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Dirichlet problem for quasi - linear elliptic equations *

Azeddine Baalal & Nedra BelHaj Rhouma

Abstract

We study the Dirichlet Problem associated to the quasilinear elliptic problem

$$-\sum_{n}^{i=1} \partial^{\partial} x_i^{\mathcal{A}_i(x,} u(x), \nabla u(x)) + \mathcal{B}(x, u(x), \nabla u(x)) = 0.$$

Then we define a potential theory related to this problem and we show that the sheaf of continuous solutions satisfies the Bauer axiomatic theory .

1 Introduction

The objective of this paper is to study the weak solutions of the following quasi - linear elliptic equation in $\mathbb{R}^d, (d \geq 2)$:

$$-\sum_{n}^{i=1} \partial^{\partial} x_i^{\mathcal{A}_i(x,} u(x), \nabla u(x)) + \mathcal{B}(x, u(x), \nabla u(x)) = 0 \tag{1.1}$$

where $\mathcal{A}_i : \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ and $\mathcal{B} : \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ are given Carath $\acute{e}$ odory functions satisfying the conditions introduced in section 2 .

An example of equation (1 . 1) is the perturbed $p-$ Laplace equation

$$-\mathrm{div}(| \nabla u |^{p-2} \nabla u) + \mathcal{B}(., u, \nabla u) = 0, \quad 1 < p < d. \tag{1.2}$$

When $p = 2$, equation (1 . 2) reduces to the perturbed Laplace equation

$$-\Delta u + \mathcal{B}(., u, \nabla u) = 0. \tag{1.3}$$

Another example included in this study is the linear equation

$$\mathcal{L}(u) = -\sum_j (\sum_i a_{ij} \partial^{\partial u}_{x_i} + d_j u) + \sum_j b_j \partial^{\partial u}_{x_j} + cu = 0,$$

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where $\mathcal{L}$ is assumed to satisfy conditions st ated in [25] (see also [1 2]) .

Equation (1 . 1) have been investigated in many interesting papers [24 , 26 , 1 1 , 2 1 , 2] . Several papers have introduced an axiomatic potential theory for the nonlinear equation (1 . 2) when $\mathcal{B} = 0$; see for example [1 1] . For equations of type (1 . 3) , see [1 , 2 , 3 , 4] .

The existence of weak solutions of (1 . 1) in variational forms was treated by means of the sub - supersolution argument [7 , 8] . Later on , Dancers / Sweers [6] , Kura [1 5] , Carl [5] , Lakshmikantham [1 0] , Papageorgiou [23] , Le / Schmitt [1 9] , and others treated the existence of weak extremal solutions of nonlinear equations of type (1 . 1) by means of the sub - supersolution method . Le [1 7]
studied the existence of extremal solutions of the problem

$$\int_\Omega A(x, \nabla u(x))(\nabla v - \nabla u)dx \geq \int_\Omega \mathcal{B}(x, u(x))(v(x) - u(x))dx, \tag{1.4}$$

for all $v \in K, u \in K$, where K is a closed convex subset of $W_0^{1,p}(\Omega)$.

Note that the solutions of (1 . 4) correspond to the obstacle problem treated in section 5 of this paper . Remark that in the references cited above , often $\mathcal{B} = \mathcal{B}(x, u(x))$ and the growth of $\mathcal{B}$ in u is less then $p - 1$ and when $\mathcal{B} = \mathcal{B}(x, u, \nabla u)$, the growth of $\mathcal{B}$ in u and ∇u is less then $p - 1$, but in our case the growth of $\mathcal{B}$ in ∇u is is allowed to go until $p - 1 + p_n$ and there is no condition on the growth
of $\mathcal{B}$ in u.

Our aim in this paper is to solve the Dirichlet problem for (1 . 1) with a continuous data boundary and to give an axiomatic of potential theory related
to the associated problem .

This paper consists of four sections . First , we recall some definitions for the (weak) subsolutions , supersolutions and solutions of the equation (1 . 1) . In particular , we prove that the supremum of two subsolutions is a subsolution
and that the infinimum of two supersolutions is also a supersolution . In section
3 , we give some conditions that allow us to have the comparison principle for sub and supersolutions . After this preparation we are able in section 4 to solve the Dirichlet problem related to the equation (1 . 1) . So at first we prove the existence of solutions to the associated variational problem , after what we solve
the Dirichlet problem for continuous data boundary . In the last section , we define a potential theory related to the equation (1 . 1) , so we obtain that the sheaf of continuous solutions of (1 . 1) satisfies the Bauer axiomatic theory [4] . We prove also that the set of all hyperharmonic functions and the set of all
hypoharmonic functions are sheaves .

Notation Throughout this paper we will use the following notation : $\mathbb{R}^d$ is the real Euclidean $d-$ space , $d \geq$ 2. For an open set U of $\mathbb{R}^d$, we denote by $C^k(U)$ the set of functions which $k-$ th derivative is continuous for k positive
integer , $C^\infty(U) = \cap_{k \geq 1} C^k(U)$ and by $C_0^\infty(U)$ the set of all functions in $C^\infty(U)$
with compact support $.L^q(E)$ is the space of all $q^{th}-$ power Lebesgue integrable
functions$_{W_0^{1,q}(U)\text{is}}$defined$_{\text{theclosure}}$ on measurable$^{\text{set}}_{\text{of}C_0^\infty(U)\text{in}}E^{.W^{1,q}(U)}_{W\quad 1,q(U)}{}^{\text{relatively}}$ is the $(1_{\text{to}}, q)_{\text{its}} -$
Sobolev$_{\text{norm.}}$spaceW_1on$_{,q'_{(}{}^{U.}_{U})}$

denotes$^{\text{denotes}}$the$_{\text{the}}$dualofW_0^1Lebesgue$^{;q(U),}_{\text{measure}}q-1.\dot{}_{\overset{=\text{of}}{q'}\,E}^{\ u\ q}For_{\vee v}$aLebesgueand$u\wedge v$measurabledesignset$E$, respectively $|^{E|}_{\text{the}}$

supremum and the infinimum of u and $v.u^+ = u \vee 0$ and $u^- = u \wedge 0$. We write $\rightharpoonup$ (resp . $\to$) to design the weak (resp . strong) convergence .

