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Strongly nonlinear parabolic initial - boundary value problems in Orlicz spaces *

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Abstract

We prove existence and convergence theorems for nonlinear parabolic problems . We also prove some compactness results in inhomogeneous Orlicz - Sobolev spaces .

1 Introduction

Let Ω be a bounded domain in $\mathbb{R}^N, T > 0$ and let

$$A(u) = \sum_{|\alpha| \leq 1} (-1)^{|\alpha|} D^\alpha A_\alpha(x, t, u, \nabla u)$$

be a Leray - Lions operator defined on $L^p(0, T; W^{1,p}(\Omega)), 1 < p < \infty$. Boccardo and Murat [5] proved the existence of solutions for parabolic initial - boundary value problems of the form

$$\frac{\partial u}{\partial t} + A(u) + g(x, t, u, \nabla u) = f \quad \text{in} \Omega \times (0, T), \tag{1.1}$$

where g is a nonlinearity with the following growth condition

$$g(x, t, s, \xi) \leq b(| s |)(c(x, t) + | \xi |^q), \quad q < p, \tag{1.2}$$

hand andwhich side satisfies f is assumed the classical(in [5])sign condition tobelong to $L^{p'}$ $g(x, t, s, \xi)s$ $(0, T; W^{-1,p'}$ $\geq (\Omega)0)$. This The right result generalizes the analogous one of Landes - Mustonen [1 4] where the nonlinearity g depends only on x, t and u. In [5] and [1 4] , the functions A_α are assumed to satisfy a polynomial growth condition with respect to u and ∇u. When trying to relax this restriction on the coefficients A_α, we are led to replace $L^p(0, T; W^{1,p}(\Omega))$ by an inhomogeneous Sobolev space $W^{1,x}L_M$ built from an Orlicz space L_M instead of L^p, where the N - function M which defines L_M is related to the actual growth of the A_α ' s . The solvability of (1 . 1) in this

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setting is proved by Donaldson [7] and Robert [1 6] in the case where $g \equiv 0$. It is our purpose in this paper , to prove existence theorems in the setting of the inhomogeneous Sobolev space $W^{1,x}L_M$ by applying some new compactness results in Orlicz spaces obtained under the assumption that the N - function $M(t)$ satisfies $\Delta'-$ condition and which grows less rapidly than $\mid t \mid^{N/(N-1)}$. These compactness results , which we are at first established in [8] , generalize those of Simon [1 7] , Landes - Mustonen [1 4] and Boccardo - Murat [6] . It is not clear whether the present approach can be further adapted to obtain the same results for general N - functions .

For related topics in the elliptic case , the reader is referred to [2] and [3] .

2 Preliminaries

Let $M : \mathbb{R}^+ \to \mathbb{R}^+$ be an N - function , i . e . M is continuous , convex , with $M(t) > 0$ for $t > 0$, $\frac{M(t)}{t} \to 0$ as $t \to 0$ and $\frac{M(t)}{t} \to \infty$ as $t \to \infty$. Equivalently , M admits the representation : $M(t) = \int_0^t a(\tau)d\tau$ where $a : \mathbb{R}^+ \to \mathbb{R}^+$ is non - decreasing , right continuous , with $a(0) = 0, a(t) > 0$ for $t > 0$ and $a(t) \to \infty$

where $\frac{a:}{\text{as}t\to\infty}\mathbb{R}+$. The $\to^{\text{N}}$ $-$function Mconjugateto$M$isdefinedby$\mathbb{R}^+$isgivenby $_a(t)=\sup\{s:a(s)\leq t\}[1,1-M(t) = \int_0^t$ 1, 12].

The N - function M is said to satisfy the Δ_2 condition if , for some $k > 0$:

$$M(2t) \leq kM(t) \quad \text{for all} t \geq 0, \tag{2.1}$$

when this inequality holds only for $t \geq t_0 > 0$, M is said to satisfy the Δ_2 condition near infinity .

Let P and Q be two N - functions . $P \ll Q$ means that P grows essentially less rapidly than Q; i . e . , for each $\varepsilon > 0$,

$$\frac{P(t)}{Q(\varepsilon t)} \to 0 \quad \text{as} t \to \infty.$$

This is the case if and only if

$$\lim_{t\to\infty} \frac{Q^{-1}(t)}{P-1(t)} = 0.$$

An N - function is said to satisfy the $\triangle' -$ condition if , for some $k_0 > 0$ and some

$$t_0 \geq 0 :$$

$$M(k_0tt') \leq M(t)M(t'), \quad \text{for all} t, t' \geq t_0. \quad (2.2)$$

It is easy to see that the $\triangle' -$ condition is stronger than the $\triangle_2 -$ condition . The following N - functions satisfy the $\triangle' -$ condition : $M(t) = t^p(\text{Log}^q t)^s$, where $1 < p < +\infty, 0 \leq s < +\infty$ and $q \geq 0$ is an integer (Log^q being the iterated of order q of the function log) .

We will extend these N - functions into even functions on all $\mathbb{R}$. Let Ω be an open subset of $\mathbb{R}^N$. The Orlicz class $\mathcal{L}_M(\Omega)$(resp . the Orlicz space $L_M(\Omega)$) is

defined as the set of (equivalence classes of) real - valued measurable functions u on Ω such that :
$\int_\Omega M(u(x))dx < +\infty$ (resp . $\int_\Omega M(\frac{u(x)}{\lambda})dx < +\infty$ for some $\lambda > 0$). Note that $L_M(\Omega)$ is a Banach space under the norm

$$\| u \| M, \Omega = \inf\{\lambda > 0 : \int_\Omega M(\frac{u(x)}{\lambda})dx \le 1\}$$

and $\mathcal{L}_M(\Omega)$ is a convex subset of $L_M(\Omega)$. The closure in $L_M(\Omega)$ of the set of bounded measurable functions with compact support in $\bar{}\Omega$ is denoted by $E_M(\Omega)$. The equality $E_M(\Omega) = L_M(\Omega)$ holds if and only if M satisfies the Δ_2 condition , for all t or for t large according to whether Ω has infinite measure or not .

The dual of $E_M(\Omega)$ can be identified with $L\underline{\hspace{4em}}_M(\Omega)$ by means of the pairing $\int_\Omega u(x)v(x)dx$, and the dual norm on $L_{\bar{}M}(\Omega)$ is equivalent to $\| . \|_{\overline{M,\Omega\cdot}}$ The space $L_M(\Omega)$ is reflexive if and only if M and $\bar{}M$ satisfy the Δ_2 condition , for all t or for t large , according to whether Ω has infinite measure or not .

