xi i \mid^p +c(x)), \tag{3.5}$$

where $b : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is a continuous increasing function and $c(x)$ is positive function which in $L^1(\Omega)$.

For the nonlinear Dirichlet boundary - value problem (1 . 2) , we st ate our main result as follows .

Theorem 3 . 1 *Under assumptions (H 1) - (H 3) and* $h \in W^{-1p'_,}(\Omega, w^*)$, *there exists a s o lution of (1 . 2) .*

Remarks . (1) Theorem 3 . 1 , generalizes to weighted case the analogous statement in [3] .

$(^{2)\mathrm{The}}_{\mathrm{ness}}$ of assumption$_{g in W_0^{1,p}(\Omega,}$ $^{(\ 2.4)}_{w).}$ appear Thus, to be when g necessary$_{\equiv 0,\mathrm{we do}}$ only$_{\mathrm{not}}$ for$_{\mathrm{need}}$ proving$_{\mathrm{assumption}}$ the bounded(

(3) If we assume that $w_0(x) \quad \equiv \quad 1$ and that there exists $\nu \in]N_P, \infty[\cap[^1_{P-1}, \infty[$ such that $w^-i^\nu \in L^1(\Omega)$ for all $i = 1, ..., N,$ (which is an integrability condition , stronger than (2 . 1)) , then

$$||| u ||| X = (\sum_{N}^{i=1} \int_\Omega \mid \partial^{u(x)}_{\partial}{}_{x_i} \mid^p w_i(x)dx)1/p$$

is a norm defined on $W_0^{1,p}(\Omega, w)$ and equivalent to (2 . 2) . Also we have that

$$W_0^{1,p}(\Omega, w) \rightarrow L^q(\Omega)$$

for$_{p1}$ $1_= p\nu^{\nu+1}_{\leq q} <^{p_1^*, p\nu}_{.}$ $_{\mathrm{Where}}$ $p^*{}^1_< Np1^{N-p1}_{N=(\nu+1)}) Np\nu^{N(\nu+1)-p\nu}_{,=\mathrm{and} q\geq 1 \mathrm{is}}$ $_{\mathrm{is}}$arbitrary$_{\mathrm{the Sobolev}}$ for $p\nu \geq N(\nu +$ conjugate$_{\mathrm{of}}$ $1)_{p1}$ where($_{\mathrm{see}}$

[6]) . Thus the hypotheses (H 1) is verified (for $\sigma \equiv 1$).

For Theorem 3 . 1 , we needed the following lemma . **Lemma**$_{quencein}$ $3.2 W_0 Assume_{,1p(\Omega, w)} such^{that} (^{H1)}_{that} u \quad \rightharpoonup^n_{and}$ $(^{H2)}_{u\ weakly}{}^{are} satisfied^{\ and}_{, in W_0^{1,p}(\Omega,} w) le_{and}$ *t* (u_n) *be a s e -*

$$\int_\Omega [a(x, u_n, \nabla u_n) - a(x, u_n, \nabla u)] \nabla(u_n - u)dx \rightarrow 0. \quad (3.6)$$

Then, $u_n \rightarrow u in W_0^{1,p}(\Omega, w).$

Proof . Let $D_n = [a(x, u_n, \nabla u_n) - a(x, u_n, \nabla u)]\nabla(u_n - u)$. Then by (3.2), D_n is a positive function and by (3.6)$D_n \to 0$ in $L^1(\Omega)$. Extracting a subsequence still denoted by u_n, and using (2 . 6) , we can write

$$\begin{cases} u_n \to u & \text{a.e.in}\Omega \\ D_n \to 0 & \text{a.e.in}\Omega. \end{cases}$$

Then , there exists a subset B of Ω, of zero measure , such that for $x \in \Omega \setminus B$,

$$| u(x) |< \infty, | \nabla u(x) |< \infty, | k(x) |< \infty, w_i(x) > 0 \text{and} u_n(x) \to u(x), D_n(x) \to$$

0 . We set $\xi_n = \nabla u_n(x), \xi = \nabla u(x)$. Then

$$D_n(x) = [a(x, u_n, \xi_n) - a(x, u_n, \xi)](\xi_n - \xi)$$

$$N \quad N$$

$$\geq \alpha \sum w_i \mid \xi_n^i \mid^p + \alpha \sum w_i \mid \xi^i \mid^p$$

$$i = 1 \quad i = 1$$

$$N \quad N$$

$$-\sum_{i=1} \beta 1 i_w{}^{/p}[k(x) + \sigma^{1/p'} \mid u_n \mid' p^q \quad + \sum_{j=1} 1 j_w{}^{/p'} \mid \xi_n^j \mid^{p-1}] \mid \xi^i \mid \tag{3.7}$$

$$N \quad N$$

$$-\sum \beta 1 i_w{}^{/p}[k(x) + \sigma^{1/p'} \mid u_n \mid p^{q\prime} + \sum 1 j_w{}^{/p'} \mid \xi^j \mid^{p-1}] \mid \xi_n^i \mid$$

$$i = 1 \quad j = 1$$

$$N \quad N \quad N$$

$$\geq \alpha \sum w_i \mid \xi_n^i \mid^p - c_x[1 + \sum 1 j_w{}^{/p'} \mid \xi_n^j \mid^{p-1} + \sum 1 i_w{}^{/p} \mid \xi_n^i \mid]$$

$$i = 1 \quad j = 1 \quad i = 1$$

where c_x is a constant which depends on x, but does not depend on n. Since $u_n(x) \to u(x)$ we have $| u_n(x) |\leq M_x$ where M_x is some positive constant . Then by a standard argument $| \xi_n |$ is bounded uniformly with respect to n; indeed (3 . 7) becomes ,

$$D_n(x) \geq \sum_{N}^{i=1} \mid \xi_n^i \mid^p (\alpha w_i - N c_x | \xi_n^i \mid^p - c x 1 i_w{}^{/p'} |_{\xi_n^i |} - c|^{x^{w\,1/p}_{\ \ i}}_{\xi_n^i |^{p-1}}).$$

If $| \xi_n |\to \infty$(for a subsequence) there exists at least one i_0 such that $| \xi_n^{i_0} |\to \infty$, which implies that $D_n(x) \to \infty$ which gives a contradiction .

