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EXISTENCE OF $\Psi-$ BOUNDED SOLUTIONS FOR A SYSTEM OF DIFFERENTIAL EQUATIONS

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Abstract . In this article , we present a necessary and sufficient condition for the existence of solutions to the linear nonhomogeneous system $x' = A(t)x + f(t)$. Under the condition stated , for every Lebesgue $\Psi-$ integrable function f there is at least one $\Psi-$ bounded solution on the interval $(0, +\infty)$.

1 . Introduction

We give a necessary and sufficient condition for the nonhomogeneous system

$$x' = A(t)x + f(t) \tag{1.1}$$

to have at least one $\Psi-$ bounded solution for every Lebesgue $\Psi-$ integrable function f, on the interval $\mathbb{R}_+ = [0, +\infty)$. Here Ψ is a continuous matrix function , instead of a scalar function , which allows a mixed asymptotic behavior of the components of the solution .

The problem of $\Psi-$ boundedness of the solutions for systems of ordinary differ - ential equations has been studied by many authors ; see for example Akinyele [1] , Constantin [3] , Avramescu [2] , Hallam [5] , and Morchalo [6] . In these papers , the function Ψ is a scalar continuous function : Increasing , differentiable , and bounded in [1] ; nondecreasing with $\Psi(t) \geq 1$ on $\mathbb{R}_+$ in [3]) .

Let $\mathbb{R}^d$ be the Euclidean $d-$ space . Elements in this space are denoted by $x = (x_1, x_2, ...x_d)^T$ and their norm by $\| x \| = \max \{| x_1 |, | x_2 |, ... | x_d |\}$. For $d \times d$ real matrices , we define the norm $| A | = \sup_{\|} x \| \leq 1 \| Ax \|$.

Let $\Psi_i : \mathbb{R}_+ \to (0, \infty), i = 1, 2, ...d$, be continuous functions , and let

$$\Psi = \mathrm{diag}[\Psi_1, \Psi_2, ...\Psi_d].$$

Then the matrix $\Psi(t)$ is invertible for each $t \geq 0$.

Definition . A function $\varphi : \mathbb{R}_+ \to R^d$ is said to be $\Psi-$ bounded on $\mathbb{R}_+$ if $\Psi(t)\varphi(t)$ is bounded on $\mathbb{R}_+$.

Definition . A function $\varphi : \mathbb{R}_+ \to R^d$ is said to be Lebesgue $\Psi-$ integrable on $\mathbb{R}_+$ if $\varphi(t)$ is measurable and $\Psi(t)\varphi(t)$ is Lebesgue integrable on $\mathbb{R}_+$.

By a solution of (1 . 1) , we mean an absolutely continuous function satisfying the system for almost all $t \geq 0$.

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 Let A be a continuous $d \times d$ real matrix and the associated linear differential
system be

$$y' = A(t)y. \tag{1.2}$$

Also let Y be the fundamental matrix of (1 . 2) with $Y(0) = I_d$, the identity $d \times d$ matrix .

Let X_1 denote the subspace of $\mathbb{R}^d$ consisting of all vectors which are values of $\Psi-$ bounded solutions of (1 . 2) at $t = 0$. Let X_2 be an arbitrary closed subspace of $\mathbb{R}^d$, supplementary to X_1. Let P_1, P_2 denote the corresponding projections of $\mathbb{R}^d$ onto X_1, X_2.

2 . The Main Results

In this section , we give the main results of this Note .

Theorem 2 . 1 . *If A is a continuous $d \times d$ real matrix , th en (1 . 1) has at least one $\Psi-$ bounded s o lution on $\mathbb{R}_+$ for every Lebesgue $\Psi-$ integrab le function f on $\mathbb{R}_+$ if and only if there is a positive constant K such that*

$$|^{|\ \Psi(t)Y(t)P_1Y^{-1}(s)\Psi^{-1}(s)|}_{\Psi(t)Y(t)P_2Y^{-1}(s)\Psi^{-1}(s)|} \leq \leq K^{K,}_{,} \quad for^{for} 0^0 \leq \leq st \leq \leq t^{,}_{s.} \tag{2.1}$$

Proof . First , we prove the " only if " part . We define the sets : $C_\Psi = \{x : \mathbb{R}_+ \to \mathbb{R}^d : x$ is $\Psi-$ bounded and continuous on $\mathbb{R}_+\}$,
$B = \{x : \mathbb{R}_+ \to \mathbb{R}^d : x$ is Lebesgue $\Psi-$ integrable on $\mathbb{R}_+\}$, $D = \{x : \mathbb{R}_+ \to \mathbb{R}^d : x$ is absolutely continuous on all intervals $J \subset \mathbb{R}_+, \Psi-$ bounded on $\mathbb{R}_+, x(0)$ in $X_2, x'(t) - A(t)x(t)$ in $B\}$.

