Homogenization of a nonlinear degenerate parabolic differential equation *

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Abstract

In this article , we study the homogenization of the nonlinear degener - ate parabolic equation

$$\partial_t b(\tfrac{x}{\varepsilon}, u_\varepsilon) - \mathrm{div}\, a(\tfrac{x}{\varepsilon}, t_\varepsilon, u_\varepsilon, \nabla u_\varepsilon) = f(x,t),$$

with mixed boundary conditions (Neumann and Dirichlet) and obtain the limit equation as $\varepsilon \to 0$. We also prove corrector results to improve the weak convergence of ∇u_ε to strong convergence .

1 Introduction

Let Ω be a bounded domain in $\mathbb{R}^n$ with Lipschitz boundary and let $T > 0$ be a constant . Let $\partial\Omega = \Gamma_1 \cup \Gamma_2$, where it is assumed that Γ_1 has positive Hausdorff measure , $H^{n-1}(\Gamma_1)$. We will denote $\Omega \times [0,T]$ by Ω_T, and $\Gamma_i \times [0,T]$ by $\Gamma_{i,T}$, $i = 1, 2.$ We consider the following initial - boundary value problem

$$\partial_t b(\tfrac{x}{\varepsilon}, u_\varepsilon x^\varepsilon_{)a(} - t^{,\varepsilon,}_{\mathrm{div}} u_\varepsilon, \nabla u_\varepsilon).\nu = 0 \quad \mathrm{on}\Gamma_{2,T}, ^{u_\varepsilon=g \quad \mathrm{on}\Gamma_{1,T},}_{a(\varepsilon^x, t_\varepsilon, u_\varepsilon, \nabla u_\varepsilon)=f(x,t)} \quad \mathrm{in}\Omega_T, \tag{1.1}$$

$$u_\varepsilon(x, 0) = u_0 \quad \mathrm{in}\Omega.$$

whose diffusion term is a monotone operator . Regarding the existence , unique - ness and regularity results for the above problem , which we will refer to as (P_ε), we refer the reader to [2] .

We are interested in the asymptotic behaviour of the problem (P_ε) as $\varepsilon \to 0.$ The homogenization of such equations with $b(y, s) \equiv s$ or $b(y, s)$ linear in s has been studied quite widely (cf . [3 , 5 , 4 , 7 , 6 , 1 6 , 9 , 1 5]) . However , the case where b is nonlinear has not been studied so much . Recently , H . Jian (cf . [1 0]) studied this problem for $b(y, s)$ of the form $b(s)$, assumed to be continuous and non - decreasing in s and satisfying the monotonicity condition . It was shown , under

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an *a priori* assumption on the boundedness of the sequence u_ε in $L^\infty(\Omega_T)$, that

the homogenized equation corresponding to this problem is

$$\partial_t b(u) - \operatorname{div} A(u, \nabla u) = f(x, t) \quad \text{in} \Omega_T,$$
$$A(u, {}_u \nabla_= u)_g . \nu = 0 \text{on} \Gamma_{1,T,}^{\text{on}} {}^{\Gamma_{2,T,}} \qquad (1.2)$$
$$u(x, 0) = 0 \quad \text{in} \Omega$$

for a suitable function A.   That is ,   the solutions u_ε of the in - homogeneous problem converge in some sense to a solution u of the homogeneous problem . They first obtain a uniform bound , with respect to ε, on ∇u_ε in $L^p(\Omega_T)$ and

hence$^{\text{on} \partial}_{\text{sequences} \partial_t b(u_\varepsilon),}$ $^{t^b(u_\varepsilon)}$ and$^{\text{at}',\varepsilon,}_{(x^a(\frac{x}{\varepsilon},}$ $\varepsilon^{u_\varepsilon}_t, {}' u_\varepsilon \nabla^{\nabla u_\varepsilon)}_{u_\varepsilon} {}_{\text{)each}}$ ${}^{\text{in}}$an have$^{\text{appropriate}}$ a weak $*$limit$^{\text{dual}}$ in space that$^{\text{Thus,}}_{\text{space,}}$ ${}^{\text{the}}_{\text{but}}$

to

complete the analysis these limits have to be identified as $\partial_t b(u)$ and $A(u, \nabla u)$, respectively .   A crucial link in showing this was the fact that $b(u_\varepsilon)$ converges strongly to $b(u)$ in some $L^q(\Omega_T)$ and this in turn was used to prove the strong convergence of u_ε to u in some $L^r(\Omega_T)$ (note that we cannot conclude the strong convergence of u_ε to u from the uniform bound on the sequence ∇u_ε in $L^p(\Omega_T)$ because the time derivative is not involved , but this information is hidden in the boundedness of $\partial_t b(u_\varepsilon)$).   This is then used to identify the limits

of the sequences $\partial_t b(u_\varepsilon)$ and $a(\varepsilon^x, t_\varepsilon, u_\varepsilon, \nabla u_\varepsilon)$.

However , for the class of problems that we consider , $b(\frac{x}{\varepsilon}, u_\varepsilon)$ can be expected to have only a weak limit in any $L^q(\Omega_T)$. This does not help in proving the strong convergence of u_ε to a u in any $L^r(\Omega_T)$, which is crucially needed for identifying

the weak limits of the sequences $\partial_t b(\frac{x}{\varepsilon}, u_\varepsilon)$ and $a(\frac{x}{\varepsilon}, t_\varepsilon, u_\varepsilon, \nabla u_\varepsilon)$.   However , we are able to prove directly that $u_\varepsilon \to u$ in some $L^r(\Omega_T)$ by adapting a technique found in   [2] .   From this we can prove that $\partial_t b(\frac{x}{\varepsilon}, u_\varepsilon)$ has as its weak limit $\partial_t b(u)$. Here , $b(s)$ denotes the average of $b(y, s)$ in the variable y in the unit cell $Y = [0, 1]^n$.   Interestingly , we show that $b(\frac{x}{\varepsilon}, u_\varepsilon) - b(\frac{x}{\varepsilon}, u)$ strongly converges

to 0 in any $L^q(\Omega_T), 0 < q < \infty$, which yields the strong convergence of $b(u_\varepsilon)$ to $b(u)$ when b is independent of the variable y.   The diffusion term in the homogenized problem is the same as in [1 0] , viz .   div $A(u, \nabla u)$ (cf .   Theorem (2 . 3)) , but we identify this using the method of two - scale convergence . We also

use the two - scale convergence method to prove the corrector results .

We prove corrector results under the strong monotonicity assumption on a which in turn , yields a corrector result for the work of H . J ian .   That is , we construct suitable strong approximations for ∇u_ε.

The layout of the paper is as follows . In Section 2 , we give the weak formu - lation for the problem (P$_\varepsilon$).   Then , we state our main results viz .   Theorem 2 . 3 and Theorem 2 . 5 .   In Section 3 , we prepare the ground for homogenization by obtaining some *a priori* estimates and by proving the strong convergence of u_ε to some u(for a subsequence) in some $L^r(\Omega_T)$. In Section 4 , we prove our main theorems .

2   Assumptions   and   Main   Results
For $p > 1, p^*$ will denote the conjugate exponent $p/(p-1)$. Let V be the space ,
$\{v \in W^{1,p}(\Omega) : v = 0 \text{ on } \Gamma_1\}$ and let V^* be the dual of V. Let $E = L^p(0,T;V)$
and let $W^{1,p}_{per}(Y)$ be the space of elements of $W^{1,p}(Y)$ having the same trace on
opposite faces of Y. We say , $u_\varepsilon \in g + E$ is a weak solution of the problem (P_ε) if it satisfies :

$$b(\tfrac{x}{\varepsilon}, u_\varepsilon) \in L^\infty(0,T;L^1(\Omega)), \partial_t b(\tfrac{x}{\varepsilon}, u_\varepsilon) \in L^{p^*}(0,T;V^*), \tag{2.1}$$

that is

$$\int_0^T < \partial_t b(\tfrac{x}{\varepsilon}, u_\varepsilon), \xi(x,t) > dt + \int_{\Omega_T} (b(\tfrac{x}{\varepsilon}, u_\varepsilon) - b(\tfrac{x}{\varepsilon}, u_0))\partial_t \xi dx dt = 0 \tag{2.2}$$

for all $\xi \in E \cap W^{1,1}(0,T;L^\infty(\Omega))$ with $\xi(T) = 0$ and

$$\int_0^T < \partial_t b(\tfrac{x}{\varepsilon}, u_\varepsilon), \xi(x,t) > dt + \int_{\Omega_T} a(\tfrac{x}{\varepsilon}, t_\varepsilon, u_\varepsilon, \nabla u_\varepsilon).\nabla \xi(x,t) dx dt$$
$$= \int_{\Omega_T} f(x,t)\xi(x,t) dx dt \quad (2.3)$$
$$\text{for all} \xi \in E.$$

