

OSCILLATION OF THIRD ORDER FUNCTIONAL DIFFERENTIAL EQUATIONS WITH DELAY

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ABSTRACT. We consider third order functional differential equations with discrete and continuous delay. We then develop several theorems related to the oscillatory behavior of these differential equations.

1. INTRODUCTION

Our goal in this paper is to study functional differential equations of the form

$$(b(t)(a(t)x'(t)))' + \sum_{i=1}^m qi(t)f(x(\sigma_i(t))) = h(t), \quad (1.1)$$

where $a, b, h \in C([t_0, \infty), \mathbb{R})$, $a(t), b(t) > 0$, $f : \mathbb{R} \rightarrow \mathbb{R}$ continuous, $\sigma_i(t) \rightarrow \infty$, as

$t \rightarrow \infty$, $i = 1, 2, \dots, m$, and

$$(b(t)(a(t)x'(t)))' + \int_c^d q(t, \xi)f(x(\sigma(t, \xi)))d\xi = 0, \quad (1.2)$$

where $a, b \in C([t_0, \infty), \mathbb{R})$, $f \in C(\mathbb{R}, \mathbb{R})$. The oscillations of solutions of third order equations were studied by Rao and Dahiya [8], Tantawy [9], Waltman [10] and

Zafer and Dahiya [11]. The results in this paper for equation (1.1) are more general comparing to Zafer and Dahiya [11]. The results for equation (1.2) are essentially

new.

As is customary, a solution of equations (1.1) and (1.2) is called oscillatory if it

has arbitrarily large zeros, otherwise it is called nonoscillatory. The solution of equations (1.1) and (1.2) is called almost oscillatory if it is oscillatory or $\lim_{t \rightarrow \infty} x^{(i)}(t) = 0$,

$i = 0, 1, 2$.

2. MAIN RESULTS

Oscillatory behavior of third order differential equations with discrete delay. Assume that $x f(x) > 0$, $x \neq 0$, $qi(t) \geq 0$ is not identically zero in any half

80 T . CANDAN & R . S . DAHIYA EJDE / CONF / 10 line of the form (τ, ∞) for some $\tau \geq 0, i = 1, 2, \dots, m$ and $\sigma_i(t) < t, \sigma'_i(t) > 0$, $i = 1, 2, \dots, m, b'(t) \geq 0$, and

$$\int_{\tau}^{\infty} \frac{dt}{b(t)} = \infty, \quad \int_{\tau}^{\infty} \frac{dt}{a(t)} = \infty. \quad (2.1)$$

Theorem 2 . 1 . Let $f(x) = x$ and $h(t) = 0$. Suppose that there exist a differentiable function $p \in C([t_0, \infty), \mathbb{R}), p(t) > 0$ such that

$$\int_{\tau}^{\infty} \left[q(t)p(t) - \frac{b(t)(p'(t))^2}{\sum_{i=1}^m \frac{(\sigma_i(t)-T)}{a(\sigma_i(t))} \sigma'_i(t) 4p(t)} \right] dt = \infty, \quad (2.2)$$

where $q(t) = \min\{q_1(t), q_2(t), \dots, q_m(t)\}$, for every $T \geq 0$, and that

$$\int_{\sigma(t)}^t \left[\int_{\sigma(t)}^r \frac{1}{a(u)} du \left(\int_u^r \frac{1}{b(v)} dv \right) \right] \sum_{i=1}^m qi(r) dr > 1, \quad (2.3)$$

where $\sigma(t) = \max\{\sigma_1(t), \sigma_2(t), \dots, \sigma_m(t)\}$. Then the equation (1.1) is oscillatory.

Proof . Let $x(t)$ be a non - oscillatory solution of (1.1). Assume $x(t)$ is eventually positive . Since $\sigma_i(t) \rightarrow \infty$ as $t \rightarrow \infty$ for $i = 1, 2, \dots, m$, there exist a $t_1 \geq t_0$ such that $x(t) > 0$ and $x(\sigma_i(t)) > 0$ for $t \geq t_1$. From (1.1), we have

$$(b(t)(a(t)x'(t)))' = - \sum_{i=1}^m qi(t)x(\sigma_i(t)). \quad (2.4)$$

Since $qi(t)$ is not negative and $x(\sigma_i(t)) > 0$ is positive for $t \geq t_1$, the right - hand side becomes non - positive . Therefore , we have

$$(b(t)(a(t)x'(t)))' \leq 0$$

for $t \geq t_1$. Thus , $x(t), x'(t), (a(t)x'(t))'$ are monotone and eventually one - signed . Now we want to show that there is a $t_2 \geq t_1$ such that for $t \geq t_2$