2 Supersolutions of (1 . 1)

Let Ω be a bounded domain in $\mathbb{R}^d (d \geq 2)$ with smooth boundary $\partial\Omega$ and let $\mathcal{L}$ be a quasi - linear elliptic differential operator in divergence form

$$\mathcal{L}(u)(x) = -\sum_{d}^{i=1} \partial^{\partial} x_i \mathcal{A}_i(x, u(x), \nabla u(x)) + \mathcal{B}(x, u(x), \nabla u(x)) \quad \text{a.e.} x \in \Omega$$

where $\mathcal{A}_i : \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ and $\mathcal{B} : \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ are given Carath é odory functions . Let $\mathcal{A} = (\mathcal{A}_1, ..., \mathcal{A}_d)$ and $1 < p < d$. We suppose that the following conditions are fulfilled : for a . e .$x \in \Omega, \forall \zeta \in \mathbb{R}$ and $\xi, \xi' \in \mathbb{R}^d$:

$$| \mathcal{A}(x, \zeta, \xi) | \leq k_0(x) + b_0(x) | \zeta |^{p-1} + a | \xi |^{p-1} \tag{P1}$$

$$(\mathcal{A}(x, \zeta, \xi) - \mathcal{A}(x, \zeta, \xi'))(\xi - \xi') > 0, \text{if} \xi \neq \xi'. \tag{P2}$$

$$\mathcal{A}(x, \zeta, \xi)\xi \geq \alpha | \xi |^p - d_0(x) | \zeta |^p - e(x) \tag{P3}$$

$$| \mathcal{B}(x, \zeta, \xi) | \leq k(x) + b(x) | \zeta |^{\alpha} + c | \xi |^r, 0 < r < (p^{p_*})', \alpha \geq 0. \tag{P4}$$

Here a, c and α are positive constants $, p' = p_{p-1}, p^* = pd_{d-p}$, while $k_0, b_0, d_0, e,$ k and b are measurable functions on Ω satisfying : $k_0 \in L^{p'}, b_0 \in Ld_{-_p}1, k \in L^q$,

$$(p^*)' < q < (^d_{p-\varepsilon} \wedge p_r) \text{and} d_0, e, b \in Ld_{-_p}\varepsilon, (0 < \varepsilon < 1).$$

$\mathcal{B}($.We$^{\text{can}}_{,u,\nabla u)}$easily$_{\in L^{(p^*)}}$show$^{\text{that}}_{\prime\text{when}}{}^{\text{if}}_{\alpha\leq}{}^{u}_{p-} \in 1_{.} W^{1,p}(\Omega)$, then $A(., u, \nabla u) \in L^{p'}$ and that **Definition** We say that a function $u \in W^{1,p}_{\text{loc}}(\Omega)$ is a (weak) solution of (1 . 1) , if

$$\mathcal{B}(., u, \nabla u) \in L^{(p^*)'}$$

$$\int_{\Omega} \mathcal{A}(., u, \nabla u) \nabla\phi + \int_{\Omega} \mathcal{B}(., u, \nabla u)\phi = 0, \tag{2.1}$$

for We$^{\text{all}\phi} \in$ say$0^{\text{that}}_{W^{1,p}}(_u\Omega_{\in}).W^{1,p}_{\text{loc}}(\Omega)$ is a supersolution (resp . subsolution) of (1 . 1) if

$$\mathcal{B}(., u, \nabla u) \in L^{(p^*)'}$$

$$\int_{\Omega} \mathcal{A}(., u, \nabla u) \nabla\phi + \int_{\Omega} \mathcal{B}(., u, \nabla u)\phi \geq 0 \quad (\text{resp.} \quad \leq 0)$$

for every nonnegative function $\phi \in W^{1,p}_0(\Omega)$.

 Note that if u is a supersolution of (1 . 1) then $-u$ is a subsolution of the equation

$$-\mathrm{div}\widehat{A}+\widehat{B}=0$$

where $\widehat{A}(x,\zeta,\xi) = -\mathcal{A}(x,-\zeta,-\xi)$ and $\widehat{B}(x,\zeta,\xi) = -\mathcal{B}(x,-\zeta,-\xi)$. Further - more , the structure of $\widehat{A}$ and $\widehat{B}$ are similar to that of $\mathcal{A}$ and $\mathcal{B}$.

We recall that if u is a bounded supersolution (resp . subsolution) , then u is upper (resp . lower) semicontinuous in Ω[21, Corollary 4 . 1 0] .

Proposition 2 . 1 *Let u and v be two subsolutions of (1 . 1) in Ω such that*

$$(\mathcal{A}(.,v,\nabla u)-\mathcal{A}(.,u,\nabla u))\nabla(v-u)\geq 0, \quad a.e. x\in\Omega.$$

Then , $\max(u,v)$ is als o a subsolution . A s imilar s tatement holds for the mini - mum of two supersolutions .

Proof . Fix ϕ in $C_0^\infty(\Omega), \phi\geq 0$. Let $\Omega_1=\{x\in\Omega : u>v\}, \Omega_2=\{x\in\Omega : u\leq v\}$ and put $I=\int_\Omega \mathcal{A}(.,u\vee v,\nabla(u\vee v))\nabla\phi = I_1+I_2$ where

$$I_1=\int_{\Omega_1}\mathcal{A}(.,u,\nabla u)\nabla\phi \quad\text{and}\quad I_2=\int_{\Omega_2}\mathcal{A}(.,v,\nabla v)\nabla\phi.$$

Let $\rho n:\mathbb{R}\to\mathbb{R}$ be such that $\rho n\in\mathcal{C}^1(\mathbb{R})$,

$$\rho n(t)=\{^1_0 \quad \mathrm{if}^{\mathrm{if}} t^t \geq \leq 0^{1/n}$$

and$_{\mathrm{see}}\rho'_n$that$^{>}0_{n_q}$on$\in]^{0,\ 1/n[.}_{W1,ploc(\Omega)}$,For$^{\mathrm{each}}_{qn\to}1_{\Omega_1}x\in$ and$\Omega\|^{\mathrm{define}}_{qn}\ \|_\infty \ \leq qn1^{(x)}_{.}$ It $=\rho n((u\succ -v$by$^{)(x)).}_{\mathrm{Lebesgue's}}$ We

Theorem of dominated convergence that $I_1=\lim_{n\to\infty}\int_{\Omega_1} qn^{\mathcal{A}}(.,u,\nabla u).\nabla\phi$ and

$$I_2=\lim_{n\to\infty}\int_{\Omega_2}(1-qn)\mathcal{A}(.,v,\nabla v).\nabla\phi.\text{Hence}$$

$$\begin{aligned}\int_\Omega qn^{\mathcal{A}}(.,u,\nabla u).\nabla\phi &= \int_\Omega\mathcal{A}(.,u,\nabla u)\nabla.(qn\phi)-\int_\Omega\mathcal{A}(.,u,\nabla u)\phi.\nabla(qn)\\ &\leq -\int_\Omega\mathcal{B}(.,u,\nabla u)(qn\phi)-\int_{\Omega_n}\mathcal{A}(.,u,\nabla u)\phi.\nabla(qn),\end{aligned}$$

where$_{\mathrm{Put}}\Omega_{nI_n}$ $=_= \{x\int^{\in}_{\Omega}{}^{\Omega\ :v}_{qn\mathcal{A}(.,}$ $<_u,\ u<\nabla^{v}_{u).\nabla\phi}$ $^{+\ 1_n\}.}_{\mathrm{and}}$ $J_n=\int_\Omega(1-qn)\mathcal{A}(.,v,\nabla v).\nabla\phi.$ Then,

similarly we have

$$\int_\Omega(1-qn)\mathcal{A}(.,v,\nabla v).\nabla\phi\leq-\int_\Omega(1-qn)\mathcal{B}(.,v,\nabla v)\phi+\int_{\Omega_n}\mathcal{A}(.,v,\nabla v)\phi.\nabla(qn).$$