For the existence of a solution for the weak formulation we make the following assumptions (
cf .   [2]) .
(A 1)   The function b (y , s) is continuous in y and $s, Y-$ periodic in y and non - decreasing
in s and $b(y, 0) = 0$.
(A 2)   There exists a constant $\theta > 0$ such that for every δ and R with $0 < \delta < R$, there exists
$C(\delta, R) > 0$ such that

$$\mid b(y, s_1) - b(y, s_2) \mid > C(\delta, M) \mid s_1 - s_2 \mid^\theta \tag{2.4}$$

for all $y \in Y$ and $s_1, s_2 \in [-R, R]$ with $\delta < \mid s_1 \mid$.
(A 3)   The mapping $(y, s, \mu, \lambda) \mapsto a(y, s, \mu, \lambda)$ defined from $\mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n$ to
$\mathbb{R}^n$ is measurable in (y, s) and continuous in (μ, λ). Further , it is assumed
that there exists three positive constants α, r, τ_0 so that for all (y, s, μ, λ) and all $\mu1, \mu2 \in \mathbb{R}$
and $\lambda, \lambda_1, \lambda_2 \in \mathbb{R}^n$ one has ,

$$a(y, s, \mu, \lambda)(\lambda) \geq \alpha \mid \lambda \mid^p \tag{2.5}$$
$$(a(y, s, \mu, \lambda_1) - a(y, s, \mu, \lambda_2))(\lambda_1 - \lambda_2) > 0, \quad \forall \lambda_1 \neq \lambda_2 \tag{2.6}$$
$$\mid a(y, s, \mu, \lambda) \mid \leq \alpha^{-1}(1+ \mid \mu \mid^{p-1} + \mid \lambda \mid^{p-1}) \tag{2.7}$$
$$\mid a(y, s, \mu1, \lambda) - a(y, s, \mu2, \lambda)$$
$$\leq \alpha^{-1} \mid \mu1 - \mu2 \mid^r (1+ \mid \mu1 \mid^{p-1-r} + \mid \mu2 \mid^{p-1-r} + \mid \lambda \mid^{p-1-r}) \tag{2.8}$$

Also it is assumed that $a(y, s, \mu, \lambda)$ is $Y - \tau_0$ periodic in (y, s) for all (μ, λ).

Assume $_{u_0 \in L^\infty(\Omega), \text{and}}^{g} \in L^{p(0,} T f_{\in}^{; W1, p(\Omega))} _{L^{p}*(\Omega_T).} \cap L^\infty(\Omega_T), \partial_t g \in L^1(0, T; L^\infty(\Omega)),$ (A4)

(A 5)   For all $y, s, \mu, \lambda_1, \lambda_2,$

$$(a(y, s, \mu, \lambda_1) - a(y, s, \mu, \lambda_2))(\lambda_1 - \lambda_2) \geq \alpha \mid \lambda_1 - \lambda_2 \mid^{p} \text{ (2.9)}$$

Remark 2 . 1   It is to be noted that (A 5) implies the conditions (2 . 5) and (2 . 6) in (A 3) , which alone are sufficient to guarantee the existence of a solution to the weak formulation of (P_ε) and for its homogenization .   (A 5) will be used only in proving the corrector result .

Remark 2 . 2   The prototype for b is a function of the form $c(y) \mid s \mid^k \text{sgn}(s)$ for some positive real number k and continuous and $Y-$ periodic function , $c(.)$, which is non - vanishing on Y.

We now state our main theorems .

Theorem 2 . 3   *Let* u_ε *be a family of s o lutions of* (P_ε). *Assume that there is a constant* $C > 0,$ *such that*

$$\sup_\varepsilon \| u_\varepsilon \| L\infty(\Omega_T) \leq C \qquad (2.10)$$

Under , the assumptions (A 1) - (A 4) , th ere exists a subsequence of ε, s till denoted b y ε, such that for al l q with $0 < q < \infty$, we have ,

$$u_\varepsilon \rightarrow u \text{ s trongly in } L^q(\Omega_T) \quad (2.11)$$

$$\nabla u_\varepsilon \rightharpoonup \nabla u \, weakly \, in \, L^p(\Omega_T) \qquad (2.12)$$

$$b(\tfrac{x}{\varepsilon}, u_\varepsilon) - b(\tfrac{x}{\varepsilon}, u) \rightarrow 0 \text{ s trongly in } L^q(\Omega_T) \quad (2.13)$$

$b(\tfrac{x}{\varepsilon}, u_\varepsilon) \rightharpoonup b(u)$ weakly in $L^q(\Omega_T)$ for $q > 1,$ (2.14) and u s o lves ,

$$\partial_t b(u) - \text{div} A(u, \nabla u) = f \quad in \Omega_T,$$
$$A(u,_u \nabla u_=)_g . \nu = 0 on \Gamma_{1,T,}^{on} {}^{\Gamma_{2,T,}}$$
$$u(x, 0) = u_0 \quad in \Omega, \qquad (2.15)$$

where b and A are defined be low by (2 . 1 6) - (2 . 1 7) .

Remark 2 . 4   Of course , the assumption (2 . 1 0) is true in special cases (see [1 2] Ch . 5) and it is reasonable on physical grounds (see [1 0]) .

The functions b and A are defined by

$$b(s) = \int_Y b(y, s) dy \qquad (2.16)$$

$$A(\mu, \lambda) = 1_{\tau_0} \int_0^{\tau_0} \int_Y a(y, s, \mu, \lambda + \nabla\Phi_{\mu,\lambda}(y, s))dyds \qquad (2.17)$$

where $\Phi_{\mu,\lambda} \in L^p(0, \tau_0; W_{per}^{1,p}(Y))$ solves the periodic boundary value problem

$$\int_0^{\tau_0} \int_Y a(y, s, \mu, \lambda + \nabla\Phi_{\mu,\lambda}(y, s)).\nabla\phi(y, s)dyds = 0 \qquad (2.18)$$

for all $\phi \in L^p(0, \tau_0; W_{per}^{1,p}(Y))$. For the existence of solutions to (2 . 1 8) , we refer
the reader to Corollary 1 . 8 ,   Ch .   3 of [1 1] .   It can be shown that $A(\mu, \lambda)$:
$\mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous and satisfies

$$| A(\mu, \lambda) |\leq \beta^{-1}(1+ | \mu |^{p-1} + | \lambda |^{p-1}) \qquad (2.19)$$
$$(A(\mu, \lambda_1) - A(\mu, \lambda_2)).(\lambda_1 - \lambda_2) > 0, \forall\lambda_1 \neq \lambda_2, \qquad (2.20)$$
$$A(\mu, \lambda).\lambda \geq \beta | \lambda |^p \qquad (2.21)$$

for a positive constant β which depends only on α, n, p, τ_0(cf .   Lemmas 2 . 4 - 2 . 6

in[8]).

   Note that in (2 . 1 2) we only have weak convergence of ∇u_ε in L^p.   We con - struct some
correctors for ∇u_ε which will improve the weak convergence (2 . 1 2)
to strong convergence .   Such results are known as corrector results in the lit -
erature of homogenization and are very useful in numerical computations .   Let $u(x, t)$ be as in
Theorem 2 . 3 and let U_1 $\in$ $L^p(\Omega_T \times (0, \tau_0); W_{per}^{1,p}(Y))$ be the
solution of the variational problem ,

$$\int_{\Omega_T} \int_Y \int_0^{\tau_0} a(y, s, u, \nabla_x u + \nabla_y U_1(x, t, y, s)).\nabla_y\Psi(x, t, y, s) \quad = 0, \qquad (2.22)$$

for all $\Psi \in L^p(\Omega_T \times (0, \tau_0); W_{per}^{1,p}(Y))$. It will be seen that there is such a function
U_1. The statement of the corrector result is as follows .