We now turn to the Orlicz - Sobolev space . $W^1L_M(\Omega)$(resp . $W^1E_M(\Omega)$) is the space of all functions u such that u and its distributional derivatives up to order 1 lie in $L_M(\Omega)$(resp .$E_M(\Omega)$). This is a Banach space under the norm

$$\| u \| 1, M, \Omega = \sum_{| \alpha | \le 1} \| D^\alpha u \| M, \Omega.$$

Thus $W^1L_M(\Omega)$ and $W^1E_M(\Omega)$ can be identified with subspaces of the product of $N+1$ copies of $L_M(\Omega)$. Denoting this product by ΠL_M, we will use the weak topologies $\sigma(\Pi L_M, \Pi E - line_M)$ and $\sigma(\Pi L_M, \Pi L_{\overline{M}}\underline{\hspace{4em}})$. The space $W_0^1E_M(\Omega)$ is defined as the (norm) closure of the Schwartz space $\mathcal{D}(\Omega)$ in $W^1E_M(\Omega)$ and the space $W_0^1L_M(\Omega)$ as the $\sigma(\Pi L_M, \Pi E - line_M)$ closure of $\mathcal{D}(\Omega)$ in $W^1L_M(\Omega)$. We say

$$\lambda > 0, \int_\Omega M(\frac{D^\alpha u_n - D^\alpha u}{\lambda})dx \to 0 \text{forall} \mid \alpha \mid \le 1. \quad \text{Thisimpliesconvergencefor}^{\sigma(\Pi L_M, \Pi L_{\overline{M}}\underline{\hspace{4em}}).\text{If}M\text{satisfiesthe}\Delta_2\text{condition}}_{\text{that}u_n\text{convergesto}u\text{forthemodularconvergencein}W}$$

has finite measure) , then modular convergence coincides with norm convergence . Let $W^{-1}L-line_M(\Omega)$(resp . $W^{-1}line{-}E_M(\Omega)$) denote the space of distributions on Ω which can be written as sums of derivatives of order ≤ 1 of functions in $L\underline{\hspace{4em}}_M(\Omega)$ (resp .$E - line_M(\Omega)$). It is a Banach space under the usual quotient norm .

If the open set Ω has the segment property , then the space $\mathcal{D}(\Omega)$ is dense in $W_0^1L_M(\Omega)$ for the modular convergence and for the topology $\sigma(\Pi L_M, \Pi L_{\overline{M}}\underline{\hspace{4em}})$ (cf . [9 , 1 0]) . Consequently , the action of a distribution in $W^{-1}L_{\bar{}M}(\Omega)$ on an element of $W_0^1L_M(\Omega)$ is well defined .

For $k > 0$, we define the truncation at height $k, T_k : \mathbb{R} \to \mathbb{R}$ by

$$T_k(s) = \{^{s}_{ks/|s|} \quad \text{if}^{\text{if}} \; |^{|} \, {}^{s|}_{s|} >^{\le} k^{k}_{.} \tag{2.3}$$

The following abstract lemmas will be applied to the truncation operators .

Lemma 2 . 1 *Let $F : \mathbb{R} \to \mathbb{R}$ be uniformly lips chitzian , with $F(0) = 0$. Let M be an N - function and le t $u \in W^1 L_M(\Omega)$ (resp . $W^1 E_M(\Omega)$). Then $F(u) \in W^1 L_M(\Omega)$(resp .$W^1 E_M(\Omega)$). Moreover , if the s e t of dis continuity points of F' is finite , then*

$$\frac{\partial}{\partial x_i} F(u) = \begin{cases} F'(u)\dfrac{\partial u}{\partial x_i} \quad a.e. \quad in\{x \in \Omega : u(x) element - slash D\} \\ 0 \quad a.e. \quad in\{x \in \Omega : u(x) \in D\}. \end{cases}$$

Lemma 2 . 2 *Let $F : \mathbb{R} \to \mathbb{R}$ be uniformly lips chitzian , with $F(0) = 0$. We sup - pose that the s e t of dis continuity points of F' is finite . Let M be an N - function , then the mapping $F : W^1 L_M(\Omega) \to W^1 L_M(\Omega)$ is s equentially continuous with respect to th e weak $*$ topology $\sigma(\Pi L_M, \Pi E - line_M)$.*

Proof By the previous lemma , $F(u) \in W^1 L_M(\Omega)$ for all $u \in W^1 L_M(\Omega)$ and

$$\| F(u) \| 1, M, \Omega \leq C \| u \| 1, M, \Omega,$$

which gives easily the result . □ Let Ω be a bounded open subset of $\mathbb{R}^N, T > 0$ and set $Q = \Omega \times]0, T[$. Let
$m \geq 1$ be an integer and let M be an N - function . For each $\alpha \in \mathbf{N}^N$, denote by
D_x^α the distributional derivative on Q of order α with respect to the variable
$x \in \mathbb{R}^N$. The inhomogeneous Orlicz - Sobolev spaces are defined as follows

$$W^{m,x} L_M(Q) = \{u \in L_M(Q) : D_x^\alpha u \in L_M(Q) \forall \mid \alpha \mid \leq m\}$$
$$W^{m,x} E_M(Q) = \{u \in E_M(Q) : D_x^\alpha u \in E_M(Q) \forall \mid \alpha \mid \leq m\}$$

The last space is a subspace of the first one , and both are Banach spaces under the norm

$$\| u \| = \sum_{\mid \alpha \mid \leq m} \| D_x^\alpha u \| M, Q.$$

We can easily show that they form a complementary system when Ω satisfies the
segment property . These spaces are considered as subspaces of the product space $\Pi L_M(Q)$ which have as many copies as there is $\alpha-$ order derivatives , $\mid \alpha \mid \leq m$. We shall also consider the weak topologies $\sigma(\Pi L_M, \Pi E - line_M)$ and $\sigma(\Pi L_M, \Pi L_{-M})$.
If $u \in W^{m,x} L_M(Q)$ then the function : $t \to u(t) = u(t, .)$ is defined on $[0, T]$ with values in $W^m L_M(\Omega)$. If , further , $u \in W^{m,x} E_M(Q)$ then the concerned function is a $W^m E_M(\Omega)-$ valued and is strongly measurable . Furthermore the
following imbedding holds : $W^{m,x} E_M(Q) \subset L^1(0, T; W^m E_M(\Omega))$. The space $W^{m,x} L_M(Q)$ is not in general separable , if $u \in W^{m,x} L_M(Q)$, we can not con - clude that the function $u(t)$ is measurable on $[0, T]$. However , the scalar function
$t \mapsto \| u(t) \| M, \Omega$ is in $L^1(0, T)$. The space $W_0^{m,x} E_M(Q)$ is defined as the (norm) closure in $W^{m,x} E_M(Q)$ of $\mathcal{D}(Q)$. We can easily show as in [1 0] that when Ω has the segment property then each element u of the closure of $\mathcal{D}(Q)$ with respect of the weak $*$ topology $\sigma(\Pi L_M, \Pi E - line_M)$ is limit , in $W^{m,x} L_M(Q)$, of some subse -
quence $(u_i) \subset \mathcal{D}(Q)$ for the modular convergence ; i . e . , there exists $\lambda > 0$ such

 207 that for all $|\alpha| \leq m$,

$$\int_Q M(\frac{D_x^\alpha u_i - D_x^\alpha u}{\lambda})dxdt \to 0 \text{as} i \to \infty,$$

this implies that (u_i) converges to u in $W^{m,x}L_M(Q)$ for the weak topology

$\sigma(\Pi L_M, \Pi L_{\bar{M}})$.Consequently

$$\overline{\mathcal{D}(Q)}^{\sigma(\Pi L_M, \Pi \overline{E}_{\bar{M}})} = \overline{\mathcal{D}(Q)}^{\sigma(\Pi L_M, \Pi \overline{L}_{\bar{M}})},$$

this space will be denoted by $W_0^{m,x}L_M(Q)$. Furthermore , $W_0^{m,x}E_M(Q) = W_0^{m,x}L_M(Q) \cap \Pi E_M$. Poincar é ' s inequality also holds in $W_0^{m,x}L_M(Q)$ i . e . there is a constant $C > 0$ such that for all $u \in W_0^{m,x}L_M(Q)$ one has

$$\sum_{|\alpha| \leq m} \| D_x^\alpha u \| M, Q \leq C \sum_{|\alpha| = m} \| D_x^\alpha u \| M, Q.$$