So , we get

$$\begin{aligned}I_n+J_n \quad &\leq \quad -\int_\Omega\mathcal{B}(.,u,\nabla u)(qn\phi)-\int_\Omega(1-qn)\mathcal{B}(.,v,\nabla v)\phi\\ &\quad +\int_{\Omega_n}(\mathcal{A}(.,v,\nabla v)-\mathcal{A}(.,u,\nabla u))\phi.\nabla(qn).\end{aligned}$$

 Using that $\nabla(qn) = \rho'_n(u-v)\nabla(u-v)$, we get

$$\begin{aligned} I_n + J_n &\leq -\int_\Omega \mathcal{B}(.,u,\nabla u)(qn\phi) - \int_\Omega (1-qn)\mathcal{B}(.,v,\nabla v)\phi \\ &\quad -\int_{\Omega_n} \rho'_n(u-v)(\mathcal{A}(.,v,\nabla v) - \mathcal{A}(.,u,\nabla u))\phi.\nabla(v-u) \\ &\leq -\int_\Omega \mathcal{B}(.,u,\nabla u)(qn\phi) - \int_\Omega (1-qn)\mathcal{B}(.,v,\nabla v)\phi. \end{aligned}$$

Finally , we have

$$\int_\Omega \mathcal{A}(.,u\vee v,\nabla(u\vee v)).\nabla\phi + \int_\Omega \mathcal{B}(.,u\vee v,\nabla(u\vee v))\phi \leq 0$$

which completes the proof . □ We say that $\mathcal{L}$ satisfies the property $(\pm)$, if for every $k > 0$ and every supersolution (resp . subsolution)u of (1 . 1) , the function $u+k$(resp .$u-k$) is also a supersolution (resp . subsolution) of (1 . 1)

Remark 2 . 1 *1) Suppose that for each* $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ *and each* $k > 0$,

$$\int (\mathcal{A}(.,u+k,\nabla u) - \mathcal{A}(.,u,\nabla u)).\nabla\phi + \int (\mathcal{B}(.,u+k,\nabla u)$$

for 2) Note every nonnegative that if $\mathcal{L}(u) = $ *function* $-\sum_j \partial^{\partial} x_j \phi \in (\sum_i W_0^{1,p}(\Omega) a_{ij} \partial^{\partial} \frac{u}{x_i} +$ *Then* $d_j u) + \mathcal{L}$ *satisfies* $(\sum_i b_i \partial \partial_{x_i} u$ *the prope*

e l lip ti c operator of s econd o rder satisfying the conditions of [1 2] , then (2 . 2) is equivalent to $(-\sum_j (d_j) + c) \geq 0$ *in the distributional s ense .*

3) Suppose that $\mathcal{A}(x,\zeta,\xi) = \mathcal{A}(x,\xi)$ *and for a . e .* $x \in \Omega$ *and* $\xi \in \mathbb{R}^d$ *the map :* $\zeta \to \mathcal{B}(x,\zeta,\xi)$ *is increasing . Then the property* $(\pm)$ *holds .*

3 Comparison principle

In this section , we will give some conditions needed for the comparison principle . This principle makes it possible to solve the Dirichlet problem and to develop a potential theory in our case .

We say that the *comparison principle* holds for $\mathcal{L}$, if for every supersolution u and every subsolution v of (1 . 1) on Ω, such that

$$\lim_{x\to y}\sup v(x) \leq \lim_{x\to y}\inf u(x)$$

for all $y \in \partial\Omega$ and both sides of the inequality are not simultaneously $+\infty$ or $-\infty$, we have $v \leq u$ a . e . in Ω.

Theorem 3 . 1 *Suppose that the operator* $\mathcal{L}$ *satisfies e ither one of the property* $(\pm)$ *and the following s trict monotony condition (s ee [22]) :*

$$(\mathcal{A}(x,\zeta,\xi) - \mathcal{A}(x,\zeta',\xi')).(\xi-\xi') + (\mathcal{B}(x,\zeta,\xi) - \mathcal{B}(x,\zeta',\xi'))(\zeta-\zeta') > 0$$

 for $(\zeta,\xi)\neq(\zeta',\xi')$.
Let u *be a supersolution and* v *be a subsolution of* (1 . 1) , *on* Ω, *such that*

$$\lim_{x\to y}\sup v(x)\leq\lim_{x\to y}\inf u(x)$$

for al l $y\in\partial\Omega$ *and both s ides of the inequality are not s imultaneously* $+\infty$ *or* $-\infty$, *then* $v\leq u$ *a . e . in* Ω.

Proof . Let $\varepsilon>0$ and K be a compact subset of Ω such that $v-u\leq\varepsilon$ on $\Omega\setminus K$, then the function $\phi=(v-u-\varepsilon)^+\in W_0^{1,p}(\Omega)$. Testing by ϕ, we obtain that

$$\begin{aligned}0 \quad \leq \quad & \int_{v>u+\varepsilon}(\mathcal{A}(.,u+\varepsilon,\nabla u)-\mathcal{A}(.,v,\nabla v))\nabla(v-u-\varepsilon)\\ &+\int_{v>u+\varepsilon}(\mathcal{B}(.,u+\varepsilon,\nabla u)-\mathcal{B}(.,v,\nabla v))(v-u-\varepsilon)\leq 0.\end{aligned}$$

Hence $\nabla(v-u-\varepsilon)^+=0$ and $(v-u-\varepsilon)^+=0$ a . e . in Ω. It follows that $v\leq u+\varepsilon$ a . e . in Ω and therefore $v\leq u$ a . e . in Ω $\square$

Corollary 3 . 2 *we suppose that* $\mathcal{A}(x,\zeta,\xi)=\mathcal{A}(x,\xi)$ *and* $\mathcal{B}(x,\zeta,\xi)=\mathcal{B}(\zeta)$ *such that the map* $\zeta\to\mathcal{B}(x,\zeta)$ *is increasing for a . e . x in* Ω. *Then , the comparison principle holds .*