$$(a(t)x'(t))' > 0. \quad (2.5)$$

Suppose this is not true , then $(a(t)x'(t))' \leq 0$. Since $qi(t), i = 1, 2, \dots, m$ are not identically zero and $b(t) > 0$, it is clear that there is $t_3 \geq t_2$ such that $b(t_3)(a(t_3)x'(t_3))' < 0$. Then , for $t > t_3$ we have

$$b(t)(a(t)x'(t))' \leq b(t_3)(a(t_3)x'(t_3))' < 0. \quad (2.6)$$

Dividing (2.6) by $b(t)$ and then integrating between t_3 and t , we obtain

$$a(t)x'(t) - a(t_3)x'(t_3) < b(t_3)(a(t_3)x'(t_3))' \int_{t_3}^t \frac{1}{b(s)} ds. \quad (2.7)$$

Letting $t \rightarrow \infty$ in (2 . 7) , and because of (2 . 1) we see that $a(t)x'(t) \rightarrow -\infty$ as $t \rightarrow \infty$. Thus there is a $t_4 \geq t_3$ such that $a(t_4)x'(t_4) < 0$. Using $(a(t)x'(t))' \leq 0$, we have

$$\begin{aligned} & \text{for } t \geq t_4 \\ a(t)x'(t) & \leq a(t_4)x'(t_4). \quad (2.8) \end{aligned}$$

If we divide (2 . 8) by $a(t)$ and integrate from t_4 to t with $t \rightarrow \infty$, the right - hand side becomes negative . Thus , we have $x(t) \rightarrow -\infty$. But this is a contradiction $x(t)$ being eventually positive and therefore it proves that (2 . 5) holds . Now we consider two cases .

Suppose $x'(t)$ is eventually positive , say $x'(t) > 0$ for $t \geq t_2$. Define the function

$$z(t) \text{ by}$$

$$z(t) = \frac{b(t)(a(t)x'(t))'}{\sum_{i=1}^m x(\sigma_i(t))} p(t).$$

It is obvious that $z(t) > 0$ for $t \geq t_2$ and $z'(t)$ is

$$z'(t) = -\frac{\sum_{i=1}^m q_i(t)x(\sigma_i(t))}{mi_{=1}x(\sigma_i(t))}p(t) + \frac{p'(t)}{p(t)}z(t) - \frac{\sum_{i=1}^m x'(\sigma_i(t))\sigma'i(t)}{mi_{=1}x(\sigma_i(t))}z(t).$$

Then

$$z'(t) \leq -q(t)p(t) + \frac{p'(t)}{p(t)}z(t) - \frac{\sum_{i=1}^m x'(\sigma_i(t))\sigma'i(t)}{\sum_{i=1}^m x(\sigma_i(t))}z(t), \quad (2.9)$$

where $q(t) = \min\{q_1(t), q_2(t), \dots, q_m(t)\}$. On the other hand, using $(b(t)(a(t)x'(t)))' \leq 0, b'(t) \geq 0$ and (2.5), we can find that

$$(a(t)x'(t))'' \leq 0. \quad (2.10)$$

Using (2.10) and the equality

$$a(t)x'(t) = a(T)x'(T) + \int_T^t (a(s)x'(s))' ds, \quad (2.11)$$

we have

$$a(t)x'(t) \geq (t - T)(a(t)x'(t))' \quad (2.12)$$

for $T \geq t_2$. Since $(a(t)x'(t))'$ is non-increasing, we obtain

$$a(\sigma_i(t))x'(\sigma_i(t)) \geq (\sigma_i(t) - T)(a(t)x'(t))' \quad \text{for } i = 1, 2, \dots, m. \quad (2.13)$$

Multiplying both sides of (2.13) by

$$\frac{\sigma'i(t)}{a(\sigma_i(t))}$$

and taking the summation from 1 to m , we have

$$\sum_m^{i=1} \sigma'i(t)x'(\sigma_i(t)) \geq \sum_m^{i=1} \frac{(\sigma_i(t) - T)}{a(\sigma_i(t))} \sigma'i(t)(a(t)x'(t))'. \quad (2.14)$$

Then, using (2.14) in (2.9), it follows that

$$z'(t) \leq -q(t)p(t) + p \frac{p'(t)}{p(t)} z(t) - \frac{\sum_{i=1}^m \frac{(\sigma_i(t) - T)}{a(\sigma_i(t))} \sigma'i(t)}{b(t)p(t)} z(t),$$

and completing the square will lead to

$$z'(t) \leq -q(t)p(t) + \frac{b(t)(p')^2(t)}{\sum_{i=1}^m \frac{(\sigma_i(t)-T)}{a(\sigma_i(t))} \sigma' i(t) 4p(t)}. \quad (2.15)$$

Integrating (2 . 1 5) between T and t and letting $t \rightarrow \infty$, we see that $\lim_{t \rightarrow \infty} z(t) = -\infty$. This contradicts $z(t)$ being eventually positive .