Theorem 2 . 5   *Let u_ε, u be as in Theorem 2 . 3 and le t U_1 be as defined above . We*
assume all th e assumptions in   Theorem 2 . 3 and furthermore ,   the s trong monotonicity
assumption (A 5) .   Then , if u, U_1 are sufficiently smooth , i . e .   be -
long to $C^1(\Omega_T)$ and $C(\Omega_T; C_{per}(0, \tau_0) \times C_{per}^1(Y))$, then
$$u_\varepsilon - u - \varepsilon U_1(x, t, x\varepsilon, t_\varepsilon) \quad \to \quad 0 \ s \ trongly \ in \ L^p(\Omega_T) and, \quad (2.23)$$
$$\nabla u_\varepsilon - \nabla u - \nabla_y U_1(x, t, x\varepsilon, t_\varepsilon) \quad \to \quad 0 \ s \ trongly \ in \ L^p(\Omega_T). \quad (2.24)$$

in$^{\textbf{Remark}}L^p(0, T2_{;W1,}^{.6}{}_{p(\Omega))}^{\text{Notethat}}as^{\text{we}}are^{\text{we}}$donot$^{\text{not}}$claiming$^{u_\varepsilon-}_{\text{havethefullgradientof}}{}^{u-\varepsilon U_1(\,x,}_{}t, U_1^{x_\varepsilon}, t_\varepsilon) \to$ with0stronglyrespectto$x$

in (2 . 24) .

3   Preliminaries

We first obtain *a priori* bounds under the assumption (2 . 1 0) .   From now on , C
will denote a generic positive constant which is independent of ε.

Lemma 3 . 1   *Let* u_ε
be a family of s o lutions of (P_ε) *and assume that (2 . 1 0)*
holds .   *Then ,*

$$\sup_\varepsilon \| \nabla u_\varepsilon \| \, Lp(\Omega_T) \quad \leq \quad C \tag{3.1}$$

$$\sup_\varepsilon \| a(\tfrac{x}{\varepsilon}, t_\varepsilon, u_\varepsilon, \nabla u_\varepsilon) \| \, Lp^*(\Omega_T) \quad \leq \quad C \tag{3.2}$$

$$\sup_\varepsilon \| \partial_t b(\tfrac{x}{\varepsilon}, u_\varepsilon) \| \, E* \quad \leq \quad C \tag{3.3}$$

Proof :   Define the function $B(.,.) : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ by

$$B(y, s) = b(y, s)s - \int_0^s b(y, \tau) d\tau \tag{3.4}$$

The following identity can be deduced as in Lemma 1 . 5 of Alt and Luckhaus [2] .

$$\int_\Omega (B(\tfrac{x}{\varepsilon}, u_\varepsilon(x, T)) - B(\tfrac{x}{\varepsilon}, u_0)) dx$$
$$= \quad \int_0^T \langle \partial_t b(\tfrac{x}{\varepsilon}, u_\varepsilon), (u_\varepsilon - g) \rangle dt - \int_{\Omega_T} (b(\tfrac{x}{\varepsilon}, u_\varepsilon) - b(\tfrac{x}{\varepsilon}, u_0)) \partial_t g \, dxdt$$
$$+ \int_\Omega (b(\tfrac{x}{\varepsilon}, u_\varepsilon(T)) - b(\tfrac{x}{\varepsilon}, u_0)) g(T) dx \tag{3.5}$$

Therefore , using (2 . 3) we obtain ,

$$\int_\Omega B(\tfrac{x}{\varepsilon}, u_\varepsilon(x, T)) dx + \int_{\Omega_T} a(\tfrac{x}{\varepsilon}, t_\varepsilon, u_\varepsilon, \nabla u_\varepsilon) . \nabla u_\varepsilon \, dxdt$$
$$= \quad \int_\Omega B(\tfrac{x}{\varepsilon}, u_0) dx + \int_{\Omega_T} a(\tfrac{x}{\varepsilon}, t_\varepsilon, u_\varepsilon, \nabla u_\varepsilon) . \nabla g \, dxdt$$
$$+ \int_{\Omega_T} f(x, t)(u_\varepsilon - g) dxdt - \int_{\Omega_T} (b(\tfrac{x}{\varepsilon}, u_\varepsilon) - b(\tfrac{x}{\varepsilon}, u_0)) \partial_t g \, dxdt$$
$$+ \int_\Omega (b(\tfrac{x}{\varepsilon}, u_\varepsilon(T)) - b(\tfrac{x}{\varepsilon}, u_0)) g(T) dx. \tag{3.6}$$

Notice that due to (2 . 1 0) , (A 1) , (A 3) and (A 4) we obtain

$$\int_\Omega B(\tfrac{x}{\varepsilon}, u_\varepsilon(x, T)) dx + \int_{\Omega_T} a(\tfrac{x}{\varepsilon}, t_\varepsilon, u_\varepsilon, \nabla u_\varepsilon) . \nabla u_\varepsilon \, dxdt$$
$$\leq C + C \| \nabla u_\varepsilon \|_{,}^{p-1} \Omega_T \| g \|_{p, \Omega_T} \tag{3.7}$$

Therefore , as B is non - negative from its definition , we get using (A 3) again that ,

$$\alpha \| \nabla u_\varepsilon \|_{,}^{p} \Omega_T \leq C + C \| \nabla u_\varepsilon \|_{,}^{p-1} \Omega_T \tag{3.8}$$

for all ε. This implies (3 . 1) . Then , (3 . 2) follows from (3 . 1) and (A 3) , while (3 . 3) follows from (3 . 1) , (3 . 2) and the weak formulation (2.3).   ◊

As a consequence of (2 . 1 0)
and the above lemma , we immediately conclude
that , for a subsequence of ε (to be denoted by ε again) ,

$$u_\varepsilon \;\rightharpoonup\; u \text{ weakly} * \text{in} L^\infty(\Omega_T), \quad (3.9)$$

$$\nabla u_\varepsilon \;\rightharpoonup\; \nabla u \text{ weakly in} L^p(\Omega_T) \quad (3.10)$$

$$b(\tfrac{x}{\varepsilon}, u_\varepsilon) \;\rightharpoonup\; b^* \text{weakly} * \text{in} L^\infty(\Omega_T) \quad (3.11)$$

$$\partial_t b(\tfrac{x}{\varepsilon}, u_\varepsilon) \;\rightharpoonup\; w \text{ weakly} * \text{in} E^*, \quad (3.12)$$

$$a(\tfrac{x}{\varepsilon}, t_\varepsilon, u_\varepsilon, \nabla u_\varepsilon) \;\rightharpoonup\; A^*(x,t) \text{weakly in} L^{p^*}(\Omega_T) \quad (3.13)$$

for some $b^* \in L^\infty(\Omega_T), w \in E^*$ and $A* \in L^{p^*}(\Omega_T)$. The t ask is to identify these quantities and
obtain the limit equation . We now prove that , for a subsequence , u_ε converges $a.e.$ to u in Ω_T
and this will form a crucial part of the present analysis . This will be found useful in identifying
b^*, w and $A \overset{*}{.}$

Lemma 3 . 2   *There exists a continuous , increasing function ω on $\mathbb{R}^+$ with $\omega(0) = 0$,
such   that ,   given   any C > $0, \delta$ > $0,$ if v_1, v_2 are   any   two functions
in*

$$\| v_i \| \infty, \Omega \le C, i = 1, 2, satisfying \qquad\qquad W^{1,p}(\Omega) \cap L^\infty(\Omega) with$$

$$\int_\Omega (b(\tfrac{x}{\varepsilon}, v_1) - b(\tfrac{x}{\varepsilon}, v_2))(v_1 - v_2) dx \le \delta \quad \forall \varepsilon > 0$$

then

$$\int_\Omega | b(\tfrac{x}{\varepsilon}, v_1) - b(\tfrac{x}{\varepsilon}, v_2) | \, dx \le \omega(\delta) \quad \forall \varepsilon > 0.$$