Let now ξ^* be a cluster point of ξ_n. We have $| \xi^* |< \infty$ and by the continuity of a with respect to the two last variables we obtain

$$(a(x, u(x), \xi^*) - a(x, u(x), \xi))(\xi^* - \xi) = 0.$$

In view of (3 . 2) we have $\xi^* = \xi$. The uniqueness of the cluster point implies
$\nabla u_n(x) \to \nabla u(x)$ a . e . in Ω. Since the sequence $a(x, u_n, \nabla u_n)$ is bounded in $\prod_{i=1}^N L^{p'}(\Omega, w^*i)$ and
$a(x, u_n, \nabla u_n) \to a(x, u, \nabla u)$ a . e . in Ω, Lemma 2 . 1 implies

$$N$$

$$a(x, u_n, \nabla u_n) \rightharpoonup a(x, u, \nabla u) \quad \text{in} \quad \prod L^{p'}(\Omega, w^*i) \text{ and a . e . in } \Omega.$$

$$i = 1$$

We set $\bar{y}n = a(x, u_n, \nabla u_n)\nabla u_n$ and $\bar{y} = a(x, u, \nabla u)\nabla u.$ As in [4 , Lemma 5] we can write

$$\bar{y}n \to \bar{y} \quad \text{in} L^1(\Omega).$$

By (3 . 3) we have

$$\alpha \sum_{N}^{i=1} w_i \mid \partial_{\partial}^{u_n}{}_{x_i} \mid^p \leq a(x, u_n, \nabla u_n)\nabla u_n.$$

Let $z_n = \sum_{i=1}^N w_i \mid \partial_{\partial}^{u_n}{}_{x_i} \mid^p, z = \sum_{i=1}^N w_i \mid \partial^{\partial u}_{x_i} \mid^p, yn = \quad \bar{y}^n_\alpha$ and $y = \quad \bar{y}_\alpha.$ Then , by Fatou ' s theorem we obtain

$$\int_\Omega 2ydx \leq \liminf_{n \to \ \infty} \int_\Omega y + yn - \mid z_n - z \mid dx$$

i . e .$0 \leq -$ lim $\sup_{n\to\infty} \int_\Omega \mid z_n - z \mid dx$ then

$$0 \leq \lim_{n\to\infty} \inf \int_\Omega \mid z_n - z \mid dx \leq \lim_{n\to\infty} \sup \int_\Omega \mid z_n - z \mid dx \leq 0,$$

this implies ,

$$\nabla u_n \to \nabla u \quad \text{in} \prod_{i = 1}^{N} L^p(\Omega, w_i),$$

which with (2 . 3) completes the present proof .

4 Proof of Theorem 3 . 1

Step (1) The approximate problem . Let

$$g\varepsilon(x, s, \xi) = 1g +^{(x, \ s, \xi)}_{\varepsilon|g(x, \ s,} \xi) \mid$$

and consider the equation

$$A(u_\varepsilon)_{u_\varepsilon} +_\in g\varepsilon(xW^{, \ u_\varepsilon, \nabla u_\varepsilon)}_{0^1, p(\Omega, \ w)} = h \tag{4.1}$$

We define the operator $G_\varepsilon : \quad X \to X^*$ by

$$\langle G_\varepsilon u, v\rangle = \int_\Omega g\varepsilon(x, u, \nabla u)vdx.$$

Thanks to H ö lder ' s inequality , for all $v \in X$ and $\phi \in X,$

$$\begin{aligned} \mid \int_\Omega g\varepsilon(x, v, \nabla v)\phi dx \mid &\leq (\int_\Omega \mid g\varepsilon(x, v, \nabla v) \mid^{q'} \sigma^- q'_q dx)1/q'(\int_\Omega \mid \phi \mid^q \sigma dx)1/q \\ &\leq 1_\varepsilon(\int_\Omega \sigma^{1-q'} dx)1/q' \parallel \phi \parallel_{q,\sigma} \leq c_\varepsilon \mid\mid\mid \phi \mid\mid\mid \end{aligned}$$

(4 . 2) For the above inequality , we have used (2 . 4) and (2 . 6) .

Lemma 4 . 1 *The operator* $A + G_\varepsilon$ $: X \to X^*$ *is bounded , coercive , hemiconti - nous , and satisfies property (M) .*

In view of Lemma 4 . 1 , Problem (4 . 1) has a solution by a classical result [1 0 , Theorem 2 . 1 and Remark 2 . 1] . Since $g\varepsilon$ verifies the sign condition and using (3 . 3) , we obtain

$$\alpha \sum_{N}^{i=1} \int_\Omega w_i \mid \partial^{u_\varepsilon}_{\partial}{}_{x_i} \mid^p \leq \langle h, u_\varepsilon \rangle$$

i . e .$\alpha \mid\mid\mid u_\varepsilon \mid\mid\mid p \leq c \parallel h \parallel X* \mid\mid\mid u_\varepsilon \mid\mid\mid$. Then

$$\mid\mid\mid u_\varepsilon \mid\mid\mid \leq \beta 0, \tag{4.3}$$

where $\beta 0$ is some positive constant . Hence , we can extract a subsequence still denoted by u_ε such that ,

$$u_\varepsilon \rightharpoonup u \text{ in } W_0^{1,p}(\Omega, w) \text{ and a . e . in } \Omega.$$