Theorem 3 . 3 *Suppose that* **i**) $[\mathcal{A}(x,\zeta,\xi)-\mathcal{A}(x,\zeta',\xi')].(\xi-\xi')\geq\gamma\mid\xi-\xi'\mid^p$ *for all* ζ,ζ' *in* $\mathbb{R}$, *for all* $\xi,\xi'\in\mathbb{R}^d$,
a . e .x in Ω *and for s ome* $\gamma>0$.

i i) *For a . e* $.x\in\Omega$ *and for al l* $\xi\in\mathbb{R}^d$, *the map* $\zeta\to\mathcal{B}(x,\zeta,\xi)$ *is increasing ,*

i ii) $\mid(\mathcal{B}(x,\zeta,\xi)-\mathcal{B}(x,\zeta,\xi')\mid\leq b(x,\zeta)\mid\xi-\xi'\mid^{p-1}$ *for a . e .* $x\in\Omega$, *for al l* $\zeta\in\mathbb{R}$ *and for al l* $\xi,\xi'\in\mathbb{R}^d$. *Where* $\sup_{|\zeta|\leq M}b(.,\zeta)\in L^s_{\mathrm{loc}}(\Omega)$, $s>d$, *for al l*

$$M>0.$$

Then th e comparison principle holds .

Proof . The main idea in this proof comes from Professor J . Maly ' . Let $\rho>0$, $M=\sup\,(v-u)$ and put $w=v-u-\rho$. Take w^+ as test function . Then , we get

$$\int_\Omega[\mathcal{A}(.,u,\nabla u)-\mathcal{A}(.,v,\nabla v)].\nabla(w^+)+\int_\Omega[\mathcal{B}(.,u,\nabla u)-\mathcal{B}(.,v,\nabla v)](w^+)\geq 0$$

and by consequence

$$\begin{aligned}\gamma\int_\Omega\mid\nabla w^+\mid^p \quad &\leq \quad \int_\Omega b(x,v)\mid\nabla w^+\mid^{p-1}w^+\\ \leq \quad C[\int_\Omega\mid\nabla w^+\mid^p]p-p1[&\int_\Omega(w^+)^{p^*}]1p_*\mid A_\rho\mid s_s^{-d}{}_d\\ &\leq \quad C\parallel\nabla w^+\parallel p^p\mid A_\rho\mid s_s^{-d}{}_d.\end{aligned}$$

where $A_\rho = \{\rho < v - u < M\}$. Hence we get $\mid A_\rho \mid \to 0$ when $\rho \to M$, which is impossible if $M > 0$. Thus , $v \leq u$ on Ω □

4 Dirichlet Problem Existence of solutions for $0 \leq \alpha \leq p-1$ and $0 \leq r \leq p-1$

Definition Let $g \in W^{1,p}(\Omega)$. We say that u is a solution of problem (P) if

$$u - g \in W_0^{1,p}(\Omega),$$

$$\int_\Omega \mathcal{A}(., u, \nabla u).\nabla\phi + \int_\Omega \mathcal{B}(., u, \nabla u)\phi = 0 \quad \forall\phi \in W_0^{1,p}(\Omega).$$

Remark 4 . 1 *Put $v = u - g$, th en u is a s o lution of th e above problem (P) if and only if v is a s o lution of*

$$u \in W_0^{1,p}(\Omega)$$

$$\int_\Omega \mathcal{A}_g(., u, \nabla u)\nabla\phi + \int_\Omega \mathcal{B}_g(., u, \nabla u)\phi = 0, \quad \forall\phi \in W_0^{1,p}(\Omega), \quad (4.1)$$

$$where \mathcal{A}_g(., u, \nabla u) = \mathcal{A}(., u + g, \nabla(u + g)) and \mathcal{B}_g(., u, \nabla u) = \mathcal{B}(., u + g, \nabla(u + g)).$$

Let $T : W_0^{1,p}(\Omega) \to W_0^{-1p',}(\Omega)$ be the operator defined by

$$\langle T(u), v\rangle = \int \mathcal{A}_g(., u, \nabla u)\nabla v + \int \mathcal{B}_g(., u, \nabla u)v \quad \forall v \in W_0^{1,p}(\Omega).$$

Next we will est ablish the existence of solution of (4 . 1) when $0 \leq \alpha \leq p-1$ and $0 \leq r \leq p-1$. Let $C = C(d, p)$ be a constant such that $\parallel u \parallel_{p*} \leq C \parallel u \parallel_p$ for every $u \in W_0^{1,p}(\Omega)$. Then , we get the following result .

Proposition 4 . 1 *Suppose that $0 \leq \alpha \leq p-1$ and $0 \leq r \leq p-1$. If Ω is small (i . e $\alpha > C(\parallel d_0 \parallel_{n/p} + \parallel b \parallel_{n/p}))$, then th e operator T is coercive .*

Proof . We have

$$\begin{aligned} \langle T(u), u\rangle &= \int \mathcal{A}(u + g, \nabla(u + g))\nabla u + \int \mathcal{B}(u + g, \nabla(u + g))u \\ \geq\ & (\alpha - C \parallel d_0 \parallel d/p - C \parallel b \parallel d/p) \parallel \nabla u \parallel p^p - H_1(\parallel u \parallel, \parallel \nabla u \parallel, \parallel g \parallel, \parallel \nabla g \parallel) \end{aligned}$$

where $C = C(d, p)$ and the growth of H_1 in $\parallel u \parallel$ and $\parallel \nabla u \parallel$ is less then $p - 1$. So , let Ω be small enough such that $\alpha > C(\parallel d_0 \parallel n/p^+ \parallel b \parallel n/p)^{\cdot}$ Hence , $\langle T\|^{(u),u\rangle}_{\nabla u\| \ p} \to +\infty$ as $\parallel \nabla u \parallel p \to +\infty$ and therefore the operator T is coercive . □

Proposition 4 . 2 *Suppose that $0 \leq \alpha \leq p-1$ and $0 \leq r \leq p-1$. Then , the operator T is pseudomonotone and satisfies th e well known property (S_+) : If $u_n \rightharpoonup u$ and* $\limsup_{n\to\infty}\langle T(u_n) - T(u), u_n - u\rangle \leq 0$, *th en $u_n \to u$.*

The proof of this proposition is found in [2 1] .

Theorem$_{has@least}4_1.3$ *Suppose$_{weaksolution}$ that $T_{in}satisfies^{the}_{W_0^{1,p}(\Omega)\cdot}$ coercive condition on Ω. Then (4 . 1)*

Proof . The operator T is pseudomonotone , bounded continuous and coercive . Hence , by [22]T is surjective . $\square$

Existence of solutions for $\alpha \geq 0$ and $p-1 < r < (p^{p_*})'$

Definition Let g be an element of $W_p^{1-1}(\partial\Omega)$.