Proof :   By the *a priori* bounds for v_1, v_2 in L^∞, we can restrict b to the domain $Y \times [-C, C]$,
where it is uniformly continuous . Now ,

$$\int_\Omega | b(\tfrac{x}{\varepsilon}, v_1) - b(\tfrac{x}{\varepsilon}, v_2) | \, dx$$

$$= \int_{\Omega \cap \{|v_1 - v_2| < \delta 2^1\}} \} \, | b(\tfrac{x}{\varepsilon}, v_1) - b(\tfrac{x}{\varepsilon}, v_2) | \, dx$$

$$+ \int_{\Omega \cap \{|v_1 - v_2| \ge \delta 2^1\}} \} \, | b(\tfrac{x}{\varepsilon}, v_1) - b(\tfrac{x}{\varepsilon}, v_2) | \, dx$$

$$\le \quad \omega_b(\delta 2^1) m(\Omega) + \delta^- 2^1 \quad \int_\Omega (b(\tfrac{x}{\varepsilon}, v_1) - b(\tfrac{x}{\varepsilon}, v_2))(v_1 - v_2) dx$$

$$\le \quad \omega_b(\delta 2^1) m(\Omega) + \delta 2^1$$

whereby ω_b is taking the $\omega(t)$ modulus $\overset{\text{of}}{.= \omega_b(t2^1) m(\Omega}\overset{\text{continuity}}{)} + t2^1.$ function for b. Thus , we obtain the
lemma $\diamondsuit$ **Lemma 3 . 3**   Let u_ε be the s o lution of (1 . 1) .   Then ,   the s equence
$\{u_\varepsilon\}_\varepsilon > 0$ is
*relatively compact in $L^\theta(\Omega_T),$ where θ is as in   (A 2) .   As a result ,   there is a
subsequence of u_ε such that ,*

$$u_\varepsilon \to u \; a . \quad e . \quad in \; \Omega_T \quad (3.14)$$

Proof :   *Step 1* : Using the arguments from [1 0] , it can be shown that

$$h^{-1} \int_0^{T-h} \int_\Omega (b(\tfrac{x}{\varepsilon}, u_\varepsilon(t+h)) - b(\tfrac{x}{\varepsilon}, u_\varepsilon(t)))(u_\varepsilon(t+h) - u_\varepsilon(t)) dx dt \le C$$

for some constant C which is independent of ε and h. *Step 2 :*   We show that

$$\int_0^{T-h} \int_\Omega \mid b(\tfrac{x}{\varepsilon}, u_\varepsilon(t+h)) - b(\tfrac{x}{\varepsilon}, u_\varepsilon(t)) \mid dx dt \to 0$$

as $h \to 0$, uniformly with respect to ε. Set , for $R > 0$ and large ,

$$
\begin{aligned}
E_{\varepsilon,R} \quad = \quad & \{ t \in (0, T-h) : \parallel u_\varepsilon(t+h) \parallel W^1, p(\Omega) + \parallel u_\varepsilon(t) \parallel W^1, p(\Omega) + \parallel g \parallel W^1, p(\Omega) \\
& + h^{-1} \int_\Omega (b(\tfrac{x}{\varepsilon}, u_\varepsilon(t+h)) - b(\tfrac{x}{\varepsilon}, u_\varepsilon(t))).(u_\varepsilon(t+h) - u_\varepsilon(t)) dx > R \}
\end{aligned}
$$

From the estimate in Step 1 , it follows that $m(E_{\varepsilon,R}) \le C/R$.   Set $E'_{\varepsilon,R}$ to be the complement of $E_{\varepsilon,R}$ in $(0, T-h)$.   Hence , for $t \in E'_{\varepsilon,R}$, by Lemma 3 . 2 , we have

$$\int_\Omega \mid b(\tfrac{x}{\varepsilon}, u_\varepsilon(t+h)) - b(\tfrac{x}{\varepsilon}, u_\varepsilon(t)) \mid dx < \omega(hR). \tag{3.15}$$

Therefore ,

$$
\begin{aligned}
& \int_0^{T-h} \int_\Omega \mid b(\tfrac{x}{\varepsilon}, u_\varepsilon(t+h)) - b(\tfrac{x}{\varepsilon}, u_\varepsilon(t)) \mid \\
= \quad & \int_{E_{\varepsilon,R}} \int_\Omega \mid b(\tfrac{x}{\varepsilon}, u_\varepsilon(t+h)) - b(\tfrac{x}{\varepsilon}, u_\varepsilon(t)) \mid \\
+ \quad & \int_{E'_{\varepsilon,R}} \int_\Omega \mid b(\tfrac{x}{\varepsilon}, u_\varepsilon(t+h)) - b(\tfrac{x}{\varepsilon}, u_\varepsilon(t)) \mid \\
& \qquad \qquad \le \quad C/R + T\omega(hR)
\end{aligned}
$$

for all ε, R and h.   Now , choose R large , fixed so that C/R is as small as we please and then let $h \to 0$ to complete the proof of Step 2 . *Step 3 :*   By assumption (A 2) , it follows from Step 2 that

$$\int_0^{T-h} \int_\Omega \mid u_\varepsilon(t+h) - u_\varepsilon(t) \mid^\theta dx dt \to 0 \text{as} h \to 0 \tag{3.16}$$

uniformly with respect to ε. *Step 4 :*   In this crucial step , we demonstrate the relative compactness of the
sequence $\{u_\varepsilon\}_\varepsilon > 0$ in $L^\theta(\Omega_T)$.   This is an argument to reduce it to the time independent case .
Set ,

$$v_\varepsilon(x, t) = \{ \quad u0^\varepsilon(x, t) \quad \text{if}^{\text{if}} t^t \in^\in \binom{0, T}{E\varepsilon, R} - \cup h)[T \underset{-}{\searrow}^E h^\varepsilon_, RT] \tag{3.17}$$

Choose , h so that T is an integral multiple of h. We have ,

$$h^1 \int_0^h \int_0^T \int_\Omega \mid u_\varepsilon(t) - \sum_{T/h}^{i=1} \chi_{((i-1)h,ih)} v_\varepsilon((i-1)h+s) \mid^\theta dxdtds$$

$$= h^1 \sum_{T/h}^{i=1} \int_{(i-1)h}^{ih} \int_{(i-1)h}^{ih} \int_\Omega \mid u_\varepsilon(t) - v_\varepsilon(s) \mid^\theta dxdtds$$

$$\leq h^1 -^h h \int_{\max(0,-s)}^{\min(T,T-s)} \int_\Omega \mid u_\varepsilon(t) - v_\varepsilon(s+t) \mid^\theta dxdtds$$

$$\leq Sup_{|s|} \leq h \int_{\max(0,-s)}^{\min(T,T-s)} \int_\Omega \mid u_\varepsilon(t) - u_\varepsilon(s+t) \mid^\theta dxdt$$

$$+ \int_{E_{\varepsilon,R} \cup (T-h,T)} \int_\Omega \mid u_\varepsilon(t) \mid^\theta dxdt$$

$$\leq Tw(hR) + C/R$$

which can be taken small , say less than μ(for all ε),   by fixing h small and $R = h^{-2^1}$. Therefore , there exists $s_\varepsilon \in (0,h)$ such that

$$\int_{\Omega_T} \mid u_\varepsilon(t) - \sum_{T/h}^{i=1} \chi_{((i-1)h,ih)}^v \varepsilon((i-1)h+s_\varepsilon) \mid^\theta dxdt$$

is small uniformly in ε.   Note that the sequences $\{v_\varepsilon((i-1)h+s_\varepsilon)\}_{\varepsilon>0}$ are independent of t ime . Therefore , it is enough to show that $\{v_\varepsilon((i-h)+s_\varepsilon)\}_\varepsilon > 0$
are relatively compact sequences in $L^\theta(\Omega_T)$ for $i = 1,...,T/h$. But , this follows from the compact inclusion of $W^{1,p}(\Omega)$ in $L^p(\Omega)$ as these sequences are bounded in $W^{1,p}(\Omega)$(by the definition of $E_{\varepsilon,R}$) for each i. $\diamond$

We end the section by recalling a fact which is quite useful in periodic ho - mogenization . Let f be a function in $L_{loc}^q(\mathbb{R}^n; C_{per}(Y))$.   Then we have the following lemma .
Lemma $\int_Y f(y,x) 3.4 dy$, $The\, for\, osal_{lq>1}^{cillatory}$ function $f(\frac{x}{\varepsilon}, x)$ converges weakly in $L_{loc}^q(\mathbb{R}^n)$ to

4   Homogenization   and   Correctors

First , we prove   (2 . 1 1) ,   (2 . 1 3)   and   (2 . 1 4)   using Lemma 3 . 3 and Lemma 3 . 4 . Then , we identify b^* and $A*$ given by (3 . 1 3) .   Finally , we prove that u satisfies the homogenized equation (2 . 1 5) .

By the *a priori* bound (2 . 1 0) and (3 . 14) , it follows by the Lebesgue dominated convergence theorem that

$$u_\varepsilon \to u\, strongly\, in\, L^q(\Omega_T), \tag{4.1}$$

for all q with $0 < q < \infty$. Thus , we have shown (2 . 1 1) and we have the following proposition .