If $x'(t)$ is eventually negative . We integrate (1 . 1) from t to ∞ and since

$$b(t)(a(t)x'(t))' > 0,$$

we have

$$-b(t)(a(t)x'(t))' + \int_t^\infty \sum_{m=1}^{i=1} q_i(r)x(\sigma_i(r))dr \leq 0. \quad (2.16)$$

82 T . CANDAN & R . S . DAHIYA EJDE / CONF / 10 Now integrating (2 . 16) from t to ∞ after dividing by $b(t)$ and using $a(t)x'(t) < 0$, will lead to

$$a(t)x'(t) + \int_t^\infty \left(\int_t^r \frac{1}{b(u)} du \right) \sum_{m=1}^{i=1} qi(r)x(\sigma_i(r))dr \leq 0. \quad (2.17)$$

Dividing (2 . 17) by $a(t)$ and integrating again from t to ∞ gives

$$\int_t^\infty \left[\int_t^r \frac{1}{a(u)} du \left(\int_u^r \frac{1}{b(v)} dv \right) \right] \sum_{m=1}^{i=1} qi(r)x(\sigma_i(r))dr \leq x(t). \quad (2.18)$$

Replacing t by $\sigma(t)$ in (2 . 18), where $\sigma(t) = \max \{ \sigma_1(t), \sigma_2(t), \dots, \sigma_m(t) \}$, will give

$$\int_{\sigma(t)}^t \left[\int_{\sigma(t)}^r \frac{1}{a(u)} du \left(\int_u^r \frac{1}{b(v)} dv \right) \right] \sum_{m=1}^{i=1} qi(r)x(\sigma_i(r))dr \leq x(\sigma(t)). \quad (2.19)$$

Using the fact that $\sigma_i(t) < t$ and $x(t)$ is decreasing in (2 . 19), we obtain

$$\int_{\sigma(t)}^t \left[\int_{\sigma(t)}^r \frac{1}{a(u)} du \left(\int_u^r \frac{1}{b(v)} dv \right) \right] \sum_{m=1}^{i=1} qi(r)dr \leq 1.$$

This is a contradiction to (2 . 3). Therefore , the proof is complete **Example 2 . 2** . Consider the following functional differential equation 2

$$(e^{-t}x')'' + \sum (2i-1)e^{-t}x(t-(i+1)\pi) = 0. \quad i=1$$

$$\text{Now } a(t) = e^{-t}, b(t) = 1, q_1(t) = e^{-t}, q_2(t) = 3e^{-t}\sigma_1(t) = t - 2\pi, \sigma_2(t) = t - 3\pi, \\ p(t) = e^t.$$

We can easily see that the conditions of Theorem 2 . 1 are satisfied . It is easy to verify that $x(t) = \cos t$ is a solution of this problem .

Theorem 2 . 3 . Let $f'(x) \geq \lambda$ for some $\lambda > 0$, and $h(t) = 0$. Suppose that there exist a differentiable function $p \in C([t_0, \infty), \mathbb{R}), p(t) > 0$ such that

$$\int^\infty \left[q(t)p(t) - \frac{b(t)(p'(t))^2}{\sum_{i=1}^m \frac{(\sigma_i(t)-T)}{a(\sigma_i(t))} \sigma'_i(t) 4\lambda p(t)} \right] dt = \infty, \quad (2.20)$$

where $q(t) = \min\{q_1(t), q_2(t), \dots, q_m(t)\}$, for every $T \geq 0$, and that

$$\lim_{t \rightarrow \infty} \sup \int_{\sigma(t)}^t \left[\int_{\sigma(t)}^r \frac{1}{a(u)} du \left(\int_u^r \frac{1}{b(v)} dv \right) \right] \sum_{m=1}^{i=1} qi(r)dr = \infty, \quad (2.21)$$

where $\sigma(t) = \max\{\sigma_1(t), \sigma_2(t), \dots, \sigma_m(t)\}$. Then the equation (1 . 1) is oscillatory .