We say that a function u is a solution of (4 . 2) with boundary value g if

$$\begin{aligned} u \in W^{1,p}(\Omega), \mathcal{B}(., u, \nabla u) \in L^{p*'}_{\text{loc}}\Omega \\ u = g \text{in} W_p^{1-1}(\partial\Omega), \\ \int_\Omega \mathcal{A}(., u, \nabla u)\nabla\phi + \int_\Omega \mathcal{B}(., u, \nabla u)\phi = 0 \quad \forall\phi \in W_0^{1,p}(\Omega). \end{aligned} \tag{4.2}$$

(For the definition and properties of the space $W_p^{1-1}(\partial\Omega)$ see e . g . [20]) .

We say that u is an upper supersolution of (4 . 2) with boundary value g if

$$\begin{aligned} u \in W^{1,p}(\Omega), \mathcal{B}(., u, \nabla u) \in L^{p*'}_{Loc}\Omega \\ u \geq g \text{in} W_p^{1-1}(\partial\Omega), \\ \int_\Omega \mathcal{A}(., u, \nabla u)\nabla\phi + \int_\Omega \mathcal{B}(., u, \nabla u)\phi \geq 0 \end{aligned}$$

for all $\phi \in W_0^{1,p}(\Omega)$ with $\phi \geq 0$.

Similarly , a lower subsolution is characterized by the reverse inequality signs in the above definition .

We recall the following result given in [1 8 , Theorem 2 . 2] .

Theorem 4 . 4 *Suppose that there exists an ordered pair $\phi \leq \psi$ of subsolution and supersolution of (4 . 2) satisfying th e following condition : There exists $k \in L^q(\Omega)$, $q > p^{*\prime}$ such that for al l $\xi \in \mathbb{R}^d$ and al l ζ with $\phi(x) \leq \zeta \leq \psi(x)$, $|^{\mathcal{B}\ (x,\ \zeta,\ \xi)|\ \leq}_{u\ \in\ W0^{1,p}(\Omega)} such^{k(x)} + cthat^{|\xi|^r}_{\phi} \leq a_u.e.x \leq^{\in}_{\psi}.$ Ω. Then , (4 . 2) has at least one s o lution*

Proposition 4 . 5 *Suppose that (4 . 2) admits a pair of bounded lower subsolution u and upper supersolution v such that $u \leq v$, then there exists a s o lution w of*

$$(4.2) such that u \leq w \leq v.$$

Proof . Let M be a positive real such that $\| u \|_\infty, \| v \|_\infty, \| g \|_\infty \leq M$. Then , for each ζ such that $u(x) - g(x) \leq \zeta \leq v(x) - g(x)$, we have $| \mathcal{B}(x, \zeta, \xi) | \leq k(x) + b(x)M^\alpha + 2^r c \mid \nabla g \mid^r + c \mid \xi \mid^r$ for a . e . $x \in \Omega$. In addition , $u($ resp . $v)$ is a lower subsolution (resp . upper supersolution) of (4 . 2) . Hence by the last Theorem , there exists a solution w of (4 . 2) such that $u \leq w \leq v$. $\square$

Corollary 4 . 6 *Suppose that al l positive constants are supersolutions and al l negative constants are subsolutions . Then for each $g \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$, there exists a bounded s o lution w of (4 . 2) such that $\| w \|_\infty \leq \| g \|_\infty$.*

 Proof . We see that $v = \| g \|_\infty$ is an upper supersolution and $u = - \| g \|_\infty$ is a lower subsolution . Hence by the Proposition given above , we get a solution

$$u \leq w \leq v \quad \square$$

4 . 1 Dirichlet Problem

In this section , we assume that $\mathcal{A}(.,0,0) = 0$ and $\mathcal{B}(.,0,0) = 0$ a . e . in Ω, that the property $(\pm)$ is satisfied , and that the comparison principle holds .

Supposeknownthat if that u is the a open set solution of Ω (is 1.regular 1) in Ω (p−regular) satisfying $u - f \in W_0^{1,p}(\Omega)$ with Then it is

$f \in W^{1,p}(\Omega) \cap C(\Omega)$, then

$$\lim_{x \to z} u(x) = f(z) \quad \forall z \in \partial\Omega.$$

$W_{\text{loc}}^{1,p}$ **Definition** (Ω) solves Let f the Dirichlet be a continuous function problem with on $\partial\Omega$. boundary value We say f if that u u is $\in C(\Omega) \cap$ a solution of (1 . 1) such that $\lim_{x \to z} u(x) = f(z)$, for all $z \in \partial\Omega$.

Theorem 4 . 7 *For each $f \in C(\partial\Omega)$, the re exists u in $C(\Omega) \cap W_{\text{loc}}^{1,p}(\Omega)$ s o lving the Dirichlet problem with boundary value f.*

Proof By the Tieze ' s extension Theorem , we can assume that $f \in C_c^\infty(\mathbb{R}^d)$. Let $(f_n)_n$ be a sequence of mollifiers of f such that $\| f_n - f \| \leq 1/2^n$ on Ω. let u_n denote the continuous solution of

$$\begin{gathered} u_n - f_n \in W_0^{1,p}(\Omega), \\ \int_\Omega \mathcal{A}(., u_n, \nabla u_n)\nabla\phi + \int_\Omega \mathcal{B}(., u_n, \nabla u_n)\phi = 0, \quad \forall \phi \in W_0^{1,p}(\Omega). \end{gathered} \tag{4.3}$$

So , by the comparison principle , $| u_n - u_m | \leq 2_n^1 + 2_m^1$. Hence , the sequence $(u_n)_n$ converges uniformly on Ω to a continuous function u. Let M be a positive real such that for all $n :$ $| f_n | + | f | \leq M$ and $| u_n | + | u | \leq M$ on Ω.