Step (2) Convergence of the positive part of u_ε. We shall prove that

$$\varepsilon_u^+ \to u^+ \text{in} W_0^{1,p}(\Omega, w) \quad \text{strongly.}$$

Let $k > 0$. Define $k_u^+ \quad = u^+ \wedge k = \min\, \{u^+, k\}$. We shall fix k, and use the notation

$$z_\varepsilon = \varepsilon_u^+ - k_u^+.$$

Assertion :

$$\lim_{\varepsilon \to 0} \sup \int_\Omega [a(x, u_\varepsilon, \nabla \varepsilon_u^+) - a(x, u_\varepsilon, \nabla k_u^+)] \nabla (\varepsilon_u^+ - k_u^+)^+ dx \leq R_k \tag{4.4}$$

where$^{R}_{W_0^{1,p}(\Omega,}{}^{k}_{w)}$ $\to$ 0andas$_{\varepsilon_z^+}$$k_\in \to W_0 +^{\infty.}_{1,p}$ (Ω,Indeed$^{,\text{byLemmas}}_{w).\text{Multiplying}}$(4.1)2.3byand$^{2.4}_{\varepsilon_z^+\text{weobtain}}$, $^{\text{wehave}}$$z_\varepsilon$ $\in$

$$\langle A u_\varepsilon, \varepsilon_z^+ \rangle + \int_\Omega g\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon) \varepsilon_z^+ dx = \langle h, \varepsilon_z^+ \rangle.$$

If $\varepsilon_z^+ > 0$, we have $u_\varepsilon > 0$ and from (3.4)$g\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon) \geq 0$, then $\langle A u_\varepsilon, \varepsilon_z^+ \rangle \leq$

$$\langle h, \varepsilon_z^+ \rangle \text{i.e.}$$

$$\int_\Omega a(x, u_\varepsilon, \nabla u_\varepsilon) \nabla \varepsilon_z^+ dx \leq \langle h, \varepsilon_z^+ \rangle.$$

Since $u_\varepsilon = \varepsilon_u^+$ in $\{x \in \Omega : \varepsilon_z^+ > 0\}$ then

$$\int_\Omega a(x, u_\varepsilon, \nabla \varepsilon_u^+) \nabla \varepsilon_z^+ dx \leq \langle h, \varepsilon_z^+ \rangle.$$

Which implies

$$\begin{aligned} &\int_\Omega [a(x, u_\varepsilon, \nabla \varepsilon_u^+) - a(x, u_\varepsilon, \nabla k_u^+)] \nabla (\varepsilon_u^+ - k_u^+)^+ dx \\ &\quad \leq - \int_\Omega a(x, u_\varepsilon, \nabla k_u^+)] \nabla (\varepsilon_u^+ - k_u^+)^+ + \langle h, \varepsilon_z^+ \rangle. \end{aligned} \tag{4.5}$$

As $\varepsilon \to 0$, we have $\varepsilon_z^+ \to (u^+ - k_u^+)^+$ a. e. in Ω. However ε_z^+ is bounded in

$$W_0^{1,p}(\Omega, w); \text{hence}$$
$$\varepsilon_z^+ \rightharpoonup (u^+ - k_u^+)^+ \quad \text{in} W_0^{1,p}(\Omega, w).$$

Since $a(x, u_\varepsilon, \nabla k_u^+) \to a(x, u, \nabla k_u^+)$ in $\prod_{i=1}^N L^{p'}(\Omega, w^* i)$, by passing to the limit in ε in (4 . 5) , we obtain (4 . 4) with

$$R_k = -\int_\Omega a(x, u, \nabla k_u^+)]\nabla(u^+ - k_u^+)^+ + \langle h, (u^+ - k_u^+)^+ \rangle.$$

Because $(u^+ - k_u^+)^+ \to 0$ in $W_0^{1,p}(\Omega, w)$ as $k \to \infty$, we have $R_k \to 0$ as $k \to \infty$.

$$-\liminf_{\varepsilon \to 0} \int_\Omega [a(x, u_\varepsilon, \nabla \varepsilon_u^+) - a(x, u_\varepsilon, \nabla k_u^+)]\nabla(\varepsilon_u^+ - k_u^+)^- dx$$

$$\text{Wehave} 0 \leq z_\varepsilon^- \leq k, \text{i.e.} \quad z_\varepsilon^- \in L^\infty(\Omega) \text{andsince}^{\text{Lemma2.2,wehave} v_\varepsilon \in W_0^{1,p}(\Omega,w).\text{Multiplying}}_{\text{Indeed,weshallusethetestfunction} v_\varepsilon = \phi\lambda(z_\varepsilon^-)} z_\varepsilon^- \in W_0^{1,p(4.1)\text{by} v_\varepsilon}{}_{\text{with}\phi\lambda(s)}(\Omega, w), \text{hence}^{\text{we}}_{=s}$$

$$\int_\Omega a(x, u_\varepsilon, \nabla u_\varepsilon)\nabla z_\varepsilon^- \phi'_\lambda(z_\varepsilon^-)dx + \int_\Omega g\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon)\phi\lambda(z_\varepsilon^-)dx =$$

Define

$$E_\varepsilon = \{x \in \Omega : \varepsilon_u^+(x) \leq k_u^+(x)\} \quad \text{and} \quad F_\varepsilon = \{x \in \Omega : 0 \leq u_\varepsilon(x) \leq k_u^+(x)\}.$$
$$\text{Since} \phi\lambda(z_\varepsilon^-) = 0 \text{in} E_\varepsilon^c,$$
$$\int_\Omega g\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon)\phi\lambda(z_\varepsilon^-)dx = \int_{E_\varepsilon} g\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon)\phi\lambda(z_\varepsilon^-)dx.$$