Proof . The beginning part of the proof is similar to the proof of Theorem 2 . 1 until we reach at two possible cases . Suppose $x'(t)$ is eventually positive . Then , we can define

$$z(t) = \frac{b(t)(a(t)x'(t))'}{\sum_{i=1}^m f(x(\sigma_i(t)))} p(t) > 0.$$

It is obvious that $z(t) > 0$ for $t \geq t_2$ and $z'(t)$ is

$$\begin{aligned} z'(t) &= -\frac{\sum_{i=1}^m qi(t)f(x(\sigma_i(t)))}{\sum_{i=1}^m f(x(\sigma_i(t)))} p(t) \\ &+ \frac{p'(t)}{p(t)} z(t) - \frac{\sum_{i=1}^m f'(x(\sigma_i(t)))x'(\sigma_i(t))\sigma'_i(t)}{\sum_{i=1}^m f(x(\sigma_i(t)))} z(t). \end{aligned}$$

$$z'(t) \leq -q(t)p(t) + \frac{p'(t)}{p(t)}z(t) - \frac{\sum_{i=1}^m f'(x(\sigma_i(t))x'(\sigma_i(t))\sigma'_i(t))}{\sum_{i=1}^m f(x(\sigma_i(t)))}z(t), \quad (2.22)$$

where $q(t) = \min \{q_1(t), q_2(t), \dots, q_m(t)\}$. On the other hand, since $(b(t)(a(t)x'(t)))' \leq 0$, (2.5) holds and $b'(t) \geq 0$, we can obtain

$$(a(t)x'(t))'' \leq 0. \quad (2.23)$$

Using (2.23) and the equality

$$a(t)x'(t) = a(T)x'(T) + \int_T^t (a(s)x'(s))' ds \quad (2.24)$$

will lead to

$$a(t)x'(t) \geq (t - T)(a(t)x'(t))'. \quad (2.25)$$

Now using non-increasing nature of $(a(t)x'(t))'$, we obtain

$$a(\sigma_i(t))x'(\sigma_i(t)) \geq (\sigma_i(t) - T)(a(t)x'(t))' \quad \text{for } i = 1, 2, \dots, m. \quad (2.26)$$

Multiplying both sides of (2.26) by

$$\frac{\sigma'_i(t)}{a(\sigma_i(t))}$$

and taking the summation from 1 to m , we have

$$\sum_{i=1}^m \sigma'_i(t)x'(\sigma_i(t)) \geq \sum_{i=1}^m \frac{(\sigma_i(t) - T)}{a(\sigma_i(t))} \sigma'_i(t)(a(t)x'(t))'. \quad (2.27)$$

Then, using (2.27) in (2.22), it follows that

$$z'(t) \leq -q(t)p(t) + \frac{p'(t)}{p(t)}z(t) - \lambda \frac{\sum_{i=1}^m \frac{(\sigma_i(t) - T)}{a(\sigma_i(t))} \sigma'_i(t)}{b(t)p(t)} z(t),$$

and then completing the square leads to

$$z'(t) \leq -q(t)p(t) + \frac{b(t)(p'(t))^2}{\lambda \sum_{i=1}^m \frac{(\sigma_i(t) - T)}{a(\sigma_i(t))} \sigma'_i(t) 4p(t)}. \quad (2.28)$$

Integrating (2.28) between T to t and letting $t \rightarrow \infty$, we see that $\lim_{t \rightarrow \infty} z(t) = -\infty$. This contradicts $z(t)$ being eventually positive.

If $x'(t)$ is eventually negative and proceeding as in the proof of Theorem 2.1 we will end up with

$$\int_{\sigma(t)}^{\infty} \left[\int_{\sigma(t)}^r \frac{1}{a(u)} du \left(\int_u^r \frac{1}{b(v)} dv \right) \right] \sum_{i=1}^m q_i(r) f(x(\sigma_i(r))) dr \leq x(\sigma(t)),$$

where $\sigma(t) = \max \{ \sigma_1(t), \sigma_1(t), \dots, \sigma_n(t) \}$. Thus we have

$$\int_{\sigma(t)}^t \left[\int_{\sigma(t)}^r \frac{1}{a(u)} du \left(\int_u^r \frac{1}{b(v)} dv \right) \right] \sum_{m=1}^i q_i(r) f(x(\sigma_i(r))) dr \leq x(\sigma(t)). \quad (2.29)$$

Using the fact that $\sigma_i(t) < t$, $f(x)$ is increasing and $x(t)$ is decreasing in (2 . 29) , we obtain

$$\int_{\sigma(t)}^t \left[\int_{\sigma(t)}^r \frac{1}{a(u)} du \left(\int_u^r \frac{1}{b(v)} dv \right) \right] \sum_{m=1}^i q_i(r) dr \leq \frac{x(\sigma(t))}{f(x(\sigma(t)))}.$$

84 T . CANDAN & R . S . DAHIYA EJDE / CONF / 1 0 Since $x(t)$ is decreasing and positive , it is approaching a finite non - negative number as $t \rightarrow \infty$. In view of (2 . 2 1) and the last equation , it is not possible that $\lim_{t \rightarrow \infty} x(t) >$