Let $G \subset G \subset \Omega$, take ϕ as a test function in (4 . 3) such that $\phi = \eta^p u_n, \eta \in C_c^\infty(\Omega), 0 \leq \eta \leq 1$ and $\eta = 1$ on G. Then

$$\begin{aligned} &\int_\Omega \mathcal{A}(., u_n, \nabla u_n)\eta^p \nabla(u_n) \\ &= -p \int_\Omega \mathcal{A}(., u_n, \nabla u_n) u_n \eta^{p-1} \nabla(\eta) - \int_\Omega \mathcal{B}(., u_n, \nabla u_n) u_n \eta^p \end{aligned}$$

 Using the assumptions on $\mathcal{A}$ and $\mathcal{B}$, we get

$$
\begin{aligned}
&\alpha\int_\Omega \eta^p \mid \nabla(u_n) \mid^p \\
\leq\ & pM\int_\Omega k_0 \mid \nabla\eta \mid + pM^p\int_\Omega b_0 \mid \nabla\eta \mid + pM\int_\Omega a \mid \nabla u_n \mid^{p-1} \eta^{p-1} \mid \nabla\eta \mid \\
&+cM\int_\Omega \mid \nabla u_n \mid^r \eta^p + \int_\Omega (M^p d_0 + Mk + M^{\alpha+1}b + e) \\
\leq\ & a(p-1)^{-1}M\varepsilon pp_-1(\int_\Omega \mid \nabla u_n \mid^p \eta^p) + crp^{-1}M\varepsilon_r^p(\int_\Omega \mid \nabla u_n \mid^p \eta^p) \\
&+C(M,\Omega,\eta,\nabla\eta).
\end{aligned}
$$

Thus , for ε small enough , we obtain

$$\int_G \mid \nabla(u_n) \mid^p \leq C(M,\Omega,\eta,\nabla\eta,\varepsilon).$$

So $(\nabla u_n)_n$ is bounded in $L^p(G)$ and therefore $(\nabla u_n)_n$ converges weakly to ∇u

$$\mathrm{in}(L^p(G))^d.$$

Fix D an open subset of G and let $\eta \in C_0^\infty(G)$ such that $0 \leq \eta \leq 1$ and $\eta = 1$ on D. Take $\psi = \eta(u_n - u)$ as test function , then

$$
\begin{aligned}
&-\int_\Omega \eta\mathcal{A}(.,u_n,\nabla u_n).\nabla(u_n-u) \\
=\ & \int_\Omega (u_n-u)\mathcal{A}(.,u_n,\nabla u_n).\nabla\eta + \int_\Omega \mathcal{B}(.,u_n,\nabla u_n)(u_n-u)\eta
\end{aligned}
$$

Since $\mathcal{A}(.,u_n,\nabla u_n)$ is bounded in $L^{p'}(G)$ and $\mathcal{B}(.,u_n,\nabla u_n)$ is bounded in $L^q(G)$,

$$
\begin{aligned}
\lim_{n\to\infty}\int_G \mathcal{A}(.,u_n,\nabla u_n)(u_n-u)\nabla\eta = 0, \\
\lim_{n\to\infty}\int_G \mathcal{B}(.,u_n,\nabla u_n)(u_n-u)\eta = 0. \\
\text{Consequently, } \lim_{n\to\infty}\int_G \mathcal{A}(.,u_n,\nabla u_n)\eta\nabla(u_n-u) = 0\text{and} \\
\lim_{n\to\infty}\int_G (\mathcal{A}(.,u_n,\nabla u_n) - \mathcal{A}(.,u_n,\nabla u))\nabla(u_n-u) = 0.
\end{aligned}
$$

To complete the proof , we need to prove that $(\nabla u_n)_n$ converges to ∇u a . e . in Ω. That is the aim of the following lemma . **Lemma 4 . 8** *Let $G \subset \Omega$ and suppose that th e s equence $(\nabla u_n)_n$ is bounded in*

$$L^p(G) and$$

$$\lim_{n\to\infty}\int_G [\mathcal{A}(.,u_n,\nabla u_n) - \mathcal{A}(.,u,\nabla u)].\nabla(u_n-u) = 0.$$

Then $\mathcal{A}(.,u_n,\nabla u_n) \to \mathcal{A}(.,u,\nabla u)$ weakly in $L^{p'}(G)$.

 Proof . Put $v_n = [\mathcal{A}(., u_n, \nabla u_n) - \mathcal{A}(., u_n, \nabla u)].\nabla(u_n - u)$. Since

$$\begin{aligned}\int_G v_n &= \int_G [\mathcal{A}(., u_n, \nabla u_n) - \mathcal{A}(., u, \nabla u)].\nabla(u_n - u) \\ &\quad - \int_G [\mathcal{A}(., u_n, \nabla u) - \mathcal{A}(., u, \nabla u)].\nabla(u_n - u),\end{aligned}$$

for a subsequence we get

$$\lim_{n\to\infty} [\mathcal{A}(., u_n, \nabla u_n) - \mathcal{A}(., u_n, \nabla u)].\nabla(u_n - u) = 0$$

a . e .$x \in G \setminus N$ with $\mid N \mid= 0$. Let $x \in G \setminus N$. By the assumptions on $\mathcal{A}$ we have

$$v_n(x) \geq \alpha \mid \nabla u_n(x) \mid^p -F(\mid \nabla u_n(x) \mid^{p-1}, \mid \nabla u(x) \mid^{p-1}).$$

Consequently , $(\nabla u_n(x))_n$ is bounded and converges to some $\xi \in \mathbb{R}^d$. It follows that $[\mathcal{A}(., u, \xi) - \mathcal{A}(., u, \nabla u)].(\xi - \nabla u) = 0$ and hence $\xi = \nabla u$. Finally we concludeweaklythat to $\mathcal{A}(. \mathcal{A}(., u_n, \nabla u_n)$ in $L^{p'}$ $\to$ $(\mathcal{A}()_G ; u, \nabla u)$ a . e . in G and $\mathcal{A}(., u_n, \nabla u_n)$converge□

that Now ∇u_n we go back $\to \nabla u$ to the e. a. in Ω proof of and $\mathcal{A}(.,$ Theorem $u_n, \nabla u_n)$ 4 $\rightharpoonup$ .7. Using $\mathcal{A}$ Lemma $(., u, \nabla u)$ in $L^{p'}$ 4. 8, we (D). conclude Hence,

$$\int_D \mathcal{A}(., u, \nabla u)\nabla\phi + \int_D \mathcal{B}(., u, \nabla u)\phi = 0 \quad \forall \phi \in C_0^\infty(\Omega).$$

Moreover , using the fact that

$$-2^1_n - 2^1_m \leq u_m - u_n \leq 2^1_n + 2^1_m \quad \forall n, m$$

we obtain

$$-2^1_n + u_n \leq u \leq 2^1_n + u_n, \quad \forall n.$$

So , we deduce that for all n and all $z \in \partial\Omega$,

$$-2^1_n + f_n(z) \leq \overset{\inf}{\lim_{x\in\Omega,}}{}_{x\to z} u(z) \leq \overset{\sup}{\lim_{x\in\Omega,}}{}_{x\to z} u(z) \leq 2^1_n + f_n(z)$$

which implies $\lim_{x\to z} u(x) = f(z)$ and completes the proof of Theorem 4.7. □

Remark 4 . 2 Using the same techniques as in the proof of Theorem 4 . 7 we can show that every increasing and lo cally bounded sequence $(u_n)_n$ of supersolu - tions of (1 . 1) in Ω is lo cally bounded in $W^{1,p}(\Omega)$ and that $u = \lim_n u_n$ is a supersolution of (1 . 1) in Ω.