Proposition 4 . 1   *We have ,*

$$b(\tfrac{x}{\varepsilon}, u_\varepsilon) - b(\tfrac{x}{\varepsilon}, u) \to 0 \ s \ tro \ ngly \ in \quad L^q(\Omega_T) \forall q, 0 < q < \infty.$$

Proof :   By the *a priori* bound (2 . 1 0) , it is enough to consider the function b on $Y \times [-M, M]$ for a large $M > 0$. As b is continuous , it is uniformly continuous on $Y \times [-M, M]$. Therefore , given $h_0 > 0$, there exists a $\delta > 0$ such that ,

$$\mid b(y, s) - b(y', s') \mid < h_0,$$
$$\text{whenever } \mid y - y' \mid + \mid s - s' \mid < \delta.$$

Now , since $u_\varepsilon \to u$ a . e   in Ω_T, by Egoroff ' s theorem , given $h_1 \ \ > 0$, there exists $E \subset \Omega_T$ such that its Lebesgue measure $m(E) \ \ < h_1$ and u_ε converges uniformly to u on $\Omega_T \setminus E \equiv E'$. Therefore , we can find $\varepsilon_1 > 0$ such that

$$\parallel u_\varepsilon - u \parallel \infty, E' < \delta \forall \varepsilon < \varepsilon_1. \quad (4.2)$$

Therefore , for $\varepsilon < \varepsilon_1$ we have ,

$$\int_{\Omega_T} \mid b(\tfrac{x}{\varepsilon}, u_\varepsilon) - b(\tfrac{x}{\varepsilon}, u) \mid^q dxdt$$

$$= \int_{E'} \mid b(\tfrac{x}{\varepsilon}, u_\varepsilon) - b(\tfrac{x}{\varepsilon}, u) \mid^q dxdt + \int_E \mid (b(\tfrac{x}{\varepsilon}, u_\varepsilon) - b(\tfrac{x}{\varepsilon}, u)) \mid^q dxdt$$

$$\leq \ h_0^q m(\Omega_T) + 2^q \sup(\mid b \mid^q) m(E)$$

$$\leq \ h_0^q m(\Omega_T) + 2^q \sup(\mid b \mid^q) h_1.$$

This completes the proof as h_0 and h_1 can be chosen arbitrarily small . $\Diamond$ **Corollary 4 . 2**   *If $b(\tfrac{x}{\varepsilon}, u_\varepsilon) = b(u_\varepsilon)$, then the above proposition shows that $b(u_\varepsilon) \to b(u) \ s \ tro \ ngly \ in \ L^q(\Omega_T)$, the result of Jian [1 0] .*   **Corollary 4 . 3**   *We have the following convergences :*

(i)$b(\tfrac{x}{\varepsilon}, u_\varepsilon)$ converges to $b(u)$ weakly in $L^q(\Omega_T)$ for any $q \in (1, \infty)$ and hence

$$b^* = b(u).$$

(ii)$\partial_t b(\tfrac{x}{\varepsilon}, u_\varepsilon) \rightharpoonup \partial_t b(u)$ weakly $*$ in E^* and thus $w = \partial_t b(u).$

Proof :   (i) We can write , $b(\tfrac{x}{\varepsilon}, u_\varepsilon) = (b(\tfrac{x}{\varepsilon}, u_\varepsilon) - b(\tfrac{x}{\varepsilon}, u)) + b(\tfrac{x}{\varepsilon}, u).$ The result now follows from Proposition 4 . 1 and Lemma 3 . 4 and (i i) follows from (i) and (3.12). $\Diamond$

Finally , we have to show that $A^* \ \ = A(u, \nabla u)$, which can be proved in a manner similar to that in [1 0] .   We present a different proof of this using the method of *two - s cale convergence* . Besides , some steps of the proof will be used in proving the corrector result .   First , we recall the definition and main results concerning the method of two - scale convergence (cf .   [1 ,   1 3 ,   1 4]) .   We set the period τ_0 in the time variable to be 1 , for convenience of notation .

Definition 4 . 4 Let
$1 < q < \infty$.   A sequence of functions $v_\varepsilon \in L^q(\Omega_T)$ is said
to two - scale converge to a function $v \in L^q(\Omega_T \times Y \times (0,1))$ if

$$\int_{\Omega_T} v_\varepsilon \psi(x,t,x\varepsilon,t_\varepsilon)dxdt \quad \varepsilon \to 0 \to \quad \int_{\Omega_T} \int_0^1 \int_Y v(x,t,y,s)\psi(x,t,y,s)dydsdxdt$$

for all $\psi \in L^{q^*}(\Omega_T; C_{per}(Y \times (0,1)))$. We write $v_\varepsilon 2_\to - sv$. **Remark 4 . 5** From the definition of
two - scale convergence ,   it is easy to see
that if v_ε is a sequence of functions in $L^q(\Omega_T)$ such that $v_\varepsilon 2_\to - sv(x,t,y,s)$, then

$$v_\varepsilon \rightharpoonup \int_0^1 \int_Y v(x,t,y,s)dyds \text{weaklyin} L^q(\Omega_T).$$

The following facts about two - scale convergence [1] will be used by us .
Theorem 4 . 6   *If v_ε is a bounded s equence in $L^q(\Omega_T)$, then there exists a func - tion*
$v \in L^q(\Omega_T \times Y \times (0,1))$ such that , up to a subsequence , $v_\varepsilon 2_\to - sv(x,t,y,s)$.
Theorem 4 . 7   *If $v_\varepsilon, \nabla v_\varepsilon$ are   bounded s equences   in $L^q(\Omega_T)$,   then there   exist*
$v \in L^q((0,T) \times (0,1); W^{1,q}(\Omega))$ and $V_1 \in L^q(\Omega_T \times (0,1); W_{per}^{1,q}(Y))$ such that ,   up to a
subsequence ,

$$v_\varepsilon \quad 2_\to - s \quad v(x,t,s),$$
$$\nabla v_\varepsilon \quad 2_\to - s \quad \nabla_x v(x,t,s) + \nabla_y V_1(x,t,y,s).$$

The following theorem [1] is useful in obtaining the limit of the product of two two - scale
convergent sequences . Let $1 < q < \infty$.
Theorem 4 . 8   *Let v_ε be a s equence in $L^q(\Omega_T)$ and w_ε be a s equence in $L^{q^*}(\Omega_T)$*
such that $v_\varepsilon 2_\to - sv$ and $w_\varepsilon 2_\to - sw$.   Further , assume that the s equence w_ε satisfies

$$\int_{\Omega_T} |w_\varepsilon|^{q^*}(x,t)dxdt \quad \varepsilon \to 0 \to \quad \int_{\Omega_T} \int_0^1 \int_Y |w(x,t,y,s)|^{q^*} dydsdxdt. \qquad (4.3)$$

Then ,

$$\int_{\Omega_T} v_\varepsilon w_\varepsilon dxdt \quad \varepsilon \to 0 \to \quad \int_{\Omega_T} \int_0^1 \int_Y v(x,t,y,s)w(x,t,y,s)dydsdxdt.$$

Definition 4 . 9   A sequence w_ε which two - scale converges and satisfies (4 . 3) is said to
be strongly two - scale convergent .
Remark 4 . 10 An example of a strongly two - scale convergent sequence is the
sequence $\psi(x,t,\varepsilon^x,t_\varepsilon)$ for any $\psi \in L_{per}^q(Y \times (0,1); C(\Omega_T))$.
We will now identify the homogenized problem corresponding to (P_ε) using the two - scale
convergence method .   To avoid the technicalities , we assume that the Dirichlet boundary data
$g = 0$.
Recalling that the solutions u_ε of the problem (P_ε) converges to u strongly in
$L^p(\Omega_T)$ and observing that we have (3 . 1) and (3 . 2) , we conclude using Theorem 4 . 7 and
Theorem 4 . 6 that

Proposition 4 . 1 1
There exist functions $U_1 \in L^p(\Omega_T \times (0,1); W^{1,p}_{per}(Y))$ *and*
$a_0 \in L^{p^*}(\Omega_T \times Y \times (0,1))$ *such that , up to a subsequence ,*

$$\nabla u_\varepsilon \quad 2 \to -s \quad \nabla_x u(x,t) + \nabla_y U_1(x,t,y,s), \tag{4.4}$$

$$a(\tfrac{x}{\varepsilon}, t_\varepsilon, u_\varepsilon, \nabla u_\varepsilon) \quad 2 \to -s \quad a_0(x,t,y,s). \tag{4.5}$$