0. Suppose $\lim_{t \rightarrow \infty} x(t) = 0$, then

$$\lim_{t \rightarrow \infty} \frac{x(\sigma(t))}{f(x(\sigma(t)))} = \lim_{t \rightarrow \infty} \frac{1}{f'(x(\sigma(t)))} = \frac{1}{f'(0)} \leq \frac{1}{\lambda}.$$

This is a contradiction to (2 . 2 1) . Therefore , the proof is complete . **Theorem 2 . 4 .** Suppose that $f'(x) \geq \lambda$ for some $\lambda > 0$ and

$$\lim_{t \rightarrow \infty} \sup \int_{\sigma(t)}^t \left[\int_{\sigma(t)}^r \frac{1}{a(u)} du \left(\int_u^r \frac{1}{b(v)} dv \right) \right] \sum_{i=1}^m q_i(r) dr = \infty. \quad (2.30)$$

In addition , suppose that there exist a continuously differentiable function $p \in C([t_0, \infty), \mathbb{R})$, $p(t) > 0$ and an oscillatory function $\psi(t)$ such that

$$\int_{\sigma(t)}^t \left[q(t)p(t) - \frac{b(t)(p'(t))^2}{\sum_{i=1}^m \frac{(\sigma_i(t)-T)}{a(\sigma_i(t))} \sigma'_i(t) 4\lambda dp(t)} \right] dt = \infty \quad (2.31)$$

for some $d \in (0, 1)$ and for every $T \geq 0$, and

$$(b(t)(a(t)\psi'(t))')' = h(t), \quad \lim_{t \rightarrow \infty} \psi^{(i)}(t) = 0, \quad i = 0, 1, 2. \quad (2.32)$$

Then the equation (1 . 1) is almost oscillatory .

Proof . Let $x(t)$ be a non - oscillatory solution of (1 . 1) . Without loss of generality we may assume that $x(t)$ is eventually positive . Consider

$$y(t) = x(t) - \psi(t). \quad (2.33)$$

Obviously $y(t)$ is eventually positive , otherwise , $x(t) < \psi(t)$ and it is a contradiction with oscillatory behavior of $\psi(t)$. We know that ,

$$(b(t)(a(t)y'(t))')' \leq 0. \quad (2.34)$$

Proceeding as in the proof of Theorem 2 . 1 , there is a $t_1 \geq 0$ such that for $t \geq t_1$

$$(a(t)y'(t))' > 0 \quad \text{and} \quad (a(t)y'(t))'' \leq 0.$$

Consider again two cases . Suppose that $y'(t)$ is eventually positive , then $y(t)$ is increasing and eventually positive . On the other hand , since $\psi(t) \rightarrow 0$ as $t \rightarrow \infty$ and $y(t) = x(t) - \psi(t)$, there exists a $t_2 \geq t_1$ such that

$$x(\sigma_i(t)) \geq dy(\sigma_i(t)) \quad \text{for} \quad t \geq t_2 \quad \text{and} \quad d \in (0, 1), \quad i = 1, 2, \dots, m.$$

Since f is an increasing function , we obtain

$$f(x(\sigma_i(t))) \geq f(dy(\sigma_i(t))) \quad \text{for } t \geq t_2, \quad i = 1, 2, \dots, m.$$

Define $z(t)$ by

$$z(t) = \frac{b(t)(a(t)y'(t))'}{\sum_{i=1}^m f(dy(\sigma_i(t)))} p(t),$$

then obviously $z(t) > 0$ for $t \geq t_2$ and $z'(t)$ is

$$\begin{aligned} z'(t) = & -\frac{\sum_{i=1}^m q_i(t)f(x(\sigma_i(t)))}{\sum_{i=1}^m f(dy(\sigma_i(t)))} p(t) + p \frac{p'(t)}{p(t)} z(t) \\ & - d \frac{\sum_{i=1}^m f'(dy(\sigma_i(t)))y'(\sigma_i(t))\sigma'_i(t)}{\sum_{i=1}^m f(dy(\sigma_i(t)))} z(t). \end{aligned}$$

$$z'(t) \leq -q(t)p(t) + \frac{p'(t)}{p(t)}z(t) - d\lambda \frac{\sum_{i=1}^m y'(\sigma_i(t))\sigma'_i(t)}{\sum_{i=1}^m f(dy(\sigma_i(t)))}z(t), \quad (2.35)$$

where $q(t) = \min\{q_1(t), q_2(t), \dots, q_m(t)\}$. We can now show that

$$\sum_{i=1}^m \sigma'_i(t)y'(\sigma_i(t)) \geq \sum_{i=1}^m \frac{(\sigma_i(t) - T)}{a(\sigma_i(t))} \sigma'_i(t)(a(t)y'(t))' \quad (2.36)$$

as in proof of Theorem 2.1. Using (2.35) and (2.36), we have

$$z'(t) \leq -q(t)p(t) + p \frac{p'(t)}{p(t)}z(t) - d\lambda \frac{\sum_{i=1}^m \frac{(\sigma_i(t)-T)}{a(\sigma_i(t))} \sigma'_i(t)}{b(t)p(t)} z(t).$$