5 Sheaf property for Superharmonic functions

The obstacle Problem

Definition Let $f, h \in W^{1,p}(\Omega)$ and let
$K_{f,h} = \{u \in W^{1,p}(\Omega) : h \leq u$ a . e . in $\Omega, u - f \in W_0^{1,p}(\Omega)\}$. If $f = h$, we denote $K_{f,h} = K_f$.

We say that a function $u \in K_{f,h}$ is a solution to the obstacle problem in $K_{f,h}$ if

$$\int_\Omega \mathcal{A}(., u, \nabla u).\nabla(v - u) + \int_\Omega \mathcal{B}(., u, \nabla u)(v - u) \geq 0$$

whenever $v \in K_{f,h}$. This function u is called solution of the problem with obstacle h and boundary value f.

Remark 5 . 1 Since $u + \phi \in K_{f,h}$ for all nonnegative $\phi \in W_0^{1,p}(\Omega)$, the solution u to the obstacle problem is always a supersolution of *(1 . 1)* in Ω. Conversely , a supersolution of *(1 . 1)* is always a solution to the obstacle problem in $K_u(D)$ for all open $D \subset D \subset \Omega$.

Theorem 5 . 1 *Let h and f be in $W^{1,p}(\Omega) \cap L^\infty(\Omega)$. If v is an upper bounded supersolution of* (4 . 2) *with boundary value f such that $v \geq h$, then there exists a so lution u to the o bstacle pro b lem in $K_{f,h}$ with $u \leq v$.*

Proof . As in [1 8] , we introduce the function

$$g(x, \zeta, \xi) = \begin{cases} \widetilde{B}(x, \zeta, \xi) & \text{if} \zeta \leq v(x) \\ \widetilde{B}(x, v, \nabla v) & \text{if} \zeta > v(x). \end{cases}$$

As in [1 3] , we define the function

$$\mathbf{a}(x, \zeta, \xi) = \begin{cases} \mathcal{A}(x, \zeta, \xi) & \text{if} \zeta \leq v(x) \\ \mathcal{A}(x, v, \nabla v) & \text{if} \zeta > v(x). \end{cases}$$

Note that $\mathbf{a}$ satisfies the conditions (P 1) , (P 2) , and (P 3) .

$L^{p'}$ A(ΩLemma)isbounded in[7, p.52]andprovesthatcontinuous. Without themap $u \to$ loss of $g(x, u, \nabla u)$fromgenerality wecan

that $r \geq p - 1$. Let $l = \max\,\{q', pp_-r\} - 1$, and define the following penalty term

$$\gamma(x, s) = [(s - v(x))^+]^l \quad \forall x \in \Omega, s \in \mathbb{R}.$$

Let $M > 0$ and consider the map $T : K_{0,h} \to W^{-1,p'}(\Omega)$ defined by

$$\langle T(u), w\rangle = \int_\Omega \mathbf{a}(., u, \nabla u)\nabla w + \int_\Omega g(., u, \nabla u)w + M \int_\Omega \gamma(., u)w.$$

 Then for any $u, w \in K_{0,h}$, we have

$$| \int_\Omega g(x,u,\nabla u)w |\leq c_1 \parallel w \parallel l+1 + c_2 \parallel \nabla u \parallel_p^r \parallel w \parallel l+1,$$

$$| \int_\Omega \gamma(x,u)w |\leq c_3 \parallel w \parallel l+1 + c_4 \parallel u \parallel_{l+1}^l \parallel w \parallel l+1,$$

and for each $u \in K_{f,h} - f$, we have

$$\int_\Omega \gamma(.,u)u \geq c_5 \parallel u \parallel l^{l+1}_{+1} - c_6.$$

An easy computation shows that for $\varepsilon > 0$,

$$\begin{aligned}(T(u),u) \quad &\geq \quad (\alpha - c_2\varepsilon) \parallel \nabla u \parallel p^p - (c \parallel u \parallel_p^p + c_1 \parallel u \parallel_{l+1}^{l+1} + c_2 c(\varepsilon) \parallel u \parallel_{l+1}^{l+1}) \\ &\qquad + M c_5 \parallel u \parallel_{l+1}^{l+1} - M c_6 - c_1 c_7.\end{aligned}$$

where $c(\varepsilon)$ is a constant which depends on ε and $c > 0$. Now , we choose M large to get the operator T coercive . Since T is bounded , pseudomonotone and continuous , then by a Theorem in [22] , there exists $w \in K_{0,h}$ such that

$$(T(w), u-w) \geq 0 \text{forall} u \in K_{0,h}.$$

Next , we show that $w \leq v$. Since $w - ((w-v) \vee 0) \in K_{0,h}$ and since v is a supersolution of (4 . 2) , it follows that

$$\int_{\{w>v\}} [\mathcal{A}(.,w,\nabla w) - \mathcal{A}(.,v,\nabla v)]\nabla(w-v) \leq M \int_{\{w>v\}} \gamma(.,w)(v-w).$$

Thus by (P 2), $(w-v)^+ = 0$ a . e . in Ω and hence , $w \leq v$ on Ω. Finally , if we take $w_1 = w + f$, we obtain a supersolution of the obstacle problem $K_{f,h}$. $\square$

Nonlinear Harmonic Space

Definition Let V be a regular set . For every $f \in C(\partial V)$, we denote by $H_V f$
the solution of the Dirichlet problem with the boundary data f.

Proposition 5 . 2 *Let f and g in $C(\partial V)$ be such that $f \leq g$. Then*

$$H_V f \leq H_V g \qquad \textbf{i)}$$

i i) *For e very $k \geq 0$, we have $H_V(k+f) \leq H_V(f) + k$ and $H_V(f) - k \leq H_V(f-k)$.*

Definition Let U be an open set . We denote by $\mathcal{U}(U)$ the set of all open , regular subsets of U which are relatively compact in U.

We say that a function u is harmonic on U, if $u \in C(U)$ and u is a solution of (1 . 1) . We denote by $\mathcal{H}(U)$ the set of all harmonic functions on U. Then ,

$$\mathcal{H}(U) = \{u \in C(U) : H_V u = u \text{forevery} V \in \mathcal{U}(U)\}.$$

A lower semicontinuous function u is said to be hyperharmonic on U, if

• $-\infty < u$

• $u \neq \infty$ in each component of U

• For each regular set $V \subset V \subset \Omega$ and for every $f \in \mathcal{H}(V) \cap C(V)$, the inequality $f \leq u$ on ∂V implies $f \leq u$ in V. We denote by $*_{\mathcal{H}}(U)$ the set of all hyperharmonic functions on U.