Further , the pair (u, U_1) *satisfies the following two - s cale homogenized pro b lem*

$$\int_0^T \langle \partial_t b(u), \phi \rangle dt + \int_{\Omega_T} \int_Y \int_0^1 a_0(x,t,y,s).(\nabla_x \phi \tag{4.6}$$

$$+ \nabla_y \Phi(x,t,y,s)) dy ds dx dt \quad = \quad \int_{\Omega_T} f\phi dx dt$$

for al l $\phi \in C_0^\infty(\Omega_T)$ *and* $\Phi \in C_0^\infty(\Omega_T; C_{per}^\infty(Y \times (0,1)))$.
Proof :   Existence of U_1, a_0 and the convergence (4 . 4) ,   (4 . 5) follow from the previous two - scale convergence theorems and by the estimates (3 . 1) and (3 . 2) . Note that , we do not get the s dependence in the first term of right hand side of (4 . 4) because of the strong convergence (4 . 1) . Now , let $\phi \in C_0^\infty(\Omega_T)$ and let $\Phi \in C_0^\infty(\Omega_T; C_{per}^\infty(Y \times (0,1)))$. We take test functions as

$$\phi_\varepsilon = \phi(x,t) + \varepsilon \Phi(x,t,x\varepsilon,t_\varepsilon)$$

in (2 . 3) . Note that ,

$$\int_0^T \langle \partial_t b(\tfrac{x}{\varepsilon}, u_\varepsilon), \phi_\varepsilon \rangle dt \quad \varepsilon \to 0 \to \quad \int_0^T \langle \partial_t b(u), \phi \rangle dt,$$

$$\int_{\Omega_T} f \phi_\varepsilon dx dt \quad \varepsilon \to 0 \to \quad \int_{\Omega_T} f\phi dx dt$$

by Corollary 4 . 3 , (i i) and the strong convergence of ϕ_ε to ϕ in $L^p(\Omega_T)$.   Also , using (4 . 5) and Theorem 4 . 8 by two - scale convergence of $\nabla \phi_\varepsilon$, we get

$$\int_{\Omega_T} a(\tfrac{x}{\varepsilon}, t_\varepsilon, u_\varepsilon, \nabla u_\varepsilon).\nabla \phi_\varepsilon dx dt$$

$$= \quad \int_{\Omega_T} a(\tfrac{x}{\varepsilon}, t_\varepsilon, u_\varepsilon, \nabla u_\varepsilon).(\nabla_x \phi(x,t) + \nabla_y \Phi(x,t,x\varepsilon,t_\varepsilon)) dx dt + o(1)$$

$$\varepsilon \to 0 \to \quad \int_{\Omega_T} \int_Y \int_0^1 a_0(x,t,y,s).(\nabla_x \phi(x,t) + \nabla_y \Phi(x,t,y,s)) dy ds dx dt.$$

Therefore , letting $\varepsilon \to 0$ *in* (2 . 3) *with* $\xi = \phi_\varepsilon$, *we get* (4.6). ◊ **Remark 4 . 12** Note that by (4 . 5) ,

$$a(\tfrac{x}{\varepsilon}, t_\varepsilon, u_\varepsilon, \nabla u_\varepsilon) \quad \rightharpoonup \quad \int_Y \int_0^1 a_0(x,t,y,s) dy ds \text{ weakly in } L^{p^*}(\Omega_T)$$

implies that $A^*(x,t) = \int_Y \int_0^1 a_0(x,t,y,s)dyds$ a . e .   in Ω_T.

Thus , by setting $\Phi = 0$ in (4 . 6) we get the homogenized equation

$$\partial_t b(u) - \mathrm{div}\, A^*(x,t) = f \mathrm{in}\Omega_T, \tag{4.7}$$

The boundary condition can be shown to be , $A * (x,t).\nu = 0$ on $\Gamma_{2,T}$, by choosing smooth test functions which vanish only $\Gamma_{1,T}$ in the previous proposition .   It

can be shown that u satisfies the initial condition , $u(x,0) = u_0(x)$, by passing to

the limit in (2 . 2) .   Thus , in order to complete the homogenization it is enough to show that $A * (x,t) = A(u, \nabla u)$ where A has been defined in (2 . 1 7) .

                We will first identify a_0. In fact , we prove the following Proposition .

Proposition 4 . 13   *Let a_0 be given by (4 . 5) and le t (u, U_1) be as in Proposition 4 . 1 1 .   Then ,*

$$a_0(x,t,y,s) = a(y,s,u, \nabla_x u(x,t) + \nabla_y U_1(x,t,y,s)) \; a . \quad e . \quad in \; \Omega_T \times Y \times (0,1).$$

Proof :   Let ϕ, Φ be as before . Let $\lambda > 0$ and $\phi 0 \in C_0^\infty(\Omega_T; C_{per}^\infty(Y \times (0,1)))^{\,n}$.

Set ,

$$
\begin{aligned}
\eta\varepsilon &:= \nabla_x \phi + (\nabla_y \Phi)(x,t,x\varepsilon,t_\varepsilon) + \lambda\phi 0(x,t,x\varepsilon,t_\varepsilon), \\
a_\varepsilon &:= a(\tfrac{x}{\varepsilon}, t_\varepsilon, u_\varepsilon, \nabla u_\varepsilon), \\
d_\varepsilon &:= a(\tfrac{x}{\varepsilon}, t_\varepsilon, \phi, \eta\varepsilon).
\end{aligned}
\tag{4.8}
$$

We have ,

$$
\begin{aligned}
J_\varepsilon &:= \int_{\Omega_T} (a_\varepsilon - d_\varepsilon).(\nabla u_\varepsilon - \eta\varepsilon) \\
&= \int_{\Omega_T} a_\varepsilon.\nabla u_\varepsilon - \int_{\Omega_T} d_\varepsilon.\nabla u_\varepsilon - \int_{\Omega_T} a_\varepsilon.\eta\varepsilon + \int_{\Omega_T} d_\varepsilon.\eta\varepsilon \\
&:= J_{1,\varepsilon} - J_{2,\varepsilon} - J_{3,\varepsilon} + J_{4,\varepsilon}
\end{aligned}
\tag{4.9}
$$

where $J_{i,\varepsilon}$ denotes the respective terms above for $i = 1, \cdots, 4$. Now ,

$$
\begin{aligned}
J_{1,\varepsilon} &= \int_{\Omega_T} a_\varepsilon.\nabla u_\varepsilon \, dxdt \\
&= -\int_0^T \langle \partial_t b(\tfrac{x}{\varepsilon}, u_\varepsilon), u_\varepsilon \rangle dt + \int_{\Omega_T} f u_\varepsilon \, dxdt \\
\varepsilon \to 0 \to \quad &-\int_0^T \langle \partial_t b(u), u \rangle dt + \int_{\Omega_T} f u \, dxdt \\
&= \int_{\Omega_T} A^*(x,t).\nabla_x u \, dxdt
\end{aligned}
\tag{4.10}
$$

where the last equality follows from (4 . 7) .   For obtaining the limit of the other terms in the right hand side of (4 . 9) we will use Theorem 4 . 8 .   For this we observe that the continuity assumptions on a and the choice of $\phi, \Phi, \phi0$ imply that the sequence

$$d_\varepsilon \quad \equiv \quad a(\tfrac{x}{\varepsilon}, t_\varepsilon, \phi, \nabla_x\phi + (\nabla_y\Phi)(x, t, x\varepsilon, t_\varepsilon) + \lambda\phi0(x, t, x\varepsilon, t_\varepsilon))$$

is $^{\text{of the form}}_{2\,-\text{scale}}$ $\varepsilon^{\text{to}}_{\psi^{(t,x,}_{\text{converges }x}}$ $'t_\varepsilon a^{)}_{(y,}$ $^{\text{for}}s, a\phi, \psi^{\in}_{\nabla_x\phi}L^{p^{per}_*}_+{}^{(Y}_{\nabla_y\Phi} \times (0+^;_{\lambda\phi0)}. {}^{1);}C(\Omega_T)\text{Also}^{)}_{\text{it}}\text{Thus}_{\text{canbe}}, d_\varepsilon\text{seen}\text{strongly}_{\text{that}}$

$\eta\varepsilon$ is strongly two - scale convergent to $\eta(x, t, y, s). = \nabla_x\phi(x, t) + \nabla_y\Phi(x, t, y, s) + \lambda\phi0(x, t, y, s)$. Thus , from these observations , Theorem 4 . 8 and (4 . 1 0) , we obtain

$$J_\varepsilon \quad \varepsilon \to 0 \to \quad \int_{\Omega_T}\int_Y\int_0^1 a_0(x, t, y, s).\nabla_x u\,dyds dxdt$$

$$- \int_{\Omega_T}\int_Y\int_0^1 a(y, s, \phi, \nabla_x\phi + \nabla_y\Phi + \lambda\phi0).(\nabla_x u + \nabla_y U_1)$$

$$- \int_{\Omega_T}\int_Y\int_0^1 a_0(x, t, y, s).(\nabla_x\phi + \nabla_y\Phi + \lambda\phi0)$$

$$+ \int_{\Omega_T}\int_Y\int_0^1 a(y, s, \phi, \nabla_x\phi + \nabla_y\Phi + \lambda\phi0).(\nabla_x\phi + \nabla_y\Phi + \lambda\phi0)$$