Thus both sides of the last inequality are equivalent norms on $W_0^{m,x}L_M(Q)$. We have then the following complementary system

$$\begin{pmatrix} W_0^{m,x}L_M(Q) & F \\ W_0^{m,x}E_M(Q) & F_0 \end{pmatrix},$$

F being the dual space of $W_0^{m,x}E_M(Q)$. It is also , except for an isomorphism , the quotient of $\Pi L_{\bar{M}}$ by the polar set $W_0^{m,x}E_M(Q)^\perp$, and will be denoted by $F = W^{-m,x}L_{\bar{M}}(Q)$ and it is shown that

$$W^{-m,x}L_{\bar{M}}(Q) = \{f = \sum_{|\alpha| \leq m} D_x^\alpha f_\alpha : f_\alpha \in L_{\bar{M}}(Q)\}.$$

This space will be equipped with the usual quotient norm

$$\| f \| = \inf \sum_{|\alpha| \leq m} \| f_\alpha \|_{\overline{M},Q}$$

where the infimum is taken on all possible decompositions

$$f = \sum_{|\alpha| \leq m} D_x^\alpha f_\alpha, \quad f_\alpha \in L_{\bar{M}}(Q).$$

The space F_0 is then given by

$$F_0 = \{f =| \sum_{\alpha| \leq m} D_x^\alpha f_\alpha : f_\alpha \in E - line_M(Q)\}$$

and is denoted by $F_0 = W^{-m,x}E - line_M(Q)$. **Remark 2 . 3** We can easily check , using [1 0 , lemma 4 . 4] , that each uniformly

lipschitzian$^{\text{mapping}}_{\text{spacesoforder}}$ $1 : W^{F_{1,x}}{}^{,\ \text{with}}_{LM(Q)}\ {}^{F(0)}_{\text{and}} = 0W_0 1$;acts$^{\text{in}}_{x_{L_M(Q)}}$).$^{\text{inhomogeneous}}$Orlicz $-$ Sobolev

3 Galerkin solutions

In this section we shall define and state existence theorems of Galerkin solutions for some parabolic initial - boundary problem .

Let Ω be a bounded subset of $\mathbb{R}^N, T > 0$ and set $Q = \Omega\times]0,T[$. Let

$$A(u) = \sum_{|\alpha| \leq m} (-1)^{|\alpha|} D_x^\alpha(A_\alpha(u))$$

be an operator such that

$A_\alpha(x,t,\xi) : \Omega \times [0,T] \times \mathbb{R}^N 0 \to \mathbb{R}$ is continuous in (t,ξ), for a . e .$x \in \Omega$ and measurable in x, for all $(t,\xi) \in [0,T] \times \mathbb{R}^N 0$, (3.1) where , N_0 is the number of all $\alpha-$ order ' s derivative , $|\alpha| \leq m$.

$| A_\alpha(x,s,\xi) | \leq \chi(x)\Phi(| \xi |)$ with $\chi(x) \in L^1(\Omega)$ and $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ increasing . (3 . 2)

$$\sum_{|\alpha| \leq m} A_\alpha(x,t,\xi)\xi_\alpha \geq -d(x,t) \text{with} d(x,t) \in L^1(Q), \quad d \geq 0. \tag{3.3}$$

Consider a function $\psi \in L^2(Q)$ and a function $\bar{u} \in L^2(\Omega) \cap W_0^{m,1}(\Omega)$. We choose an orthonormal sequence $(\omega_i) \subset \mathcal{D}(\Omega)$ with respect to the Hilbert space $L^2(\Omega)$ such that the closure of (ω_i) in $C^m(\overline{\Omega})$ contains $\mathcal{D}(\Omega)$. $C^m(\overline{\Omega})$ being

the space of functions which are m times continuously differentiable on $\overline{\Omega}$. For

$$V_n = \text{span}\langle\omega_1, ..., \omega_n\rangle \text{and}$$

$$\| u \| C^1, m(Q) = \sup\{| D_x^\alpha u(x,t) |, | \frac{\partial u}{\partial t}(x,t) | : | \alpha | \leq m, (x,t) \in Q\}$$

we have

$$\mathcal{D}(Q) \subset \overline{\{\cup_{n=1}^\infty C^1([0,T],V_n)\}}^{C^{1,m}(Q)}$$

this implies that for ψ and $\bar{u}$, there exist two sequences (ψ_n) and $(\bar{u}_n)$ such that

$$\psi_n \in C^1([0,T],V_n), \quad \psi_n \to \psi \text{in} L^2(Q). \tag{3.4}$$

$$\bar{u}_n \in V_n, \quad \bar{u}_n \to \bar{u} \text{in} L^2(\Omega) \cap W_0^{m,1}(\Omega). \tag{3.5}$$

Consider the parabolic initial - boundary value problem

$$\frac{\partial u}{\partial t} + A(u) = \psi \text{in} Q, \quad D_x^\alpha u = 0 \text{on} \partial\Omega \times]0,T[, \text{for all} |\alpha| \leq m-1, \quad u(0) = \bar{u} \text{in} \Omega. \tag{3.6}$$

In the sequel we denote $A_\alpha(x,t,u,\nabla u, ..., \nabla^m u)$ by $A_\alpha(x,t,u)$ or simply by $A_\alpha(u)$.

 Definition 3 . 1 A function $u_n \in C^1([0,T],V_n)$ is called Galerkin solution of
(3 . 6) if

$$\int_\Omega \frac{\partial u_n}{\partial t}\varphi dx + \int_\Omega | \sum_{\alpha|\leq m} A_\alpha(u_n).D_x^\alpha \varphi dx = \int_\Omega \psi_n(t)\varphi dx$$

for all $\varphi \in V_n$ and all $t \in [0,T]; u_n(0) = \text{-}u_n.$

We have the following existence theorem .

Theorem 3 . 2 ([13]) *Under conditions (3 . 1) - (3 . 3) , there exists at least one Galerkin s o lutio n of (3 . 6) .*

Consider now the case of a more general operator

$$A(u) = \sum_{| \alpha |\leq m}(-1)^{|\alpha|} D_x^\alpha(A_\alpha(u))$$

where instead of (3 . 1) and (3 . 2) we only assume that

$$A_\alpha(x,t,\xi) : \Omega \times [0,T] \times \mathbb{R}^N 0 \to \mathbb{R} \text{ is continuous in } \xi, \quad \text{for a . e . } (x,t) \in Q \text{ and measurable in } (x,t) \text{ for all } \xi \in \mathbb{R}^N 0. \quad (3.7)$$

$$| A_\alpha(x,s,\xi) | \leq C(x,t)\Phi(| \xi |) \text{with} C(x,t) \in L^1(Q). \quad (3.8)$$

We have also the following existence theorem **Theorem**$_{isinL^1(0,}3.T^3_;(^{[14\ ])}_{n_V)and}$ *There exists a function u_n in $C([0,T],V_n)$ such that* $\frac{\partial u_n}{\partial t}$

$$\int_{Q_\tau} \frac{\partial u_n}{\partial t}\varphi dxdt + \int_{Q_\tau} | \sum_{\alpha|\leq m} A_\alpha(x,t,u_n).D_x^\alpha \varphi dxdt = \int_{Q_\tau} \psi_n \varphi dxdt$$

for al l $\tau \in [0,T]$ *and all* $\varphi \in C([0,T],V_n)$, *where* $Q_\tau = \Omega \times [0,\tau]; u_n(0) = \text{-}u_n.$