When $u_\varepsilon \leq 0$, we have $g\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon) \leq 0$ and since $\phi\lambda(z_\varepsilon^-) \geq 0$, we obtain

$$\int_{E_\varepsilon} g\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon)\phi\lambda(z_\varepsilon^-)dx$$
$$\leq \int_{F_\varepsilon} g\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon)\phi\lambda(z_\varepsilon^-)dx$$
$$\leq \int_{F_\varepsilon} b(\mid u_\varepsilon \mid)[\sum_N^{i=1} w_i \mid \partial_\partial^{u_\varepsilon}{}_{x_i} \mid^p + c(x)]\phi\lambda(z_\varepsilon^-)dx$$
$$\leq b(k) \int_{F_\varepsilon} [\sum_N^{i=1} w_i \mid \partial_\partial^{u_\varepsilon}{}_{x_i} \mid^p + c(x)]\phi\lambda(z_\varepsilon^-)dx$$
$$\leq b({}_\alpha k) \int_{F_\varepsilon} a(x, u_\varepsilon, \nabla u_\varepsilon)\nabla u_\varepsilon \phi\lambda(z_\varepsilon^-)dx + b(k) \int_{F_\varepsilon} c(x)\phi\lambda(z_\varepsilon^-).$$

As in $[3, Theorem 1.1]$, we can show that

$$
\begin{aligned}
-2^1 \int_\Omega [a(x,u_\varepsilon,\nabla \varepsilon_u^+) - a(x,u_\varepsilon,\nabla k_u^+)]\nabla(\varepsilon_u^+ - k_u^+)^- \\
\leq \int_\Omega [a(x,u_\varepsilon,\nabla u_\varepsilon) - a(x,u_\varepsilon,\nabla \varepsilon_u^+)]\nabla k_u^+ \phi'_\lambda(k_u^+)dx + \langle -h, \phi\lambda(z_\varepsilon^-)\rangle \\
+ \int_\Omega a(x,u_\varepsilon,\nabla k_u^+)\nabla z_\varepsilon^- \phi'_\lambda(z_\varepsilon^-)dx + b(_\alpha k)\int_\Omega a(x,u_\varepsilon,\nabla \varepsilon_u^+)\nabla k_u^+ \phi\lambda(z_\varepsilon^-)dx \\
+b(_\alpha k)\int_\Omega a(x,u_\varepsilon,\nabla k_u^+)\nabla(\varepsilon_u^+ - k_u^+)\phi\lambda(z_\varepsilon^-)dx + b(k)\int_\Omega c(x)\phi\lambda(z_\varepsilon^-)dx,
\end{aligned}
$$

for $\lambda = b_{4\alpha^2}(k)^2$. For short notation, we rewrite the above inequality as

$$I_{\varepsilon k} \leq I_{\varepsilon k}^1 + I_{\varepsilon k}^2 + I_{\varepsilon k}^3 + I_{\varepsilon k}^4 + I_{\varepsilon k}^5.$$

Now, we extract a subsequence that satisfies the following two conditions :

$$a(x,u_\varepsilon,\nabla u_\varepsilon) \rightharpoonup \gamma 1 \quad \text{and} \quad a(x,u_\varepsilon,\nabla \varepsilon_u^+) \rightharpoonup \gamma 2 \quad \text{in} \prod_{i=1}^{N} L^{p'}(\Omega, w^* i). \tag{4.7}$$

Lemma 4.2 *For k fixed, as $\varepsilon \to 0$, the following statements hold :*

$$
\begin{aligned}
(a) I_{\varepsilon k}^1 \to I_k^1 = \int_\Omega [\gamma 1 - \gamma 2]\nabla k_u^+ \phi'_\lambda(k_u^+)dx + \langle -h, \phi\lambda((u^+ - k_u^+)^-)\rangle \\
(b) I_{\varepsilon k}^2 \to I_k^2 = \int_\Omega a(x,u,\nabla k_u^+)\nabla((u^+ - k_u^+)^-)\phi'_\lambda((u^+ - k_u^+)^-) \\
(c) I_{\varepsilon k}^3 \to I_k^3 = b(_\alpha k)\int_\Omega \gamma 2^{\nabla u_k^+}\phi\lambda((u^+ - k_u^+)^-)dx \\
(d) I_{\varepsilon k}^4 \to I_k^4 = b(_\alpha k)\int_\Omega a(x,u,\nabla k_u^+)\nabla(u^+ - k_u^+)\phi\lambda((u^+ - k_u^+)^-)dx \\
(e) I_{\varepsilon k}^5 \to I_k^5 = b(k)\int_\Omega c(x)\phi\lambda((u^+ - k_u^+)^-)dx
\end{aligned}
$$

In view of Lemma 4.2, $(u^+ - k_u^+)^- = 0$ and $\phi\lambda(0) = 0$, we have

$$\limsup_{\varepsilon \to 0} I_{\varepsilon k} \leq I_k^1 + I_k^2 + I_k^3 + I_k^4 + I_k^5 = \int_\Omega [\gamma 1(x) - \gamma 2(x)]\nabla k_u^+ \phi'_\lambda(k_u^+)dx.$$

Moreover, if $u_\varepsilon < 0$ we have $(u_\varepsilon)_k^+ = 0$, hence,

$$(a(x,u_\varepsilon,\nabla u_\varepsilon) - a(x,u_\varepsilon,\nabla \varepsilon_u^+))(u_\varepsilon)_k^+ = 0 \quad \text{a.e.}$$

which implies $(\gamma 1(x) - \gamma 2(x))k_u^+ = 0$, and so $\limsup_{\varepsilon \to 0} I_{\varepsilon k} \leq 0$; thus, (4.6) follows .