An upper semicontinuous function u is said to be hypoharmonic on U, if

• $u < +\infty$

• $u \neq \infty$ in each component of U

• For each regular set $V \subset V \subset \Omega$ and each $f \in \mathcal{H}(V) \cap C(V)$, the inequality $f \geq u$ on ∂V implies $f \geq u$ in V. We denote by $\mathcal{H}_*(U)$ the set of all hypoharmonic functions on U.

Proposition 5 . 3 *Let* $u \in *_{\mathcal{H}}(U)$ *and* $v \in \mathcal{H}_*(U)$, *then for each* $k \geq 0$ *we have*

$$u + k \in *_{\mathcal{H}}(U) and v - k \in \mathcal{H}_*(U).$$

Proposition 5 . 4 *Let* u *be a superharmonic function and* v *be a subharmonic function on* U *such that*

$$\lim_{x}^{\sup v(x)} \to_z \leq \lim \inf_{x \to z} u(x)$$

for al l $z \in \partial U$, *and bo th s ides of the previous inequality are not s imultaneously* $+\infty$ *or* $-\infty$, *then* $v \leq u$ *in* U.

Proof . Let $x \in U$ and $\varepsilon > 0$. Choose a regular open set $V \subset V \subset U$ such that $x \in V$ and $v < u + \varepsilon$ on ∂V. Let $(\phi i) \in C^{\infty}(\Omega)$ be a decreasing sequence converging to v in V. Then $\phi i \leq u + \varepsilon$ on ∂V for i large . Let $h = H_V(\phi i)$, then $v \leq h \leq u + \varepsilon$ on V. By letting $\varepsilon \to 0$, we get $v(x) \leq u(x)$. $\square$

Theorem 5 . 5 *The space* $(\mathbb{R}^d, \mathcal{H})$ *satisfies the Bauer convergence property .*

Proof . Let $(u_n)_n$ be an increasing sequence in $\mathcal{H}(U)$ locally bounded . By Theorem 4 . 1 1 in [2 1] , for every $V \subset V \subset U$, the set $\{u_n(x), x \in V, n \in \mathbb{N}\}$ is equicontinuous . Then the sequence converges lo cally and uniformly in U to a continuous function u. Take $\varepsilon > 0$, since $u - \varepsilon \leq u_n \leq u + \varepsilon$, we get

$$H_V(u) - \varepsilon \leq u_n \leq H_V(u) + \varepsilon \text{and} \therefore H_V(u) = u \quad \square$$

Theorem 5 . 6 *Suppose that the conditions in subsection 4 . 1 are satisfied* , $k_0 = e = k = 0$ *and* $\alpha \geq p - 1$. *Then* $(\mathbb{R}^d, \mathcal{H})$ *is a nonlinear Bauer harmonic space .*

Proof . It is clear that $\mathcal{H}$ is a sheaf of continuous functions and by Theorem 4 . 7 there exists a basis of regular sets stable by intersection . The Bauer convergence property is fulfilled by Theorem 5 . 5 . Since $k_0 = e = k = 0$ and $\alpha \geq p-1$, we have the following form of the Harnack inequality (e . g . [2 1] , [26] or [24]) : For every non empty open set U in $\mathbb{R}^d$, for every constant $M > 0$ and every compact K in U, there exists a constant $C = C(K, M)$ such hat

$$\sup_K u \leq C \inf_K u$$

for every $u \in \mathcal{H}^+(U)$ with $u \leq M$. It follows that the sheaf $\mathcal{H}$ is non degenerate .

□

Theorem 5 . 7 *Suppose that the condition of s trict monotony holds . Let $u \in \mathcal{H}^*(\Omega) \cap L^\infty(\Omega)$. Then u is a supersolution on U.*

Proof . Let $V \subset V \subset \Omega$. Let $(\phi i)i$ be an increasing sequence in $C_c^\infty(\Omega)$ such that $u = \sup_i \phi i$ on V. Let

$$K_{\phi i} = \{w \in W^{1,p}_{\text{loc}}(\Omega) : \phi i \leq w, \quad w - \phi i \in W_0^{1,p}(V)\}.$$

We know by Theorem 5 . 1 that there exists a solution u_i to the obstacle problem $K_{\phi i}$ such that $\| u_i \|_\infty \leq \| \phi i \|_\infty$. We claim that $(u_i)i$ is increasing . In fact

$$u_i \wedge u_{i+1} \in K_{\phi i}, \text{then}$$

$$\int_{\{u_i > u_{i+1}\}} (\mathcal{A}(., u_i, \nabla u_i) - \mathcal{A}(., u_{i+1}, \nabla u_{i+1}))\nabla(u_{i+1} - u_i)$$
$$+ \int_{\{u_i > u_{i+1}\}} (\mathcal{B}(., u_i, \nabla u_i) - \mathcal{B}(., u_{i+1}, \nabla u_{i+1}))(u_{i+1} - u_i) \quad \geq 0.$$

Hence $\nabla(u_{i+1} - u_i)^+ = 0$ a . e . which yields that $u_i \leq u_{i+1}$ a . e . in V.

On the other hand , for each i the function u_i is a solution of (1 . 1) in $D_i := \{\phi i \quad < u_i\}$. Indeed , let $\psi \in C_c^\infty(W)$, $W \subset W \subset D_i$, and $\varepsilon > 0$ such that $\varepsilon \| \psi \| \leq \inf_W (u_i - \phi i)$. Then , we get $u_i + \varepsilon\psi \in K_{\phi i}$ and

$$\int_W \mathcal{A}(., u_i, \nabla u_i).\nabla\psi + \int_W \mathcal{B}(., u_i, \nabla u_i)\psi = 0.$$

Since

$$\liminf_{x \to y} u(x) \geq u(y) \geq \phi i(y) = \lim_{x \to y} \overset{u}{} i(x)$$

for all $y \in \partial D_i$, it yields , by the comparison principle , that $u \geq u_i$ in D_i. Hence $u \geq u_i$ in D. Thus $u = \lim_{i\to\infty} \phi i \leq \lim_{i\to\infty} u_i \leq u$. Finally , using Remark 4 . 2 we complete the proof . □

Theorem 5 . 8 *Suppose that the condition of s trict monotonicity ho lds . Then $*_{\mathcal{H}}$ is a shea f $-$ period*

The proof of this theorem is the same as in [2 , Theorem 4 . 2] .

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Azeddine Baalal
D *é* partement de Math *é* matiques et d ' Informatique , Facult *é* des Sciences A *ï* n Chock , Km 8 Route El Jadida BP 5366 M *â* arif , Casablanca , Maroc . E - mail : baalal @ facsc - achok . ac . ma

Nedra BelHaj Rhouma
Institut Pr é paratoire aux Etudes d ' Ing é nieurs de Tunis ,
2 Rue Jawaher Lel Nehru , 1 8 Montfleury , Tunis , Tunisie .
E - mail : Nedra . BelHajRhouma @ ipeit . rnu . tn