It is well - known that C_Ψ is a real Banach space with the norm

$$\| x \| C_\Psi = \sup_{t \geq 0} \| \Psi(t)x(t) \| .$$

Also , it is well - known that B is a real Banach space with the norm

$$\| x \| B = \int_0^\infty \| \Psi(t)x(t) \| \, dt.$$

The set D is obviously a real linear space and

$$\| x \| D = \sup_{t \geq 0} \| \Psi(t)x(t) \| \quad + \quad \| x' - A(t)x \| B$$

is a norm on D.

Now , we show that $(D, \| \cdot \quad \| D)$ is a Banach space . Let $(x_n)_n$ be a fundamental sequence in D. Then , $(x_n)_n$ is a fundamental sequence in C_Ψ. Therefore , there exists a continuous and bounded function $x : \mathbb{R}_+ \to \mathbb{R}^d$ such that

$$\lim_{n\to\infty} \Psi(t)x_n(t) = x(t), \quad \text{uniformlyon}\mathbb{R}_+.$$

$$\text{Denote}\bar{x}(t) = \Psi^{-1}(t)x(t) \in C_\Psi.\text{From}$$

$$\| x_n(t) - \bar{x}(t) \| \quad \leq | \Psi^{-1}(t) || \Psi(t)x_n(t) - x(t) \|,$$

it follows that $\lim_{n\to\infty} x_n(t) \quad = \bar{x}(t)$, uniformly on every compact of $\mathbb{R}_+$. Thus ,

$$\bar{x}(0) \in X_2.$$

On the other hand , $(f_n(t))$, where $f_n(t) = \Psi(t)(x_n'(t) - A(t)x_n(t))$, is a funda - mental sequence in L, the Banach space of all vector functions which are Lebesgue integrable on $\mathbb{R}_+$ with the norm

$$\| f \| = \int_0^\infty \| \Psi(t)f(t) \| \, dt.$$

Thus , there is a function f in L such that

$$\lim_{n\to\infty} \int_0^\infty \| f_n(t) - f(t) \| \, dt = 0.$$

Putting $\bar{f}(t) = \Psi^{-1}(t)f(t)$, it follows that $\bar{f}(t) \in B$ For a fixed , but arbitrary , $t \geq 0$, we have

$$\begin{aligned}
\bar{x}(t) - \bar{x}(0) &= \overset{(}{\lim_{n\to\infty}} x_n(t) - x_n(0)) \\
&= \lim_{n\to\infty} \int_0^t x_n'(s)ds \\
&= \lim_{n\to\infty} \int_0^t [(x_n'(s) - A(s)x_n(s)) + A(s)x_n(s)]ds \\
&= \lim_{n\to\infty} \int_0^t \{\Psi^{-1}(s)[f_n(s) - f(s)] + \bar{f}(s) + A(s)x_n(s)\}ds \\
&= \int_0^t [\bar{f}(s) + A(s)\bar{x}(s)]ds.
\end{aligned}$$

It follows that $\bar{x}'(t) - A(t)\bar{x}(t) = \bar{f}(t) \in B$ and $\bar{x}(t)$ is absolutely continuous on all intervals $J \subset \mathbb{R}_+$. Thus , $\bar{x}(t) \in D$. From $\lim_{n\to\infty} \Psi(t)x_n(t) = \Psi(t)\bar{x}(t)$, uniformly on $\mathbb{R}_+$ and

$$\lim_{n\to\infty} \int_0^\infty \| \Psi(t)[(x_n'(t) - A(t)x_n(t)) - (\bar{x}_{(t)}' - A(t)\bar{x}(t))] \| \, dt = 0,$$

it follows that $\lim_{n\to\infty} \| x_n - \bar{x} \| D = 0$. Thus , $(D, \| \cdot \quad \| D)$ is a Banach space . Now , we define

$$T : D \to B, \quad Tx = x' - A(t)x.$$

Clearly , T is linear and bounded , with $\| T \| \leq 1$. Let $Tx = 0$. Then , $x' = A(t)x, x \in D$. This shows that x is a $\Psi-$ bounded solution of (1 . 2) . Then , $x(0) \in X_1 \cap X_2 = \{0\}$. Thus , $x = 0$, such that the operator T is one - to - one .