Assertion :

$$\varepsilon_u^+ \to u^+ \quad \text{in} W_0^{1,p}(\Omega, w) \quad \text{strongly.} \quad (4.8)$$

As in [3, theorem 1.1], from (4.4) and (4.6), we have

$$\begin{aligned}\limsup_{\varepsilon\to 0}\int_\Omega [a(x,u_\varepsilon,\nabla\varepsilon_u^+) - a(x,u_\varepsilon,\nabla u^+)]\nabla(\varepsilon_u^+ - u^+)\\ \le R_k + \int_\Omega[\gamma 2(x) - a(x,u,\nabla k_u^+)]\nabla(k_u^+ - u^+).\end{aligned}$$

Letting $k \to \infty$ and using lemma 3.2 we obtain (4.8).

Step (3) Convergence of the negative part of u_ε. As in the preceding step, we shall prove that $u_\varepsilon^- \to u^-$ in $W_0^{1,p}(\Omega,w)$ strongly. (4.9)

Assertion :

$$\limsup_{\varepsilon\to 0}\int_\Omega -[a(x,u_\varepsilon,-\nabla u_\varepsilon^-) - a(x,u_\varepsilon,-\nabla u_k^-)]\nabla(u_\varepsilon^- - u_k^-)^+ dx \le \tilde{R}_k, \quad (4.10)$$

where $\tilde{R}_k \to 0$ as $k \to +\infty$. Indeed, when we define $u_k^- = u^- \wedge k, y\varepsilon = u_\varepsilon^- - u_k^-$, and multiply (4.1) by y_ε^+, we obtain

$$\int_\Omega a(x,u_\varepsilon,\nabla u_\varepsilon)\nabla y_\varepsilon^+ dx + \int_\Omega g\varepsilon(x,u_\varepsilon,\nabla u_\varepsilon) y_\varepsilon^+ dx = \langle h, y_\varepsilon^+\rangle.$$

Since $y_\varepsilon^+ > 0$ implies $u_\varepsilon < 0$, from (3.4) we have $g\varepsilon(x,u_\varepsilon,\nabla u_\varepsilon) \le 0$. Hence $g\varepsilon(x,u_\varepsilon,\nabla u_\varepsilon)y_\varepsilon^+ \le 0$ a.e. in Ω. Then

$$\int_\Omega a(x,u_\varepsilon,\nabla u_\varepsilon)\nabla y_\varepsilon^+ dx \ge \langle h, y_\varepsilon^+\rangle.$$

Since $u_\varepsilon = -u_\varepsilon^-$ on the set $\{x\in\Omega : y_\varepsilon^+ > 0\}$, we can write

$$\int_\Omega a(x,u_\varepsilon,-\nabla u_\varepsilon^-)\nabla y_\varepsilon^+ dx \ge \langle h, y_\varepsilon^+\rangle,$$

which implies

$$\begin{aligned}-\int_\Omega[a(x,u_\varepsilon,-\nabla u_\varepsilon^-) - a(x,u_\varepsilon,-\nabla u_k^-)]\nabla(u_\varepsilon^- - u_k^-)^+ dx\\ \le \int_\Omega a(x,u_\varepsilon,-\nabla u_k^-)\nabla(u_\varepsilon^- - u_k^-)^+ - \langle h, y_\varepsilon^+\rangle.\end{aligned}$$

As $\varepsilon \to 0$ we have $y_\varepsilon^+ \to (u^- - u_k^-)^+$ a.e. in Ω. Since y_ε^+ is bounded in $W_0^{1,p}(\Omega,w), y_\varepsilon^+ \rightharpoonup (u^- - u_k^-)^+$ in $W_0^{1,p}(\Omega,w)$(for k fixed). Passing to the limit in ε we obtain (4.10) with

$$\tilde{R}_k = \int_\Omega a(x,u,-\nabla u_k^-)\nabla(u^- - u_k^-)^+ - \langle h,(u^- - u_k^-)^+\rangle.$$

Because $(u^- - u_k^-)^+ \to 0$ in $W_0^{1,p}(\Omega,w)$ as $k\to\infty$ we obtain that $\tilde{R}_k \to 0$ as

$$k \to \infty.$$

Assertion :

$$\limsup_{\varepsilon\to 0}\int_\Omega [a(x,u_\varepsilon,-\nabla u_\varepsilon^-)-a(x,u_\varepsilon,-\nabla u_k^-)]\nabla(u_\varepsilon^- - u_k^-)^- dx \le 0. \quad (4.11)$$

This can be done as in (4 . 6) by considering a test function $v_\varepsilon = \phi\lambda(y_\varepsilon^-)$. Finally combining (4 . 1 0) and (4 . 1 1) , we deduce as in (4 . 8) the assertion (4 . 9) .
Step (4) Convergence of u_ε. From (4 . 8) and (4 . 9) , we deduce that for a subse - quence ,

$u_\varepsilon \to u$ in $W_0^{1,p}(\Omega,w)$ and a . e . in Ω (4.12) $\nabla u_\varepsilon \to \nabla u$ a . e . in Ω, (4.13)
which implies

$$gg^{\varepsilon(x,u_\varepsilon,\nabla u_\varepsilon)\to\ \ g\ \ (x,\ u,\ \nabla u)}_{\varepsilon(x,u_\varepsilon,\nabla\ \ \ u_\varepsilon)u_\varepsilon\to g(x,\ u,\ \ \nabla u)u} \quad \mathrm{a}^{.\mathrm{e}.}_{\mathrm{a}}{}_{.\mathrm{e}.}\mathrm{in}_{\mathrm{in}}\Omega_{\Omega.} \tag{4.14}$$