4 Strong convergence of truncations

In this section we shall prove a convergence theorem for parabolic problems which allows us to deal with approximate equations of some parabolic initial - boundary problem in Orlicz spaces (see section 6) . Let Ω, be a bounded subset of $\mathbb{R}^N$ with the segment property and let $T > 0, Q = \Omega \times]0,T[$. Let M be an N - function satisfying a Δ' condition and the growth condition

$$M(t) \ll \quad | t | \frac{N}{N-1}$$

and let P be an N - function such that $P \ll M$. Let $A : W^{1,x}L_M(Q) \to W^{-1,x}L - line_M(Q)$ be a mapping given by

$$A(u) = -\text{div} a(x,t,u,\nabla u)$$

 where $a(x,t,s,\xi)$: $\Omega\times[0,T]\times\mathbb{R}\times\mathbb{R}^N \rightarrow \mathbb{R}^N$ is a Carath $\acute{e}$ odory function
satisfying for a . e . $(x,t)\in\Omega\times]0,T[$ and for all $s\in\mathbb{R}$ and all $\xi,\xi^*\in\mathbb{R}^N$:

$$|\, a(x,t,s,\xi)\,| \leq c(x,t)+k_1\overline{\quad} P^{-1}M(k_2\,|\, s\,|)+k_3\overline{\quad} M^{-1}M(k_4\,|\,\xi\,|) \quad (4.1)$$

$$[a(x,t,s,\xi)-a(x,t,s,\xi^*)][\xi-\xi^*]>0 \quad \text{if}\xi\neq\xi^* \quad (4.2)$$

$$\alpha M(\frac{|\,\xi\,|}{\lambda})-d(x,t)\leq a(x,t,s,\xi)\xi \quad (4.3)$$

where$c(x,t)\in E-line_M(Q), c\geq 0, d(x,t)\in L^1(Q), k_1,k_2,k_3,k_4\in\mathbb{R}^+and\alpha,\lambda\in$

$\mathbb{R}_*^+$. Consider the nonlinear parabolic equations

$$\frac{\partial u_n}{\partial t}-\text{div}a(x,t,u_n,\nabla u_n)=f_n+gn \quad \text{in}\mathcal{D}'(Q) \qquad (4.4)$$

and assume that :

$$u_n \rightharpoonup u \quad \text{weakly in } W^{1,x}L_M(Q) \text{ for } \sigma(\Pi L_M,\Pi E_{\text{-}M}), \qquad (4.5)$$

$$f_n\rightarrow f \quad \text{stronglyin}W^{-1,x}E-line_M(Q), \qquad (4.6)$$

$$gn\rightharpoonup g \quad \text{weaklyin}L^1(Q). \qquad (4.7)$$

We shall prove the following convergence theorem .

Theorem 4 . 1 *Assume that (4 . 1) - (4 . 7) hold . Then , for any* $k>0$, *th e trunca - tion of* u_n *at he ight* k*(s ee (2 . 3) for the definition of th e truncation) satisfies*

$$\nabla T_k(u_n)\rightarrow\nabla T_k(u) \quad \textit{s tro ngly in } (L_M^{\text{loc}}(Q))^N. \quad (4.8)$$

Remark 4 . 2 An elliptic analogous theorem is proved in Benkirane - Elmahi [2] .

Remark 4 . 3 Convergence (4 . 8) allows , in particular , to extract a subsequence n' such that :

$$\nabla u_{n'}\rightarrow\nabla u \quad \text{a . e . in } Q.$$

Then by lemma 4 . 4 of [9] , we deduce that

$$a(x,t,u_{n'},\nabla u_{n'})\rightharpoonup a(x,t,u,\nabla u) \quad \text{weakly in } L_{\text{-}M}(Q))^N \text{ for } \sigma(\Pi L-line_M,\Pi E_M).$$

Proof of Theorem 4 . 1 **Step** 1 : For each $k>0$, define $S_k(s)=\int_0^s T_k(\tau)d\tau$. Since T_k is continuous, for all $w\in W^{1,x}L_M(Q)$ we have $S_k(w)\in W^{1,x}L_M(Q)$ and $\nabla S_k(w)=T_k(w)\nabla w$. So that, by mollifying as in [6], it is easy to see that for all $\varphi\in\mathcal{D}(Q)$ and all $v\in W^{1,x}L_M(Q)$ with $\frac{\partial v}{\partial t}\in W^{-1,x}L_{-M}(Q)+L^1(Q)$, we have

$$\langle\langle\frac{\partial v}{\partial t},\varphi T_k(v)\rangle\rangle=-\int_Q\frac{\partial\varphi}{\partial t}S_k(v)dxdt. \qquad (4.9)$$

where $\langle\langle,\rangle\rangle$ means for the duality pairing between $W_0^{1,x}L_M(Q)+L^1(Q)$ and $W^{-1,x}L-line_M(Q)\cap L^\infty(Q)$. Fix now a compact set K with $K\subset Q$ and a function

Abdelhak Elmahi 2 1 1 $\varphi K \in \mathcal{D}(Q)$ such that $0 \leq \varphi K \leq 1$ in Q and $\varphi K = 1$ on K. Using in (4 . 4)
$v_n = \varphi K(T_k(u_n) - T_k(u)) \in W^{1,x}L_M(Q) \cap L^\infty(Q)$ as test function yields

$$\begin{aligned}
&\langle\langle \frac{\partial u_n}{\partial t}, \varphi K^T k(u_n)\rangle\rangle - \langle\langle \frac{\partial u_n}{\partial t}, \varphi K^T k(u)\rangle\rangle \\
&+ \int_Q \varphi K^a(x,t,u_n,\nabla u_n)[\nabla T_k(u_n) - \nabla T_k(u)]dxdt \\
&+ \int_Q (T_k(u_n) - T_k(u))a(x,t,u_n,\nabla u_n)\nabla \varphi K dxdt \\
&= \langle\langle f_n, v_n\rangle\rangle + \langle\langle gn, v_n\rangle\rangle.
\end{aligned} \tag{4.10}$$

Since $u_n \in W^{1,x}L_M(Q)$ and $\frac{\partial u_n}{\partial t} \in W^{-1,x}L_{-M}(Q) + L^1(Q)$ then by (4 . 9) ,

$$\langle\langle \frac{\partial u_n}{\partial t}, \varphi K^T k(u_n)\rangle\rangle = -\int_Q \frac{\partial \varphi K}{\partial t} S_k(u_n)dxdt.$$

On$^{\text{the}}_{gn\text{is}}$bounded$^{\text{otherhand}}inL^{\text{since}_1}(Q)^{(u_n)\text{is}}_{\text{and}}$bounded$^{\text{in}}_{\text{soin}\mathcal{M}(Q)}and^{W^{1,x}L_M(Q)}_{h_n = \text{div}}$ $^{\text{and}}a(x^{\partial}_{,}-u_n\partial tt, u^{=}_{n,}h_n\nabla^{+g}_{u_n)+}{}^{n\text{ while}}f_n$is

boundedin$W$Consequently$^{-1,xL_{-M}(Q),}_{,T_k(u_n)}$ $^{\text{then}}_{\to T_k(u)}$ and by [8$^{,\text{ Corollary}}_{Sk}$ $^{1],}_{(u_n)\to S_k(u)}u_n \to$ in$_{L_M}u$lo c strongly(Q).
So that$^{\text{in}L^{\text{loc}}_M(Q).}$

$$\int_Q \frac{\partial \varphi K}{\partial t} S_k(u_n)dxdt \to \int_Q \frac{\partial \varphi K}{\partial t} S_k(u)dxdt$$

and also $\int_Q (T_k(u_n) - T_k(u))a(x,t,u_n,\nabla u_n)\nabla\varphi K dxdt \to 0$ as $n \to \infty$. Further - more $\langle\langle f_n, v_n\rangle\rangle \to 0$, by (4 . 6) . Since $gn \in L^1(Q)$ and $T_k(u_n) - T_k(u) \in L^\infty(Q)$,

$$\langle\langle gn, \varphi K(T_k(u_n) - T_k(u))\rangle\rangle = \int_Q gn\varphi K(T_k(u_n) - T_k(u))dxdt$$

which tends to 0 by Egorov ' s theorem .