Completing the square in the above equation leads to

$$z'(t) \leq -q(t)p(t) + \frac{b(t)(p')^2(t)}{\sum_{i=1}^m \frac{(\sigma_i(t)-T)}{a(\sigma_i(t))} \sigma'_i(t)4\lambda dp(t)}. \quad (2.37)$$

Integrating (2.37) from T to t and letting $t \rightarrow \infty$, we see that $\lim_{t \rightarrow \infty} z(t) = -\infty$. This contradicts $z(t)$ being eventually positive.

Now suppose $y'(t)$ is eventually negative. Since y is eventually positive and decreasing, $\lim_{t \rightarrow \infty} y(t) = c$, where c is a nonnegative number. Therefore, $\lim_{t \rightarrow \infty} x(t) = c$. Integrating (1.1) three times as we did in the proof of Theorem 2.1, we will end up with

$$\int_{\sigma(t)}^{\infty} \left[\int_{\sigma(t)}^r \frac{1}{a(u)} du \left(\int_u^r \frac{1}{b(v)} dv \right) \right] \sum_{i=1}^m qi(r)f(x(\sigma_i(r)))dr \leq y(t),$$

where $\sigma(t) = \max \{\sigma_1(t), \sigma_1(t), \dots, \sigma_n(t)\}$. Thus we have

$$\int_{\sigma(t)}^t \left[\int_{\sigma(t)}^r \frac{1}{a(u)} du \left(\int_u^r \frac{1}{b(v)} dv \right) \right] \sum_{i=1}^m qi(r)f(x(\sigma_i(r)))dr \leq y(t). \quad (2.38)$$

Hence, we conclude that $\lim_{t \rightarrow \infty} \inf x(t) = 0$. But $x(t)$ is monotone, so we have $\lim_{t \rightarrow \infty} x(t) = 0$. Thus $c = 0$ and by (2.32) and (2.33) $\lim_{t \rightarrow \infty} x^{(i)}(t) = 0$, $i = 0, 1, 2$, which means that $x(t)$ is almost oscillatory. This completes the proof.

Oscillatory behavior of third order differential equations with continuous deviating arguments. Suppose that the following conditions hold unless stated otherwise

$$(a) \quad a(t) > 0, b(t) > 0, b'(t) \geq 0, \int_{t_0}^{\infty} \frac{dt}{a(t)} = \infty, \int_{t_0}^{\infty} \frac{dt}{b(t)} = \infty,$$

$$(b) \quad q(t, \xi) \in C([t_0, \infty) \times [c, d], \mathbb{R}), q(t, \xi) > 0,$$

$$(c) \quad \frac{f(x)}{x} \geq \epsilon > 0, \text{ for } x \neq 0, \epsilon \text{ is a constant},$$

(d) $\sigma(t, \xi) \in C([t_0, \infty) \times [c, d], \mathbb{R}), \sigma(t, \xi) < t, \xi \in [c, d], \sigma(t, \xi)$ is nondecreasing with respect to t and ξ and

$$\lim_{t \rightarrow \infty} \min_{\xi \in [c, d]} \sigma(t, \xi) = \infty.$$

Theorem 2 . 5 . *If*

$$\int_{t_1}^{\infty} \int_c^d q(s, \xi) d\xi ds = \infty \quad (2.39)$$

and

$$\epsilon \int_{g(t)}^t \left[\int_{g(t)}^r \frac{1}{a(u)} du \left(\int_u^r \frac{1}{b(v)} dv \right) \right] \int_c^d q(r, \xi) d\xi dr > 1, \quad (2.40)$$

where $g(t) = \sigma(t, d)$. Then the equation (1 . 2) is oscillatory .

Proof . Suppose that $x(t)$ is non - oscillatory solution of (1 . 2) . Without loss of generality we may assume that $x(t)$ is eventually positive . (If $x(t)$ is eventually negative solution , it can be proved by the same arguments) . From (1 . 2) , we have

$$(b(t)(a(t)x'(t))')' = - \int_c^d q(t, \xi) f(x(\sigma(t, \xi))) d\xi. \quad (2.41)$$

Proceeding as in the proof of Theorem 2 . 1 , we have

$$\begin{aligned} (b(t)(a(t)x'(t))')' &\leq 0, \\ (a(t)x'(t))' &> 0 \quad \text{and} \quad (a(t)x'(t))'' \leq 0 \end{aligned}$$

for large enough t . Thus , $x(t)$, $x'(t)$ and $(a(t)x'(t))'$ are monotone and eventually one - signed . From condition (c),

$$f(x(\sigma(t, \xi))) \geq \epsilon x(\sigma(t, \xi)) > 0.$$

Therefore ,

$$0 \geq (b(t)(a(t)x'(t))')' + \epsilon \int_c^d q(t, \xi) x(\sigma(t, \xi)) d\xi. \quad (2.42)$$

Now consider again two cases .