On the other hand , multiplying (4 . 1) by u_ε and using (3 . 3) , (3 . 4) , (4 . 2) , (4 . 3) we obtain

$$0 \le \int_\Omega g\varepsilon(x,u_\varepsilon,\nabla u_\varepsilon)u_\varepsilon dx \le \tilde{\beta}, \tag{4.15}$$

where $\tilde{\beta}$ is some positive constant . For any measurable subset E of Ω and any $m > 0$, we have

$$\int_E \mid g\varepsilon(x,u_\varepsilon,\nabla u_\varepsilon)\mid dx = \int_{E\cap X_m^\varepsilon} \mid g\varepsilon(x,u_\varepsilon,\nabla u_\varepsilon)\mid dx + \int_{E\cap Y_m^\varepsilon} \mid g\varepsilon(x,u_\varepsilon,\nabla u_\varepsilon)\mid dx$$

where

$$X_m^\varepsilon = \{x\in\Omega :\mid u_\varepsilon(x)\mid \le m\}, \quad Y_m^\varepsilon = \{x\in\Omega :\mid u_\varepsilon(x)\mid > m\} \tag{4.16}$$

From this and (3 . 5) , (4 . 1 5) , (4 . 1 6) , we have

$$\begin{aligned}\int_E \mid g\varepsilon(x,u_\varepsilon,\nabla u_\varepsilon)\mid dx &\le \int_{E\cap X_m^\varepsilon} \mid g\varepsilon(x,u_\varepsilon,\nabla u_\varepsilon)\mid dx + 1_m \int_\Omega g\varepsilon(x,u_\varepsilon,\nabla u_\varepsilon)u_\varepsilon dx \\ &\le b(m)\int_E (\sum_N^{i=1} w_i \mid \partial_\partial^{u_\varepsilon}{}_{x_i} \mid^p + c(x)) + \tilde{\beta}_m^1.\end{aligned}$$

Since the sequence (∇u_ε) converges strongly in $\prod_{i=1}^N L^p(\Omega,w_i)$, then above inequality implies the equi - integrability of $g\varepsilon(x,u_\varepsilon,\nabla u_\varepsilon)$. Thanks to (4 . 1 4) and Vitali ' s theorem ,

$$g\varepsilon(x,u_\varepsilon,\nabla u_\varepsilon) \to g(x,u,\nabla u) \quad \mathrm{stronglyin} L^1(\Omega). \tag{4.17}$$

From (4 . 1 2) and (4 . 1 7) we can pass to the limit in

$$\langle Au_\varepsilon, v\rangle + \int_\Omega g\varepsilon(x,u_\varepsilon,\nabla u_\varepsilon)v = \langle h,v\rangle$$

and we obtain

$$\langle Au, v\rangle + \int_\Omega g(x,u,\nabla u)v = \langle h, v\rangle \quad \forall v \in W_0^{1,p}(\Omega, w)\cap L^\infty(\Omega). \tag{4.18}$$

Moreover , since $g\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon)u_\varepsilon \geq 0$ a . e . in Ω, by (4 . 1 4) , (4 . 1 5) and Fatou ' s lemma , we have $g(x,u,\nabla u)u \in L^1(\Omega)$. It remains to show that ,

$$\langle Au, u\rangle + \int_\Omega g(x,u,\nabla u)u = \langle h, u\rangle.$$

Put $v = u_k$ in (4 . 1 8) where u_k is the truncation of u. Then

$$\langle Au - h, u_k\rangle \to \langle Au - h, u\rangle$$

and

$$g(x,u,\nabla u)u_k \to g(x,u,\nabla u)u \mathrm{in} L^1(\Omega).$$

Using Lebesgue ' s dominated convergence theorem , since

$$\mid g(x,u,\nabla u)u_k \mid \leq \mid g(x,u,\nabla u) \mid\mid u \mid \in L^1(\Omega)$$

we conclude that $g(x,u,\nabla u)u_k \to g(x,u,\nabla u)u$ a . e . in Ω.

Proof of Lemma 4 . 1 We set $B_\varepsilon = A + G_\varepsilon$. Using (3 . 1) and H ö lder ' s inequal - ity we can show that A is bounded [5] . Thanks to (4 . 2) we have B_ε bounded . The coercivity follows from (3 . 3) and (3 . 4) . To show that B_ε is hemicontinous , let $t \to t_0$ and prove that

$$\langle B_\varepsilon(u+tv), \tilde{w}\rangle \to \langle B_\varepsilon(u+t_0v), \tilde{w}\rangle \text{ as } t \to t_0 \quad \text{for all } u, v, \tilde{w} \in X.$$

Since for a . e . $x \in \Omega, a_i(x, u+tv, \nabla(u+tv)) \to a_i(x, u+t_0v, \nabla(u+t_0v))$ as $t \to t_0$, thanks to the growth condition (3 . 1) , Lemma 2 . 1 implies

$$a_i(x,u+tv,\nabla(u+tv)) \rightharpoonup a_i(x,u+t_0v,\nabla(u+t_0v)) \quad \mathrm{in} L^{p'}(\Omega, 1{i_w}^{p'}{}_-) \quad \mathrm{as} t \to t_0.$$

Finally for all $\tilde{w} \in X$,

$$\langle A(u+tv), \tilde{w}\rangle \to \langle A(u+t_0v), \tilde{w}\rangle \quad \mathrm{as} t \to t_0.$$

for$^{\mathrm{On}}$the$_{\mathrm{a.e.}}$other$_{x\in}\Omega^{\mathrm{hand},}_{.}$ $_{\mathrm{Also}}$ $^{g\varepsilon(x,\ u+tv,\ \nabla(u+tv))}_{(g\varepsilon(x,u\ +\ \ tv+\nabla(u}$ $\to +^{g\varepsilon(x,}_{tv}$ $_{)))t}u + \mathrm{is}^{t_0v,}_{\mathrm{bounded}}\nabla(u +_{\mathrm{in}} t_{0L}^{v)prime-parenright}$ $^{\mathrm{as}}_{q(\Omega,}t^{\to t}_{\sigma\ 1-q}0_{\prime)}$

because

$$\int_\Omega \mid g\varepsilon(x,u+tv,\nabla(u+tv)) \mid^{q'} \sigma^{1q'_-} \leq (\tfrac{1}{\varepsilon})^{q'} \int_\Omega \sigma^{1q'_-} \leq c_\varepsilon,$$

then Lemma 2 . 1 gives

$$g\varepsilon(x,u+tv,\nabla(u+tv)) \rightharpoonup g\varepsilon(x,u+t_0v,\nabla(u+t_0v)) \quad \mathrm{in} L^{q'}(\Omega, \sigma^{1-q'}) \quad \mathrm{as} t \to t_0.$$