Note that by setting $\phi = 0$ in (4 . 6) we get ,

$$\int_{\Omega_T}\int_Y\int_0^1 a_0(x, t, y, s).\nabla_y\Phi(x, t, y, s)dyds dxdt \quad = \quad 0 \qquad (4.11)$$

for any $\Phi \in C_0^\infty(\Omega_T; C_{per}^\infty(Y \times (0, 1)))$.   Thus , the above limit can be rewritten as

$$\lim_{\varepsilon\to 0} J_\varepsilon \quad = \quad \int_{\Omega_T}\int_Y\int_0^1 a_0(x, t, y, s).(\nabla_x u - \nabla_x\phi - \lambda\phi0)dyds dxdt$$

$$- \int_{\Omega_T}\int_Y\int_0^1 a(y, s, \phi, \nabla_x\phi + \nabla_y\Phi + \lambda\phi0)$$

$$\times (\nabla_x u + \nabla_y U_1 - \nabla_x\phi - \nabla_y\Phi - \lambda\phi0).$$

Now , letting $\phi \to u$ strongly in $L^p(0, T; V)$ and $\Phi \to U_1$ in $L^p(\Omega_T \times (0, 1); W_{per}^{1,p}(Y))$ strongly we get ,

$$\phi \to u^{\Phi\to U_1}_{\lim}\lim_{\varepsilon\to 0} J_\varepsilon = \int_{\Omega_T}\int_Y\int_0^1 (a(y, s, u, \nabla_x u + \nabla_y U_1 + \lambda\phi0) - a_0(x, t, y, s)).\lambda\phi0,$$

(4 . 1 2) where we have used the continuity properties of a. On the other hand ,

$$J_\varepsilon \quad = \quad \int_{\Omega_T}(a_\varepsilon - a(\tfrac{x}{\varepsilon}, t_\varepsilon, u_\varepsilon, \eta\varepsilon)).(\nabla u_\varepsilon - \eta\varepsilon)dxdt$$

$$+ \int_{\Omega_T} (a(\tfrac{x}{\varepsilon}, t_\varepsilon, u_\varepsilon, \eta\varepsilon) - a(\tfrac{x}{\varepsilon}, t_\varepsilon, \phi, \eta\varepsilon)).(\nabla u_\varepsilon - \eta\varepsilon) dx dt$$

$$.= \quad L_{1,\varepsilon} + L_{2,\varepsilon},$$

where $L_{i,\varepsilon}, i = 1, 2$ denotes the respective terms above .   By the monotonicity assumption (2.6), $L_{1,\varepsilon} \geq 0$. Therefore , $J_\varepsilon \geq L_{2,\varepsilon}$. Now , by (2 . 8) and generalized H ö lder ' s inequality ,

$$
\begin{aligned}
\mid L_{2,\varepsilon} \mid \quad &\leq \quad \int_{\Omega_T} \mid a(\tfrac{x}{\varepsilon}, t_\varepsilon, u_\varepsilon, \eta\varepsilon) - a(\tfrac{x}{\varepsilon}, t_\varepsilon, \phi, \eta\varepsilon) \mid . \mid \nabla u_\varepsilon - \eta\varepsilon \mid dx dt \\
&\leq \quad \alpha^{-1} \parallel u_\varepsilon - \phi \parallel_p^r (m(\Omega_T) p - 1 - pr + \parallel u_\varepsilon \parallel_p^{p-1-r} + \parallel \phi \parallel_p^{p-1-r} \\
&\qquad\qquad\qquad + \parallel \eta\varepsilon \parallel p^{p-1-r)} \parallel \nabla u_\varepsilon - \eta\varepsilon \parallel_p .
\end{aligned}
$$

Therefore ,

$$
\begin{aligned}
J_\varepsilon \quad &\geq \quad L_{2,\varepsilon} \\
&\geq \quad -\alpha^{-1} \parallel u_\varepsilon - \phi \parallel_p^r (m(\Omega_T) p - 1 - pr + \parallel u_\varepsilon \parallel_p^{p-1-r} + \parallel \phi \parallel_p^{p-1-r} \\
&\qquad\qquad + \parallel \eta\varepsilon \parallel p^{p-1-r)} \parallel \nabla u_\varepsilon - \eta\varepsilon \parallel_p \\
&\geq \quad -\alpha^{-1} \parallel u_\varepsilon - \phi \parallel_p^r (C + \parallel \phi \parallel_p^{p-1-r} + \parallel \eta\varepsilon \parallel_p^{p-1-r})(C + \parallel \eta\varepsilon \parallel_p)
\end{aligned}
$$

since $u_\varepsilon, \nabla u_\varepsilon$ are bounded in $L^p(\Omega_T)$.   We now use the fact that $\eta\varepsilon$ is strongly two - scale convergent to η, defined before , to obtain the limit as $\varepsilon \to 0$ in the above inequality and we get

$$\geq \quad -\alpha^{-1} \parallel u - \phi \parallel_p^r (C + \parallel \phi \parallel_p^{p-1-r} + \parallel \eta \parallel_p^{p-1-r})(C + \parallel \eta \parallel_{p, \ \Omega_T \times Y \times (0,1)}^{)} \quad \lim_{\varepsilon \to 0} J_\varepsilon$$

Now letting $\phi \to u$ and $\Phi \to U_1$ as before , we get

$$\lim_{\substack{\phi \to u \\ \Phi \to U_1}} \quad \lim_{\varepsilon \to 0} J_\varepsilon \geq 0. \tag{4.13}$$

Therefore , from (4 . 1 2) and (4 . 1 3) , we get

$$\int_{\Omega_T} \int_Y \int_0^1 (a(y, s, u, \nabla_x u + \nabla_y U_1 + \lambda\phi 0) - a_0(x, t, y, s)).\lambda\phi 0 \, dy \, ds \, dx \, dt \geq 0$$

(4 . 1 4) for all $\lambda > 0$ and for all $\phi 0 \in C_0^\infty(\Omega_T; C_{per}^\infty(Y \times (0,1)))^n$.   Dividing the above inequality and letting $\lambda \to 0$, we get using the continuity of a, that

$$\int_{\Omega_T} \int_Y \int_0^1 (a(y, s, u, \nabla_x u + \nabla_y U_1) - a_0(x, t, y, s)).\phi 0 \, dy \, ds \, dx \, dt \geq 0 \tag{4.15}$$

$\text{for}_{L^p(\Omega T}^{\text{all}} \phi_\times^0 \in Y \times C(_{0,}^{\infty 0(\Omega_T;} \quad _n \text{ we} C^\infty per_{\text{get}}(Y_{a_0(x,t,}^{\times (0,}, 1)))_y^n, s) = \text{By}_{a(y,s,u}^{\text{the density of}} , \nabla_x u(x,t) \text{these} + \nabla_y U_{1(x,t,y,s)}^{\text{functions}}$
$\qquad\qquad\qquad _{1))} \qquad ,$

$$\text{a . e .} \quad \text{in } \Omega_T \times Y \times (0, 1). \quad \Diamond$$

Proof of Theorem 2 . 3 :   The proof follows from Proposition 4 . 1 1 , Remark 4 . 1 2 , Proposition 4 . 1 3 , (2 . 1 7) and (4.11).   ◊

  We now prove corrector results .   First , we prove a certain corrector result without any smoothness assumption on (u, U_1).   Then we deduce Theorem 2 . 5 from this corrector result .
  Let $\delta > 0$ and choose $\phi \in C_0^1(\Omega_T), \Phi \in C_0(\Omega_T; C_{per}(0,1) \times C_{per}^1(Y))$ ap - proximating u, U_1 respectively , viz .

$$\| \phi - u \| \, Lp(0, T; W^1, p(\Omega)) \le \delta \qquad (4.16)$$

$$\| \, \Phi - U_1 \, \|_{Lp(\Omega_T \times (0,1); W_p} \, 1_{e}^{p} \, {}_r(Y)) \le \delta.. \qquad (4.17)$$