Suppose that $x'(t)$ is eventually positive , say $x'(t) > 0$ for $t > t_2$. Now we can choose a constant $k > 0$ such that $x(k) > 0$. By (d), there exist a sufficiently large T such that $\sigma(t, \xi) > k$ for $t > T, \xi \in [c, d]$. Therefore ,

$$x(\sigma(t, \xi)) \geq x(k).$$

Thus ,

$$(b(t)(a(t)x'(t))')' + \epsilon x(k) \int_c^d q(t, \xi) d\xi \leq 0. \quad (2.43)$$

Integrating this last equation from t_1 to t , we get

$$b(t)(a(t)x'(t))' \leq b(t_1)(a(t_1)x'(t_1))' - \epsilon x(k) \int_{t_1}^t \int_c^d q(s, \xi) d\xi ds. \quad (2.44)$$

Taking the limit of both sides as $t \rightarrow \infty$ and using (2 . 39) , the last inequality above leads to a contradiction to $(a(t)x'(t))' > 0$. Now suppose $x'(t)$ is eventually

negative . Proceeding as in the proof of Theorem 2 . 1 and integrating equation (1 . 2) three times , we get

$$\int_t^\infty \left[\int_t^r \frac{1}{a(u)} du \left(\int_u^r \frac{1}{b(v)} dv \right) \right] \int_c^d q(r, \xi) f(x(\sigma(r, \xi))) d\xi dr \leq x(t) \quad (2.45)$$

Using (c) in (2 . 45) , we obtain

$$\int_t^\infty \left[\int_t^r \frac{1}{a(u)} du \left(\int_u^r \frac{1}{b(v)} dv \right) \right] \epsilon \int_c^d q(r, \xi) x(\sigma(r, \xi)) d\xi dr \leq x(t). \quad (2.46)$$

$$\epsilon \int_{g(t)}^t \left[\int_{g(t)}^r \frac{1}{a(u)} du \left(\int_u^r \frac{1}{b(v)} dv \right) \right] \int_c^d q(r, \xi) x(\sigma(r, \xi)) d\xi dr \leq x(g(t)). \quad (2.47)$$

Since $x(t)$ is decreasing and positive,

$$\epsilon \int_{g(t)}^t \left[\int_{g(t)}^r \frac{1}{a(u)} du \left(\int_u^r \frac{1}{b(v)} dv \right) \right] \int_c^d q(r, \xi) d\xi dr \leq 1.$$

This is a contradiction to (2.40). Therefore, the proof is complete. **Example 2.6.** Consider the following functional differential equation

$$x''' + \int_{2/7\pi}^{1/2\pi} \frac{2e^{-1/\xi}}{\xi^2} x\left(t - \frac{1}{\xi}\right) d\xi = 0$$

so that $a(t) = 1, b(t) = 1, f(x) = x, q(t, \xi) = \frac{2e^{-1/\xi}}{\xi^2}, \sigma(t, \xi) = t - \frac{1}{\xi}$. We can easily see that the conditions of Theorem 2.5 are satisfied. It is easy to verify that $x(t) = e^{-t} \sin t$ is a solution of this problem.

Theorem 2.7. Suppose (2.40) holds. In addition to that suppose there exist $p \in$

$$C([t_0, \infty), \mathbb{R}), p(t) > 0 \text{ such that } \int_{t_0}^{\infty} \left[\Gamma(t)p(t) - \frac{a(\sigma(t, c))b(t)(p'(t))^2}{(\sigma(t, c) - T)\sigma'(t, c)4p(t)} \right] dt = \infty, \quad (2.48)$$

where $\Gamma(t) = \epsilon \int_c^d q(t, \xi) d\xi$. Then the equation (1.2) is oscillatory.