Since $\tilde{w} \in L^q(\Omega, \sigma)$ for all $\tilde{w} \in X$,

$$\langle G_\varepsilon(u + tv), \tilde{w} \rangle \to \langle G_\varepsilon(u + t_0 v), \tilde{w} \rangle \quad \text{as} t \to t_0.$$

Next we show that B_ε satisfies property (M) ; i . e . for a sequence u_j in X satisfy - ing : (i)$u_i \rightharpoonup u$ in X, (ii)$B_\varepsilon u_j \rightharpoonup \chi$ in X^*, and (ii i) lim $\sup_{j \to \infty} \langle B_\varepsilon u_j, u_j - u \rangle \leq$ 0 , we have $\chi = B_\varepsilon u$. Indeed , by H ö lder ' s inequality and (2 . 6) ,

$$\int_\Omega g\varepsilon(x, u_j, \nabla u_j)(u_j - u)$$

$$\leq (\int_\Omega \mid g\varepsilon(x, u_j, \nabla u_j) \mid^{q'} \sigma^{-q'/q} dx) 1/q' (\int_\Omega \mid u_j - u \mid^q \sigma dx) 1/q$$

$$\leq 1_\varepsilon (\int_\Omega \sigma - q' q dx)^{1/q'} \parallel u_j - u \parallel_{q,\sigma} \to 0 \quad \text{as} j \to \infty,$$

i . e ., $\langle G_\varepsilon u_j, u_j - u \rangle \to 0$ as $j \to \infty$. Combining the last convergence with (i ii) , we obtain

$$\lim \sup \langle Au_j, u_j - u \rangle \leq 0.$$
$$j \to \infty$$

And by the pseudo - monotonicity of A[5, Prop . 1] , we have $Au_j \rightharpoonup Au$ in X^* and $\lim_{j \to \infty} \langle Au_j, u_j - u \rangle = 0$. On the other hand ,

$$0 =_j^{\lim} \to_\infty \int_\Omega a(x, u_j, \nabla u_j) \nabla (u_j - u) dx$$

$$=_j^{\lim} \to_\infty \int_\Omega (a(x, u_j, \nabla u_j) - a(x, u_j, \nabla u)) \nabla (u_j - u) dx$$

$$+ \int_\Omega a(x, u_j, \nabla u) \nabla (u_j - u) dx.$$

The last integral in the right hand tends to zero since $a(x, u_j, \nabla u) \to a(x, u, \nabla u)$ in $\prod_{i=1}^N L^{p'}(\Omega, 1 i_w{}^{-p'})$ as $j \to \infty$; hence , by Lemma 3 . 2 we have $\nabla u_j \to \nabla u$ a . e . in Ω. Then $g\varepsilon(x, u_j, \nabla u_j) \to g\varepsilon(x, u, \nabla u)$ a . e . in Ω as $j \to \infty$. And since

$$\mid g\varepsilon(x, u_j, \nabla u_j) \sigma'^{\frac{1-q'}{q}} \mid \leq 1_\varepsilon \sigma'^{\frac{1-q'}{q}} \in L^{q'}(\Omega) \quad (\text{dueto}(2.4),$$

by Lebesgue ' s dominated convergence theorem , we obtain

$$g\varepsilon(x, u_j, \nabla u_j) \to g\varepsilon(x, u, \nabla u) \quad \text{in} L^{q'}(\Omega, \sigma^{1-q'}) \quad \text{as} j \to \infty,$$

which with (2 . 6) imply $\int_\Omega g\varepsilon(x, u_j, \nabla u_j) v dx \to \int_\Omega g\varepsilon(x, u, \nabla u) v dx$ as $j \to \infty$, for all $v \in X$, i . e ., $G_\varepsilon u_j \rightharpoonup G_\varepsilon u$ in X^*. Finally ,

$$B_\varepsilon u_j = Au_j + G_\varepsilon u_j \rightharpoonup Au + G_\varepsilon u = B_\varepsilon u = \chi \text{in} X^*.$$

Proof of Lemma 4 . 2 Part (a) follows from $\nabla\phi\lambda(k_u^+) \in \prod_{i=1}^N L^p(\Omega, w_i)$ and (4 . 7) . Using Lemma 2.1, $\nabla(\phi\lambda(z_\varepsilon^-)) \rightharpoonup \nabla(\phi\lambda(u^+ - k_u^+)^-)$ in $\prod_{i=1}^N L^p(\Omega, w_i)$; then part (b) follows since $a(x, u_\varepsilon, \nabla k_u^+) \to a(x, u, \nabla k_u^+)$ in $\prod_{i=1}^N L^{p'}(\Omega, w^*i)$.

To prove part (c) , we have

$\partial_\partial^{+u_k}{}_{x_i}\phi\lambda(z_\varepsilon^-)1i_w{}^{/p} \to \partial_\partial^{+u_k}{}_{x_i}\phi\lambda((u^+ - k_u^+)^-)1i_w{}^{/p}$ a . e . in Ω and

$$| \partial_\partial^{+u_k}{}_{x_i}\phi\lambda(z_\varepsilon^-)1i_w{}^{/p} |^p \leq \tilde{\beta} | \partial_\partial^{+u_k}{}_{x_i}1i_w{}^{/p} |^p \in L^1(\Omega),$$

where $\tilde{\beta}$ is a positive constants . Then , by Lebesgue ' s dominated convergence theorem we have

$$\partial_\partial^{+u_k}{}_{x_i}\phi\lambda(z_\varepsilon^-) \to \partial_\partial^{+u_k}{}_{x_i}\phi\lambda((u^+ - k_u^+)^-) \quad \text{in} L^p(\Omega, w_i),$$

i . e . $\nabla k_u^+\phi\lambda(z_\varepsilon^-) \to \nabla k_u^+\phi\lambda((u^+ - k_u^+)^-)$ in $\prod_{i=1}^N L^p(\Omega, w_i)$. Then by (4 . 7) we obtain part (c) .