Define , $\eta \varepsilon$ as in (4 . 8) with $\lambda = 0$. Then we have the following lemma . **Lemma 4 . 14**   *Let $\delta > 0$ be fixed .*   *Fix* ϕ, Φ *as above .*   *Under the s trong mono - tonicity assumption (A 5) , we have*

$$\limsup_{\varepsilon \to 0} \| \, \nabla u_\varepsilon - \eta \varepsilon \, \|_{p,} \, \Omega_T \le O(\delta^{r}{}_p^0) \quad (4.18)$$

$$where r_0 = \min(r, 1).$$

Proof :   We will use some of the calculations from Proposition 4 . 1 3 .   For that we observe that the regularity that we have now taken for ϕ, Φ would have been sufficient in the proof of that proposition also .   Let J_ε be as in the proof of Proposition 4 . 1 3 . We have , by the strong monotonicity condition (A 5) ,

$$\alpha \| \, \nabla u_\varepsilon - \eta \varepsilon \, \|_{p, \Omega_T}^p \quad \le \quad \int_{\Omega_T} (a(\tfrac{x}{\varepsilon}, t_\varepsilon, u_\varepsilon, \nabla u_\varepsilon) - a(\tfrac{x}{\varepsilon}, t_\varepsilon, u_\varepsilon, \eta \varepsilon)).(\nabla u_\varepsilon - \eta \varepsilon) dx dt$$

$$.= \quad K_\varepsilon \quad (4.19)$$

Now ,

$$= \quad \int_{\Omega_T} ([a(\tfrac{x}{\varepsilon}, t_\varepsilon, u_\varepsilon, \nabla u_\varepsilon) - a(\tfrac{x}{\varepsilon}, t_\varepsilon, \phi, \eta \varepsilon)] \qquad K_\varepsilon$$

$$+ [a(\tfrac{x}{\varepsilon}, t_\varepsilon, \phi, \eta \varepsilon) - a(\tfrac{x}{\varepsilon}, t_\varepsilon, u, \eta \varepsilon)]$$

$$+ [a(\tfrac{x}{\varepsilon}, t_\varepsilon, u, \eta \varepsilon) - a(\tfrac{x}{\varepsilon}, t_\varepsilon, u_\varepsilon, \eta \varepsilon)]).(\nabla u_\varepsilon - \eta \varepsilon) dx dt$$

$$\le \quad J_\varepsilon + \alpha^{-1} \| u - \phi \|_p^r [(m(\Omega_T) p - 1 - pr + \| u \|_p^{p-1-r} + \| \phi \|_p^{p-1-r} + \| \eta \varepsilon \|_p^{p-1-r})$$

$$\times (\sup \| \nabla u_\varepsilon \|_p + \| \eta \varepsilon \|_p)]$$

$$\varepsilon$$

$$+ \alpha^{-1} \| u_\varepsilon - u \|_p^r [(m(\Omega_T) p - 1 - pr + \| u_\varepsilon \|_p^{p-1-r} + \| u \|_p^{p-1-r} + \| \eta \varepsilon \|_p^{p-1-r})$$

$$\times (\sup \| \nabla u_\varepsilon \|_p + \| \eta \varepsilon \|_p)]$$

$$\varepsilon$$

$$\le \quad J_\varepsilon + C \delta^r (C + \| \eta \varepsilon \|_p^{p-1-r})(C + \| \eta \varepsilon \|_p)$$

$$+ C \| u - u_\varepsilon \|_p^r (C + \| u_\varepsilon \|_p^{p-1-r} + \| \eta \varepsilon \|_p^{p-1-r})(C + \| \eta \varepsilon \|_p).$$

$$\leq \quad \lim_{\varepsilon \to 0} J_\varepsilon + C\delta^r(C+ \parallel \nabla_x\phi + \nabla_y\Phi \parallel_p^{p-1-r}).(C+ \parallel \nabla_x\phi + \nabla_y\Phi \parallel_p) \qquad \lim_{\varepsilon \to 0} \sup K_\varepsilon$$

$$\leq \quad \lim_{\varepsilon \to 0} J_\varepsilon + C\delta^r,$$

where the last constant C is independent of δ for $0 < \delta \leq 1$, as the norms of ϕ, Φ are close to the norms of u, U_1 respectively . Also , we know from the proof of Proposition 4 . 1 3 that

$$\lim_{\varepsilon \to 0} J_\varepsilon \quad = \quad \int_{\Omega_T} \int_Y \int_0^1 a(y, s, u, \nabla_x u + \nabla_y U_1).(\nabla_x u - \nabla_x \phi) dy ds dx dt$$

$$- \int_{\Omega_T} \int_Y \int_0^1 a(y, s, \phi, \nabla_x \phi + \nabla_y \Phi).(\nabla_x u - \nabla_x \phi + \nabla_y U_1 - \nabla_y \Phi)$$

$$\leq \quad C \parallel \nabla_x u - \nabla_x \phi \parallel p^{(1+} \quad \parallel u \parallel p^{p-1} + \parallel \nabla_x u \parallel_p^{p-1} + \quad \parallel \nabla_y U_1 \parallel p^{p-1)}$$

$$+C(\parallel \nabla_x u - \nabla_x \phi \parallel p + \parallel \nabla_y U_1 - \nabla_y \Phi \parallel_p)$$

$$\times (1+ \parallel \phi \parallel p^{p-1} + \parallel \nabla_x \phi \parallel_p^{p-1} + \parallel \nabla_y \Phi \parallel p^{p-1)}$$

$$\leq \quad C\delta$$

by the choice of ϕ, Φ. Thus ,

$$\lim_{\varepsilon \to 0} \sup \parallel \nabla u_\varepsilon - \eta\varepsilon \parallel p^p \leq \lim_{\varepsilon \to 0} J_\varepsilon + C\delta^r \leq C(\delta + \delta^r) \leq C\delta^r 0$$

for $0 < \delta \leq 1$. Hence the lemma . $\Diamond$ Under the stronger continuity assumption on a, viz .

$$\mid a(y, s, \mu, \lambda_1) - a(y, s, \mu, \lambda_2) \mid \leq \mid \lambda_1 - \lambda_2 \mid^r (1+ \mid \mu \mid^{p-1-r} + \mid \lambda_1 \mid^{p-1-r} + \mid \lambda_2 \mid^{p-1-r})$$

$$(4 . 20)$$

for all $(y, s, \mu, \lambda_1, \lambda_2)$, we have the following corollary .

Corollary 4 . 15   *Assume (4 . 20) .*   *Then , we have*

$$\lim_{\varepsilon \to 0} \parallel a(\tfrac{x}{\varepsilon}, t_\varepsilon, u_\varepsilon, \nabla u_\varepsilon) - a(\tfrac{x}{\varepsilon}, t_\varepsilon, u, \eta\varepsilon) \parallel p^* \leq 0_{\delta^{r2}}^p \tag{4.21}$$

Proof :   Note that ,

$$\int_{\Omega_T} \mid a(\tfrac{x}{\varepsilon}, t_\varepsilon, u_\varepsilon, \nabla u_\varepsilon) - a(\tfrac{x}{\varepsilon}, t_\varepsilon, u, \eta\varepsilon \mid^{p^*} dx dt$$

$$\leq \quad C \int_{\Omega_T} \mid a(\tfrac{x}{\varepsilon}, t_\varepsilon, u_\varepsilon, \nabla u_\varepsilon) - a(\tfrac{x}{\varepsilon}, t_\varepsilon, u, \nabla u_\varepsilon) \mid^{p^*} dx dt$$

$$+C \int_{\Omega_T} \mid a(\tfrac{x}{\varepsilon}, t_\varepsilon, u, \nabla u_\varepsilon) - a(\tfrac{x}{\varepsilon}, t_\varepsilon, u, \eta\varepsilon) \mid^{p^*} dx dt$$

$$\leq \quad C \parallel u_\varepsilon - u \parallel p_r^{p,\Omega_T} + \parallel \nabla u_\varepsilon - \eta\varepsilon \parallel_p^p \; {}_r^{;\Omega_T} (C+ \parallel \eta\varepsilon \parallel_{p,\Omega_T}^{p-1-r})$$

where in the last inequality
we have used (2 . 8) and (4 . 20) and the fact that the
sequences $u_\varepsilon, \nabla u_\varepsilon$ are bounded . Letting $\varepsilon \to 0$, using Theorem 2 . 5 , we get

$$\lim_{\varepsilon \to 0} \| a(\tfrac{x}{\varepsilon}, t_\varepsilon, u_\varepsilon, \nabla u_\varepsilon) - a(\tfrac{x}{\varepsilon}, t_\varepsilon, u, \eta\varepsilon) \| p^* \quad \leq \quad 0_{\delta^{r2}}^p (C + \| \nabla_x \phi + \nabla_y \Phi \|_p^{p-1-r})$$
$$\leq \quad C0_{\delta^{r2}}^p.$$

This completes the proof . $\diamond$

Proof of Theorem 2 . 5 :   If u, U_1 are sufficiently smooth we can take $\phi = u$ and $\Phi = U_1$ in
the proof of the previous lemma and (2 . 24) follows as we can
t ake $\delta \equiv 0$.   The convergence in (2 . 23) is obvious from the strong convergence
of u_ε to u in $L^p(\Omega_T)$. $\diamond$

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