bounded$^{\text{Since}}\varphi^{K^Tk(u)}_{\text{term in }W-1,xL_{-M}(Q)}$ $^{\text{belongs}}$ $^{\text{to}}_{\text{and}}$ $W0^{1,xL}$ $^{M(Q)}_{\text{of }gn}\cap L^\infty$which$^{(Q)}_{\text{weakly}}$ $^{\text{while}}$converges$\frac{\partial u_n}{\partial t}^{\text{is}}$ in$^{\text{the}}L^{\Sigma\,1}(Q)of_1$a

has

$$\langle\langle \frac{\partial u_n}{\partial t}, \varphi K^T k(u)\rangle\rangle \to \langle\langle \frac{\partial u}{\partial t}, \varphi K^T k(u)\rangle\rangle = -\int_Q \frac{\partial \varphi}{\partial t} S_k(u)dxdt.$$

We have thus proved that

$$\int_Q \varphi K^a(x,t,u_n,\nabla u_n)[\nabla T_k(u_n) - \nabla T_k(u)]dxdt \to 0 \quad \text{as} n \to \infty. \tag{4.11}$$

Step 2 : Fix a real number $r > 0$ and set $Q_{(r)} = \{x \in Q : | \nabla T_k(u) | \leq r\}$ and denote by χr the characteristic function of $Q_{(r)}$. Taking $s \geq r$ one has :

$$
\begin{aligned}
0 \leq \int_{Q_{(r)}} & \varphi K[a(x,t,u_n,\nabla T_k(u_n)) - a(x,t,u_n,\nabla T_k(u))] \\
& \times[\nabla T_k(u_n) - \nabla T_k(u)]dxdt \\
\leq \int_{Q_{(s)}} & \varphi K[a(x,t,u_n,\nabla T_k(u_n)) - a(x,t,u_n,\nabla T_k(u))] \\
& \times[\nabla T_k(u_n) - \nabla T_k(u)]dxdt \\
= \int_{Q_{(s)}} & \varphi K[a(x,t,u_n,\nabla T_k(u_n)) - a(x,t,u_n,\nabla T_k(u)\chi s)] \\
& \times[\nabla T_k(u_n) - \nabla T_k(u)\chi s]dxdt \\
\leq \int_{Q} & \varphi K[a(x,t,u_n,\nabla T_k(u_n)) - a(x,t,u_n,\nabla T_k(u)\chi s)] \qquad (4.12) \\
& \times[\nabla T_k(u_n) - \nabla T_k(u)\chi s]dxdt \\
= \int_{Q} & \varphi K^a(x,t,u_n,\nabla u_n)[\nabla T_k(u_n) - \nabla T_k(u)]dxdt \\
- \int_{Q} & \varphi K[a(x,t,u_n,\nabla u_n) - a(x,t,u_n,\nabla T_k(u_n))] \\
& \times[\nabla T_k(u_n) - \nabla T_k(u)\chi s]dxdt \\
+ \int_{Q} & \varphi K^a(x,t,u_n,\nabla u_n)[\nabla T_k(u) - \nabla T_k(u)\chi s]dxdt \\
- \int_{Q} & \varphi K^a(x,t,u_n,\nabla T_k(u)\chi s)[\nabla T_k(u_n) - \nabla T_k(u)\chi s]dxdt.
\end{aligned}
$$

Now pass to the limit in all terms of the right - hand side of the above equation . By (4 . 1 1) , the first one tends to 0 . Denoting by χG_n the characteristic function of $G_n = \{(x,t) \in Q : | u_n(x,t) | > k\}$, the second term reads

$$\int_Q \varphi K[a(x,t,u_n,\nabla u_n) - a(x,t,u_n,0)]\chi G_n^{\nabla T} k(u)\chi s dxdt \quad (4.13)$$

whichby(4.1$^{\text{tends}}$)and$^{\text{to}}$(4^{0}5since)[$^a_{\text{while}}$ $^{(x,\ t,\ u_n,\nabla u_n)}_{\chi G_n \nabla T_k(u)\chi s}$ $- a(x,t,u_n$converges$_{\text{strongly}}$0)]is,$^{\text{by}}_{\text{bounded}}$$^{\text{in}}_{\text{in}(E_M(Q))}$ $^{(L\ -M(Q))\ N}_{N\ \text{to}0}$

Lebesgue ’ s theorem . The fourth term of (4 . 1 2) tends to

$$- \int_Q \varphi K^a(x,t,u^{(s)}_{\varphi \underset{=}{K} \, {}^{a(x,}\int_{Q\setminus Q} t,u,\nabla T_k(u)\chi s} {}^{)[\nabla T_k(u)}_{,0)\nabla T_k(u)} - \nabla T_k dx^{(u)\chi s]}_{dt} dxdt \quad (4.14)$$

while$^{\text{since}}$$a(x\nabla^{,\ t,\ u_n,\ \nabla T_k(u)\chi s)}_{T_k(un) - \nabla T_k(u)\chi s}$tends$^{\text{strongly}}_{\text{converges}}$toweakly$ato^{(x,\ t,\ u,\ \nabla T_k(u)\chi s)}_{\nabla T k(u) - \nabla T_k(u)\chi s}$ $^{\text{in}}$in$^{(E________\ M(Q))^N}_{(L \qquad M(Q))^N}$

for$\sigma(\Pi L_M, \Pi E - line_M)$.

Since $a(x,t,u_n,\nabla u_n)$ is bounded in $(L\underline{\qquad}_M(Q))^N$ one has (for a subsequence still denoted by u_n)

$a(x,t,u_n,\nabla u_n) \rightharpoonup h$ weakly in $(L_{\text{-}M}(Q))^N$ for $\sigma(\Pi L_{\text{-}M}, \Pi E_M)$. (4.15) Finally , the third term of the right - hand side of (4 . 1 2) tends to

$$\int_{Q\setminus Q_{(s)}} \varphi K^{h\nabla T} k(u)dxdt. \tag{4.16}$$

We have , then , proved that

$$\begin{aligned} 0 \leq \overset{\sup}{\lim_{n}}{}_{\to\infty} \int_{Q_{(r)}} & \varphi K[a(x,t,u_n,\nabla T_k(u_n)) - a(x,t,u_n,\nabla T_k(u))] \\ & \times[\nabla T_k(u_n) - \nabla T_k(u)]dxdt \\ \leq \int_{Q\setminus Q_{(s)}} & \varphi K[h - a(x,t,u,0)]\nabla T_k(u)dxdt. \end{aligned} \tag{4.17}$$

Using the fact that $[h - a(x,t,u,0)]\nabla T_k(u) \in L^1(\Omega)$ and letting $s \to +\infty$ we

$$\text{get, since} \mid Q\setminus Q_{(s)} \mid \to 0,$$

$$\int_{Q_{(r)}} \varphi K[a(x,t,u_n,\nabla T_k(u_n)) - a(x,t,u_n,\nabla T_k(u))][\nabla T_k(u_n) - \nabla T_k(u)]dxdt$$