Proof. Suppose that $x(t)$ is non-oscillatory solution of (1.2). We can assume that

$x(t)$ is eventually positive. The case of $x(t)$ is eventually negative can be proved by the same arguments. Proceeding as in the proof of Theorem 2.1, we have

$$\begin{aligned} (b(t)(a(t)x'(t)))' &\leq 0, \\ (a(t)x'(t))' &> 0 \quad \text{and} \quad (a(t)x'(t))'' \leq 0. \end{aligned}$$

Thus, $x(t), x'(t)$ and $(a(t)x'(t))'$ are monotone and eventually one-signed. From

$$\begin{aligned} &\text{condition (c),} \\ &f(x(\sigma(t, \xi))) \geq \epsilon x(\sigma(t, \xi)) > 0. \\ &(b(t)(a(t)x'(t)))' + \epsilon \int_c^d q(t, \xi) x(\sigma(t, \xi)) d\xi \leq 0. \end{aligned} \quad (2.49)$$

If $x'(t)$ is eventually positive, then we can define

$$z(t) = \frac{b(t)(a(t)x'(t))'}{x(\sigma(t, c))} p(t).$$

It is obvious that $z(t) > 0$ for $t \geq t_2$ and $z'(t)$ is

$$z'(t) = \frac{(b(t)(a(t)x'(t)))'}{x(\sigma(t, c))} p(t) + p \frac{p'(t)}{p(t)} z(t) - \frac{x'(\sigma(t, c))\sigma'(t, c)}{x(\sigma(t, c))} z(t). \quad (2.50)$$

From proof of Theorem 2.1, we have

$$a(t)x'(t) \geq (t - T)(a(t)x'(t))'.$$

Since $(a(t)x(t))'$ is non - increasing , we have

$$a(\sigma(t, c))x'(\sigma(t, c)) \geq (\sigma(t, c) - T)(a(t)x'(t))',$$

$$x'(\sigma(t, c)) \geq \frac{(\sigma(t, c) - T)(a(t)x'(t))'}{a(\sigma(t, c))} \quad (2.51)$$

Plug (2.51) in (2.50), then we obtain

$$z'(t) = \frac{(b(t)(a(t)x'(t))')'}{x(\sigma(t, c))} p(t) + p \frac{p'(t)}{p(t)} z(t) - \frac{(\sigma(t, c) - T)\sigma'(t, c)}{p(t)b(t)a(\sigma(t, c))} z^2(t).$$

Completing the square leads to

$$z'(t) \leq -\Gamma(t)p(t) + \frac{b(t)a(\sigma(t, c))(p'(t))^2}{(\sigma(t, c) - T)\sigma'(t, c)4p(t)}. \quad (2.52)$$

Integrating (2.52) from T to t and letting $t \rightarrow \infty$, we see that $\lim_{t \rightarrow \infty} z(t) = -\infty$. This contradicts $z(t)$ being eventually positive.

If $x'(t)$ is eventually negative, the proof is exactly the same as in the second part of the proof of previous Theorem.

REFERENCES

- [1] D. D. Bainov and D. P. Mishev, Oscillation Theory for Neutral Differential Equations with Delay, *Adam Hilger* (1981).
- [2] T. Candan and R. S. Dahiya, Oscillation of third order nonlinear differential equations with and without delay, *Int. J. Diff. Eq. and Appl.* **4** (2), (2001), 125 - 139.
- [3] T. Candan and R. S. Dahiya, Oscillatory behavior of n -th order nonlinear neutral differential equations with forcing terms, *Math. J. Toyama Univ.* **24**, (2001), 9 - 19.
- [4] R. S. Dahiya, Oscillation of third order differential equations with and without delay, *Differential Equations and Nonlinear Mechanics* **5** **28** (2001), 75 - 87.
- [5] I. Györi and G. Ladas, Oscillation Theory of Delay Differential Equations with Applications, *Clarendon Press, Oxford* (1991).
- [6] I. T. Kiguradze, On the oscillation of solutions of the equation $d^m u/dt^m + a(t)u^m \operatorname{sign} u = 0$, *Mat. Sb.* **65** (1965), 172 - 187.
- [7] G. S. Ladde, V. Lakshmikantham and B. G. Zhang, Oscillation Theory of Differential Equations with Deviating Arguments, *Marcel Dekker, Inc.* (1987).
- [8] V. Sree Hari Rao and R. S. Dahiya, Properties of solutions of a class of third - order linear differential equations, *Period. Math. Hungarica* **20** (3) (1989), 177 - 184.
- [9] A. S. Tantawy, "Oscillation of Third and Higher Order Functional Differential Equations", *Ph. D Thesis, ISU* (1989). [10] P. Waltman, Oscillation criteria for third - order nonlinear differential equations, *Pacific J. Math.* **18** (1966), 385 - 389. [11] A. Zafer and R. S. Dahiya, Oscillatory and Asymptotic Behavior of Solutions of Third Order Delay Differential Equations With Forcing Terms, *Differential Equations and Dynamical Systems* **1** (2), (1993) 123 - 136.

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