Now , let $f \in B$ and let $x(t)$ be the $\Psi-$ bounded solution of the system (1 . 1) . Let $z(t)$ be the solution of the Cauchy problem

$$z' = A(t)z + f(t), \quad z(0) = P_2x(0).$$

Then , $x(t) - z(t)$ is a solution of (1 . 2) with $P_2(x(0) - z(0)) = 0$, i . e .$x(0) - z(0) \in X_1$. It follows that $x(t) - z(t)$ is $\Psi-$ bounded on R_+. Thus , $z(t)$ is $\Psi-$ bounded on $\mathbb{R}_+$. It follows that $z(t) \in D$ and $Tz = f$. Consequently , the operator T is onto .

From a fundamental result of Banach : " If T is a bounded one - to - one linear operator from Banach space onto another , then the inverse operator T^{-1} is also bounded , we have that there is a positive constant $K = \| T^{-1} \| \quad -1$ such that , for $f \in B$ and for the solution $x \in D$ of (1 . 1) ,

$$\sup_{t\geq 0} \| \Psi(t)x(t) \| \leq K \int_0^\infty \| \Psi(t)f(t) \| .$$

 For $s \geq 0, \delta > 0, \xi \in \mathbb{R}^d$, we consider the function $f : \mathbb{R}_+ \to \mathbb{R}^d$,

$$f(t) = \begin{cases} \Psi^{-1}(t)\xi, & \text{for} s \leq t \leq s+\delta \\ 0, & \text{elsewhere.} \end{cases}$$

Then , $f \in B$ and $\| f \| B = \delta \| \xi \|$. The corresponding solution $x \in D$ is

$$x(t) = \int_s^{s+\delta} G(t,u)du,$$

where

$$G(t,u) = \{ {}^{Y(t)P_1Y^{-1}(u),}_{-Y(t)P_2Y^{-1}(u),} \quad \text{for}^{\text{for}} 0^0 \leq \leq ut \leq \leq t_u.$$

Clearly , G is continuous except on the line $t = u$, where it has a j ump discontinuity . Therefore ,

$$\| \Psi(t)x(t) \| = \quad \| \int_s^{s+\delta} \Psi(t)G(t,u)\Psi^{-1}(u)\xi du \| \leq K\delta \| \xi \| .$$

It follows that

$$\| \Psi(t)G(t,s)\Psi^{-1}(s)\xi \| \leq K \| \xi \| .$$

Hence ,

$$| \Psi(t)G(t,s)\Psi^{-1}(s) | \leq K,$$

which is equivalent with (2 . 1) . By continuity , (2 . 1) remains true also in the case

$$t = s.$$

Now , we prove the " if " part . We consider the function

$$x(t) = \int_0^t Y(t)P_1Y^{-1}(s)f(s)ds - \int_t^\infty Y(t)P_2Y^{-1}(s)f(s)ds, t \geq 0,$$

where f is a Lebesgue $\Psi-$ integrable function on $\mathbb{R}_+$ It is easy to see that $x(t)$ is a $\Psi-$ bounded solution on $\mathbb{R}_+$ of (1 . 1) . The proof is now complete . $\square$

Remark . By taking $\Psi(t) = I_d$ in Theorem 2 . 1 , the conclusion in [4 , Theorem 2 , Chapter V] follows .

Theorem 2 . 2 . *Suppose that :*

(1) *The fundamental matrix* $Y(t)$ *of* (1 . 2) *satisfies the conditions :*

$$\text{(a)} \quad \lim_{t\to\infty} \Psi(t)Y(t)P_1 = 0;$$

$$\text{(b)} \quad | \Psi(t)Y(t)P_1Y^{-1}(s)\Psi^{-1}(s) \leq K, for 0 \leq s \leq t,$$

$$| \Psi(t)Y(t)P_2Y^{-1}(s)\Psi^{-1}(s) | \leq K, for 0 \leq t \leq s,$$

where K *is a positive constant and* P_1 *and* P_2 *are as in the Introduction*

(2) *The function* $f : \mathbb{R}_+ \to \mathbb{R}^d$ *is Lebesgue* $\Psi-$ *integra b le on* $\mathbb{R}_+$. *Then , every* $\Psi-$ *bounded s o lution* $x(t)$ *of* (1 . 1) *is such that*

$$\lim_{t\to\infty} \| \Psi(t)x(t) \| = 0.$$

Proof . Let $x(t)$ be a $\Psi-$ bounded solution of (1 . 1) . There is a positive constant M such that $\| \Psi(t)x(t) \| \leq M$, for all $t \geq 0$. We consider the function

$$y(t) = x(t) - Y(t)P_1x(0) - \int_0^t Y(t)P_1Y^{-1}(s)f(s)ds + \int_t^\infty Y(t)P_2Y^{-1}(s)f(s)ds$$

for all t $\geq$ 0.