To prove part (d) , we have

$$a_i(x, u_\varepsilon, \nabla k_u^+)\phi\lambda((\varepsilon_u^+ - k_u^+)^-)p^{p'}_{1w_i-\prime} \to a_i(x, u, \nabla k_u^+)\phi\lambda((u^+ - k_u^+)^-)w_i^{1-p^{p'}_\prime}$$

a . e . in Ω, and

$$| a_i(x, u_\varepsilon, \nabla k_u^+)\phi\lambda((\varepsilon_u^+ - k_u^+)^-)w'^{p\,i}{}_{1-p'} |^{p'} \leq M | a_i(x, u_\varepsilon, \nabla k_u^+) |^{p'} 1i_w{}^{p'}{}_-.$$

Then the generalized Lebesgue ' s dominated convergence theorem implies

$$a_i(x, u_\varepsilon, \nabla k_u^+)\phi\lambda((\varepsilon_u^+ - k_u^+)^-) \to a_i(x, u, \nabla k_u^+)\phi\lambda((u^+ - k_u^+)^-) \quad \text{in} L^{p'}(\Omega, w^*i).$$

lows$^{\text{Since}}\nabla^{(\varepsilon_u^+}_{\text{from}} | c^-(x^{u+_k)}{}_{)\phi\lambda((u^+ \quad -+_{u_k})^-)|} \rightharpoonup \nabla^{(u^+ \quad -+_{u_k})}$ in$^{L^p(\Omega, \quad w_i)}_{\in L^1 \quad (\Omega)\text{and}}$weconcludeLebesgue′spart$^{\text{(d).}}_{\text{dominated}}$Part$^{\text{(e)fol}-}_{\text{convergence}}$

theorem .

5 Example

Some ideas of this example come from [5] . Let Ω be a bounded domain of $\mathbb{R}^N(N \geq 1)$, satisfying the cone condition . Let us consider the Carath $\acute{e}$ odory functions :

$$a_i(x, s, \xi) = w_i | \xi i |^{p-1} \operatorname{sgn}(\xi i) \quad \text{for} i = 1, ..., N$$

$$g(x, s, \xi) = \operatorname{sgn}(s) \sum_{i=1}^{N} w_i | \xi i |^{p},$$

where $w_i(x)$ are a given weight functions strictly positive almost everywhere in Ω. We shall assume that the weight functions satisfy ,

$$w_i(x) = w(x), \quad x \in \Omega, \quad \text{forall} i = 0, ..., N.$$

Then , we consider the Hardy inequality (2.5) in the form ,

$$\Big(\int_\Omega |\, u(x) \,|^q \, \sigma(x) dx\Big)^{1/q} \le c\Big(\int_\Omega |\, \nabla u(x) \,|^p \, w\Big)^{1/p}.$$

It is easy to show that the $a_i(x, s, \xi)$ are Carathéodory functions satisfying the growth condition (3.1) and the coercivity (3.3) . Also the Carathéodory function $g(x, s, \xi)$ satisfies the conditions (3.4) and (3.5) . On the other hand , the monotonicity condition is verified . In fact ,

$$\sum_{i=1}^{N} (a_i(x, s, \xi) - a_i(x, s, \hat{\xi}))(\xi i - \hat{\xi} i)$$

$$= w(x) \sum_{i=1}^{N} (|\, \xi i \,|^{p-1} \operatorname{sgn} \xi i - |\, \hat{\xi} i \,|^{p-1} \operatorname{sgn} \hat{\xi} i)(\xi i - \hat{\xi} i) > 0$$

for almost all $x \in \Omega$ and for all $\xi, \hat{\xi} \in \mathbb{R}^N$ with $\xi \ne \hat{\xi}$, since $w > 0$ a . e . in Ω. In particular , let us use the special weight functions w and σ expressed in terms of the distance to the boundary $\partial\Omega$. Denote $d(x) = \text{dist}\ (x, \partial\Omega)$ and set

$$w(x) = d^\lambda(x), \quad \sigma(x) = d^\mu(x).$$

In this case , the Hardy inequality reads

$$\Big(\int_\Omega |\, u(x) \,|^q \, d^\mu(x) dx\Big) 1/q \le c\Big(\int_\Omega |\, \nabla u(x) \,|^p \, d^\lambda(x) dx\Big) 1/p.$$

The corresponding imbedding is compact if : (i) For , $1 < p \le q < \infty$,

$$\lambda < p - 1, \quad N_q - N_p + 1 \ge 0, \quad \mu_q - \lambda_p + N_q - N_p + 1 > 0, \quad (5.1)$$

$$(\text{ii}) \text{For} 1 \le q < p < \infty,$$

$$\lambda < p - 1, \quad \mu_q - \lambda_p + 1_q - 1_p + 1 > 0, \quad (5.2)$$

(iii) For $q > 1$,

$$\mu(q' - 1) < 1. \tag{5.3}$$

Remarks .

1 . Condition (5.1) or (5.2) are sufficient for the compact imbedding (2.6) to hold ; see for example [5 , Example 1] , [6 , Example 1.5] , and [12 , Theorems 19.17 , 19.22] .

2 . Condition (5.3) is sufficient for (2.4) to hold [9 , pp . 40 - 41] . Finally , the hypotheses of Theorem 3.1 are satisfied . Therefore , (1.2) has at least one solution .

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