(4 . 1 8) which approaches 0 as $n \to \infty$. Consequently

$$\int_{Q_{(r)}\cap K} [a(x,t,u_n,\nabla T_k(u_n)) - a(x,t,u_n,\nabla T_k(u))][\nabla T_k(u_n) - \nabla T_k(u)]dxdt \to 0$$

as $n \to \infty$. As in [2] , we deduce that for some subsequence $\nabla T_k(u_n) \to \nabla T_k(u)$ a . e . in $Q_{(r)} \cap K$. Since r, k and K are arbitrary , we can construct a subsequence (diagonal in r, in k and in j, where (K_j) is an increasing sequence of compacts sets covering Q), such that

$$\nabla u_n \to \nabla u \quad \text{a . e . in } Q. \tag{4.19}$$

Step 3 : As in [2] we deduce that

$$\int_Q \varphi K^a(x,t,u_n,\nabla u_n)\nabla T_k(u_n)dxdt \to \int_Q \varphi K^a(x,t,u,\nabla u)\nabla T_k(u)dxdt$$

as $n \to \infty$, and that

$$a(x,t,u_n,\nabla T_k(u_n))\nabla T_k(u_n) \to a(x,t,u,\nabla T_k(u))\nabla T_k(u) \text{stronglyin} L^1(K).$$

(4 . 20)

This implies that (see [2] if necessary) : $\nabla T_k(u_n) \to \nabla T_k(u)$ in $(L_M(K))^N$ for the modular convergence and so strongly and convergence (4 . 8) follows .

Note that in convergence (4 . 8) the whole sequence (and not only for a subse - quence) converges since the limit $\nabla T_k(u)$ does not depend on the subsequence .

5 Nonlinear parabolic problems

Now , we are able to establish an existence theorem for a nonlinear parabolic initial - boundary value problems . This result which specially applies in Orlicz spaces generalizes analogous results in of Landes - Mustonen [14] . We start by giving the statement of the result .

Let Ω be a bounded subset of $\mathbb{R}^N$ with the segment property , $T>0$, and $Q=\Omega\times]0,T[$. Let M be an N - function satisfying the growth condition

$$M(t) \ll \quad |\ t\ |\ \frac{N}{N-1},$$

and the $\triangle' -$ condition . Let P be an N - function such that $P \quad \ll M$. Consider an operator $A: W_0^{1,x}L_M(Q) \rightarrow W^{-1,x}L\underline{\qquad}_M(Q)$ of the form

$$A(u) = -\mathrm{div} a(x,t,u,\nabla u) + a_0(x,t,u,\nabla u) \quad (5.1)$$

where$a$: $\Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^N \quad \rightarrow \mathbb{R}^N$ anda_0 : $\Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^N \quad \rightarrow \quad \mathbb{R}$

are Carath $\acute{e}$ odory functions satisfying the following conditions , for a . e . $(x,t) \in \Omega \times [0,T]$ for all $s \in \mathbb{R}$ and $\xi \neq \xi^* \in \mathbb{R}^N$:

$$||_{a \quad 0(x,t,\ s,\ \xi)|}^{a(x,\ t \quad ,\ s,\ \xi)|} \leq \leq_{c(x,\ t)}^{c(x,\ t)} + k_{+k_1}^{1--P_M^1(k_2|s \qquad |)} {}_{-M^{-1}_{M(k_2}|s|)} + k_{+k_3}^{3--M_M^1(k_4|\xi|),} \underline{\qquad} M^{-1}{}_{P(k_4|\xi|),} \quad (5.2)$$

$$[a(x,t,s,\xi) - a(x,t,s,\xi^*)][\xi - \xi^*] > 0, \quad (5.3)$$

$$a(x,t,s,\xi)\xi + a_0(x,t,s,\xi)s \geq \alpha M(\frac{|\ \xi\ |}{\lambda}) - d(x,t) \quad (5.4)$$

where$c(x,t) \in E - line_M(Q), c \geq 0, d(x,t) \in L^1(Q), k_1,k_2,k_3,k_4 \in \mathbb{R}^+and\alpha, \lambda \in$

$\mathbf{R}_*^+$. Furthermore let

$$f \in W^{-1,x}E - line_M(Q) \tag{5.5}$$

We shall use notations of section 3 . Consider , then , the parabolic initial - boundary value problem

$$u\frac{\partial u}{\partial t(x,_t} {}^{+}{}_{)=\ 0\ \ \mathrm{on}\partial\Omega\times]0,T[}^{A(u)=\ f\mathrm{in}\ \ Q} \tag{5.6}$$
$$u(x,0) = \psi(x)\mathrm{in}\Omega.$$

where ψ is a given function in $L^2(\Omega)$. We shall prove the following existence theorem .

Theorem$_{solution u}5_{\in}.1$ $Assume W_{0^{1,x}L_{M(Q)}) \cap (_{L^2}}^{that} {}^{(5.2)-(5.5\ \ )\ hold.}_{(Q)\cap C(\ \ [0,T],} Then_{L^2(\Omega))of}^{there} exists (5.6), @inleast_{the following} 1 weak$

s ense :

$$-\int_Q u\frac{\partial\varphi}{\partial t}dxdt_+ + [\int_\Omega \int_{Qa_0(}^{u(t} {}^{)\varphi(t)dx]_0^T}_{x,\ t,\ \ u,} \nabla u).\varphi + \int_Q dx_{dt}a_=(x,\langle^{t,}_{f},u,\varphi_\rangle\nabla u).\nabla\varphi dxdt \tag{5.7}$$

 for al l $\varphi \in C^1([0,T], L^2(\Omega))$.

$\subset L^1(0,T;W^{-1,1}(\Omega))$. Then$u \in$ withcontinuityoftheimbedding. $\frac{\partial u}{\partial t} \in W^{-1,x}L-line_M(Q)^{C([0,T],W^{-1,1}(\Omega))}_{\textbf{Remark}5.2\text{In}}$ (5 .6), wehave$u \in W_0^{1,x}L_M(Q) \subset W^{1,1}(0,T;W^{-1,1}($

sibly after modification on a set of zero measure , continuous from $[0,T]$ into $W^{-1,1}(\Omega)$ in such a way that the third component of (5 . 6) , which is the initial condition , has a sense .

Proof of Theorem 4 . 1 It is easily adapted from the proof given in [1 4] . For convenience we suppose that $\psi = 0$. For each n, there exists at least one solution u_n of the following problem (see Theorem 3 . 3 for the existence of u_n) :

$$\begin{aligned} u_n \in C([0,T], V_n), \quad \frac{\partial u_n}{\partial t} \in L^1(0,T;V_n), \quad u_n(0) = \psi_n \equiv 0 \quad \text{and,} \\ \text{forall}\tau \in [0,T], \quad \int_{Q_\tau} \frac{\partial u_n}{\partial t}\varphi dxdt + \int_{Q_\varepsilon} a(x,t,u_n,\nabla u_n).\nabla\varphi dxdt \\ + \int_{Q_\varepsilon} a_0(x,t,u_n,\nabla u_n).\varphi dxdt = \int_{Q_\varepsilon} f_n\varphi dxdt, \quad \forall\varphi \in C([0,T],V_n). \end{aligned} \tag{5.8}$$

where $fk \subset \cup_{n=1}^{\infty} C([0,T],V_n)$ with $fk \to f$ in $W^{-1,x}E - line_M(Q)$. Putting $\varphi = u_n$ in (5 . 8) , and using (5 . 2) and (5 . 4) yields

$$\| a_0(x,t,u_{n\ ,\ \nabla u_n)}^{\|un\|W_0^1\ ,xL_M(Q)} \|L\_\_\_\_\_\_M(Q) \leq \leq C^{C,}\ \|^{u_n}_{\text{and}}\|^{\|L\ \infty(0,T;L^2(\Omega))}_{a(x,\ t,\ u_n,\ \nabla u_n)} \leq C\|_{L\_\_\_\_\_\_M(Q)} \leq C. \tag{5.9}$$

Hence , for a subsequence

$u_n \rightharpoonup u$ weakly in $W_0^{1,x}L_M(Q)$ for $\sigma(\Pi L_M, \Pi E - line_M)$ and weakly in $L^2(Q)$,

$$a_0(x,t,u_n,\nabla u_n) \rightharpoonup h_0, a(x,t,u_n,\nabla u_n) \rightharpoonup h\text{in}L_{\_M}(Q)\text{for}\sigma(\Pi L - line_{M,}\Pi E_M)$$

(5 . 1 0) where $h_0 \in L_{\_M}(Q)$ and $h \in (L_{\_M}(Q))^N$. As in [1 4] , we get that for some subsequence $u_n(x,t) \to u(x,t)$ a . e . in Q (it suffices to apply Theorem 3 . 9
instead of Proposition 1 of [14]) . Also we obtain

$$-\int_Q u\frac{\partial\varphi}{\partial t}dxdt + [\int_\Omega u(t)\varphi(t)dx]_0^T + \int_Q h\nabla\varphi dxdt + \int_Q h_0\varphi dxdt = \langle f,\varphi\rangle,$$

for all $\varphi \in C^1([0,T];\mathcal{D}(\Omega))$. The proof will be completed , if we can show that

$$\int_Q (h\nabla\varphi + h_0\varphi)dxdt = \int_Q (a(x,t,u,\nabla u)\nabla\varphi + a_0(x,t,u,\nabla u)\varphi)dxdt \tag{5.11}$$

for all $\varphi \in C^1([0,T];\mathcal{D}(\Omega))$ and that $u \in C([0,T], L^2(\Omega))$. For that , it suffices to show that

$$\lim_{n\to\infty} \int_Q (a(x,t,u_n,\nabla u_n)[\nabla u_n - \nabla u] + a_0(x,t,u_n\nabla u_n)[u_n - u])dxdt \leq 0. \tag{5.12}$$

 Indeed , suppose that (5 . 1 2) holds and let $s > r > 0$ and set $Q^r = \{(x,t) \in Q :$
$\mid \nabla u(x,t) \mid \leq r\}$. Denoting by χs the characteristic function of Q^s, one has

$$
\begin{aligned}
0 &\leq \int_{Q^r} [a(x,t,u_n,\nabla u_n) - a(x,t,u_n,\nabla u)][\nabla u_n - \nabla u]dxdt \\
&\leq \int_{Q^s} [a(x,t,u_n,\nabla u_n) - a(x,t,u_n,\nabla u)][\nabla u_n - \nabla u]dxdt \\
&= \int_{Q^s} [a(x,t,u_n,\nabla u_n) - a(x,t,u_n,\nabla u.\chi s)][\nabla u_n - \nabla u.\chi s]dxdt \\
&\leq \int_{Q} [a(x,t,u_n,\nabla u_n) - a(x,t,u_n,\nabla u.\chi s)][\nabla u_n - \nabla u.\chi s]dxdt \\
&= \int_Q a_0(x,t,u_n,\nabla u_n)(u_n - u) - \int_Q a(x,t,u_n,\nabla u_n.\chi s)[\nabla u_n - \nabla u.\chi s]dxdt \\
&\quad + \int_Q [a(x,t,u_n,\nabla u_n)(\nabla u_n - \nabla u) + a_0(x,t,u_n,\nabla u_n)(u_n - u)]dxdt \\
&\quad + \int_{Q\setminus Q^s} a(x,t,u_n,\nabla u_n)\nabla u dxdt.
\end{aligned}
$$

(5 . 1 3) The first term of the right - hand side tends to 0 since $(a_0(x,t,u_n,\nabla u_n))$ is bounded in $L_{-M}(Q)$ by (5 . 2) and $u_n \to u$ strongly in $L_M(Q)$. The sec - ond term tends to $\int_{Q\setminus Q^s} a(x,t,u_n,0)\nabla u dxdt$ since $a(x,t,u_n,\nabla u_n.\chi s)$ tends strongly in $(_M^{E-line}(Q))^N$ to $a(x,t,u,\nabla u.\chi s)$ and $\nabla u_n \rightharpoonup \nabla u$ weakly in $(L_M(Q))^N$

for $\int_{Q\setminus}^{\sigma(\Pi} {}_{Q^s}^{L_M,} h\nabla u dx \Pi E{-}line_{Mdt}^{)}$ since The third term satisfies $a(x,t,u_n,\nabla \; u_n) \rightharpoonup (5.1_h 2)$ while the weakly in fourth (term line $-L_M(Q)$) tends to N for

$\sigma(\Pi L_{-M}, \Pi E_M)$ and M satisfies the $\triangle_2 -$ condition . We deduce then that

$$
\begin{aligned}
0 \leq \lim_{n\to\infty} \sup \int_{Q^s} &[a(x,t,u_n,\nabla u_n) - a(x,t,u_n,\nabla u)][\nabla u_n - \nabla u]dxdt \\
&\leq \int_{Q\setminus Q^s} [h - a(x,t,u,0)]\nabla u dxdt \to 0 \quad \text{as} s \to \infty.
\end{aligned}
$$

and so , by (5 . 3) , we can construct as in [2] a subsequence such that $\nabla u_n \to \nabla u$ a . e . in Q. This implies that $a(x,t,u_n,\nabla u_n) \to a(x,t,u,\nabla u)$ and that $a_0(x,t,u_n,\nabla u_n) \to a_0(x,t,u,\nabla u)$ a . e . in Q. Lemma 4 . 4 of [9] shows that $h = a(x,t,u,\nabla u)$ and $h_0 = a_0(x,t,u,\nabla u)$ and (5 . 1 1) follows . The remaining of the proof is exactly the same as in [14]. □

Corollary 5 . 3 *The function u can be us ed as a t esting function in (5 . 6) i . e .*

$$
\frac{1}{2}[\int_\Omega (u(t))^2 dx]_0^\tau + \int_{Q_\tau} [a(x,t,u,\nabla u).\nabla u + a_0(x,t,u,\nabla u)u]dxdt = \int_{Q_\tau} f u dxdt
$$

for al l $\tau \in [0,T]$. The proof of this corollary is exactly the same as in [14] .

6 Strongly nonlinear parabolic problems

In this last section we shall state and prove an existence theorem for strongly nonlinear parabolic initial - boundary problems with a nonlinearity $g(x,t,s,\xi)$ having growth less than $M(|\xi|)$. This result generalizes Theorem 2 . 1 in Boccardo - Murat [5] . The analogous elliptic one is proved in Benkirane - Elmahi [2] .

The notation is the same as in section 5 . Consider also assumptions (5 . 2) - (5 . 5) to which we will annex a Carath é odory function $g : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ satisfying , for a . e . $(x,t) \in \Omega \times [0,T]$ and for all $s \in \mathbb{R}$ and all $\xi \in \mathbb{R}^N$:

$$g(x,t,s,\xi)s \geq 0 \tag{6.1}$$

$$| g(x,t,s,\xi) | \leq b(| s |)(c'(x,t) + R(| \xi |)) \tag{6.2}$$

where $c' \in L^1(Q)$ and $b : \mathbb{R}^+ \to \mathbb{R}^+$ and where R is a given N - function such that $R \ll M$. Consider the following nonlinear parabolic problem

$$\frac{\partial u}{\partial t} +^{A(u\ )}_{u} {}_{(x,t)} +_{=} g(x0,_{\text{on}} t, u, \nabla u)\partial_{\Omega\times} = (0f_{,T)}\text{in}_{,}Q, \tag{6.3}$$

$$u(x,0) = \psi(x) \quad \text{in}\Omega.$$

We shall prove the following existence theorem .

Theorem 6 . 1 *Assume that (5 . 1) - (5 . 5) , (6 . 1) and (6 . 2) hold . Then , there exists*
at least one distributional s o lution of (6 . 3) .

Proof It is easily adapted from the proof of theorem 3 . 2 in [2] Consider first

$$gn(x,t,s,\xi) = \frac{g(x,t,s,\xi}{1+\frac{1}{n}g(x,t,s}$$

$\text{and}(_{5.4)}\text{put}^{A_n(u\ )}_{\text{sothat},\text{by}} = A(u) + gn\text{Theorem}^{(x,}_{5.1,\text{there}}t, u, \nabla_{\text{exists}}u), \text{wesee@}_{\text{least}}\text{that}_1 A_{n\text{solution}}\text{satisfies}_{u_n}\text{conditions}^{(5.2)-}_{\in W_0^{1,x}L_M}$

of the approximate problem

$$\frac{\partial u_n}{\partial t} + A^{(u_n)}_{u_n}{}_{(x,\ +_t)}\ g^n_{=}({}_0x, t_{\text{on}}, un, \partial^{\nabla u_n)}_{\Omega\times]0,} = T[^f \quad \text{in}Q \tag{6.4}$$

$$u_n(x,0) = \psi(x) \quad \text{in}\Omega$$

and , by Corollary 5 . 3 , we can use u_n as testing function in (6 . 4) . This gives

$$\int_Q [a(x,t,u_n,\nabla u_n).\nabla u_n + a_0(x,t,u_n,\nabla u_n).u_n]dxdt \leq \langle f, u_n\rangle$$

and thus (u_n) is a bounded sequence in $W_0^{1,x}L_M(Q)$. Passing to a subsequence if necessary , we assume that

$$u_n \rightharpoonup u \quad \text{weakly in } W_0^{1,x}L_M(Q) \text{ for } \sigma(\Pi L_M, \Pi E-line_M) \tag{6.5}$$

 for some $u \in W_0^{1,x} L_M(Q)$. Going back to (6 . 4) , we have

$$\int_Q gn(x,t,u_n,\nabla u_n) u_n dxdt \leq C.$$

We shall prove that $gn(x,t,u_n,\nabla u_n)$ are uniformly equi - integrable on Q. Fix $m > 0$. For each measurable subset $E \subset Q$, we have

$$\begin{aligned} &\int_E \mid gn(x,t,u_n,\nabla \\ &\leq \int_{E\cap\{|u_n|\leq m\}} \mid gn(x,t,u_n,\nabla u_n) \mid + \int_{E\cap\{|u_n|>m\}} \mid gn(x,t,u_n,\nabla \\ &\leq b(m)\int_E [c'(x,t) + R(\mid \nabla u_n \mid)]dxdt + \frac{1}{m}\int_{E\cap\{|u_n|>m\}} \mid gn(x,t,u_n,\nabla u_n) \mid \\ &\leq b(m)\int_E [c'(x,t) + R(\mid \nabla u_n \mid)]dxdt + \frac{1}{m}\int_Q u_n gn(x,t,u_n,\nabla u_n) \\ &\leq b(m)\int_E c'(x,t)dxdt + b(m)\int_E R(\frac{\mid \nabla u_n \mid}{\lambda'})dxdt \end{aligned}$$

there$^{\text{Let}\varepsilon} > 0,$ thereexistsδ_1 $>^{\text{is}}$ $0m >$ such0_{that}such$^{\text{that}}_{b(m)\int_E}\frac{C}{m}{}^{<}{}_{c''}(x, \frac{\varepsilon}{3}{}^{\cdot}{}_{t)} dx^{\text{Furthermore},}{}_{dt<}$ $\frac{\varepsilon}{3}.$ On$^{\text{since}}$the$c'' \in$ otherL^1_{har}

let $\mu > 0$ such that $\parallel \nabla u_n \parallel M,Q \leq \mu, \forall n.$ Since $R \ll M$, there exists a constant $K_\varepsilon > 0$ depending on ε such that

$$b(m)R(s) \leq M(\frac{\varepsilon}{6}\frac{s}{\mu}) + K_\varepsilon$$

for all $s \geq 0$. Without loss of generality , we can assume that $\varepsilon < 1$. By convexity we deduce that

$$b(m)R(s) \leq \frac{\varepsilon}{6} M(\frac{s}{\mu}) + K_\varepsilon$$

for all $s \geq 0$. Hence

$$\begin{aligned} b(m)\int_E R(\frac{\mid \nabla u_n \mid}{\lambda'})dxdt &\leq \frac{\varepsilon}{6}\int_E M(\frac{\mid \nabla u_n \mid}{\mu})dxdt + K_\varepsilon \mid E \mid \\ &\leq \frac{\varepsilon}{6}\int_Q M(\frac{\mid \nabla u_n \mid}{\mu})dxdt + K_\varepsilon \mid E \mid \\ &\leq \frac{\varepsilon}{6} + K_\varepsilon \mid E \mid . \end{aligned}$$

When $\mid E \mid \leq \varepsilon/(6K_\varepsilon)$, we have

$$b(m)\int_E R(\frac{\mid \nabla u_n \mid}{\lambda'})dxdt \leq \frac{\varepsilon}{3}, \quad \forall n.$$

Consequently , if $\mid E \mid < \delta = \inf\,(\delta_1, \frac{\varepsilon}{6K_\varepsilon})$ one has

$$\int_E \mid gn(x,t,u_n,\nabla u_n) \mid dxdt \leq \varepsilon, \quad \forall n,$$

 this shows that the $gn(x,t,u_n,\nabla u_n)$ are uniformly equi - integrable on Q. By
Dunford - Pettis ' s theorem , there exists $h \in L^1(Q)$ such that

$$gn(x,t,u_n,\nabla u_n) \rightharpoonup h \quad \text{weakly in} L^1(Q). \tag{6.6}$$

Applying then Theorem 4 . 1 , we have for a subsequence , still denoted by u_n,

$$u_n \to u, \nabla u_n \to \nabla u \text{ a . e . in } Q \text{ and } u_n \to u \text{ strongly in } W_0^{1,x} L_M^{\text{loc}}(Q). \tag{6.7}$$

We$^{\text{deduce}}_{\sigma(\Pi L_{-M},}$ $^{\text{that}}_{\Pi L_M}$)$a$and$^{(x,}_{\text{since}}$ ^{t,u} $\frac{n\partial_{u_n}}{\partial t}$ $^{\nabla u}$ $^{n)}_{\to}$ $\frac{\partial u}{\partial t}$$^{\rightharpoonup}_{\text{in}}$ $^{a(x,}_{\mathcal{D}'(Q)}$ $^{t,u,}_{\text{then}}$ $^{\nabla u)}$passing$^{\text{weakly}}$in$_{\text{to}}$($^{L}_{\text{the}}$ $M^{(Q)}$limit$)^{N}_{\text{in}}$($^{\text{for}}$6.4)

as $n \to +\infty$, we obtain

$$\frac{\partial u}{\partial t} + A(u) + g(x,t,u,\nabla u) = f \quad \text{in} \mathcal{D}'(Q).$$

This completes the proof of Theorem 6 . 1 .

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