From the hypotheses , it follows that the function $y(t)$ is a $\Psi-$ bounded solution of (1 . 2) . Then , $y(0) \in X_1$. On the other hand , $P_1 y(0) = 0$. Therefore , $y(0) = P_2 y(0) \in X_2$. Thus , $y(0) = 0$ and then $y(t) = 0$ for $t \geq 0$.

Thus , for $t \geq 0$ we have

$$x(t) = Y(t)P_1 x(0) + \int_0^t Y(t)P_1 Y^{-1}(s)f(s)ds - \int_t^\infty Y(t)P_2 Y^{-1}(s)f(s)ds.$$

Now , for a given $\varepsilon > 0$, there exists $t_1 \geq 0$ such that

$$\int_t^\infty \parallel \Psi(s)f(s) \parallel ds < \frac{\varepsilon}{2K}, \quad \text{for} t \geq t_1.$$

Moreover , there exists $t_2 > t_1$ such that , for $t \geq t_2$,

$$\mid \Psi(t)Y(t)P_1 \mid \leq \frac{\varepsilon}{2}[\parallel x(0) \parallel + \int_0^{t_1} \parallel Y^{-1}(s)f(s) \parallel ds] - 1.$$

Then , for $t \geq t_2$ we have

$$\begin{aligned} \parallel \Psi(t)x(t) \parallel \quad &\leq \mid \Psi(t)Y(t)P_1 \mid \parallel x(0) \parallel \quad + \int_0^{t_1} \mid \Psi(t)Y(t)P_1 \mid \parallel Y^{-1}(s)f(s) \parallel ds \\ &+ \int_{t_1}^t \mid \Psi(t)Y(t)P_1 Y^{-1}(s)\Psi^{-1}(s) \mid \parallel \Psi(s)f(s) \parallel ds \\ &+ \int_t^\infty \mid \Psi(t)Y(t)P_2 Y^{-1}(s)\Psi^{-1}(s) \mid \parallel \Psi(s)f(s) \parallel ds \\ &\leq \mid \Psi(t)Y(t)P_1 \mid [\parallel x(0) \parallel \quad + \int_0^{t_1} \quad \parallel Y^{-1}(s)f(s) \parallel ds] \\ &+K \int_{t_1}^\infty \parallel \Psi(s)f(s) \parallel ds < \varepsilon. \end{aligned}$$

This shows that $\lim t \to \infty \parallel \Psi(t)x(t) \parallel \quad = 0$. The proof is now complete . □

Remark . Theorem 2 . 2 generalizes a result in Constantin [3] .

Note that Theorem 2 . 2 is no longer true if we require that the function f be $\Psi-$ bounded on $\mathbb{R}_+$, instead of condition (2) of the Theorem . Even if the function f is such that

$$\lim t \to \infty \parallel \Psi(t)f(t) \parallel \quad = 0,$$

Theorem 2 . 2 does not apply . This is shown by the next example . **Example .** Consider the linear system (1 . 2) with $A(t) = O_2$. Then $Y(t) = I_2$ is a fundamental matrix for (1 . 2) . Consider

$$\Psi(t) = (\frac{1}{t{+}01} \quad t0_+1)$$

We have $\Psi(t)Y(t) = \Psi(t)$, such that

$$P_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad p2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

It follows that the first hypothesis of the Theorem is satisfied with $K = 1$. When we take $f(t) = (\sqrt{t+1}, (t+1)^{-2})^T$, then $\lim_{t\to\infty} \parallel \Psi(t)f(t) \parallel \quad = 0$. On the other hand , the solutions of the system (1 . 1) are

$$x(t) = (\frac{2}{3}^{(t}_{-}{}^{+}_{-t_+}1^{1}_{1)}{}^{3/2+}_{+c}{}_{2}c_1)$$

 It follows that the solutions of the system (1 . 1) are $\Psi-$ unbounded on $\mathbb{R}_+$.

Remark . When in the above example we consider

$$f(t) = ((t+1)^{-1}, (t+1)^{-3})_{,}^{T}$$

then we have

$$\int_0^\infty \parallel \Psi(t) f(t) \parallel dt = 1.$$

On the other hand , the solutions of the system (1 . 1) are

$$x(t) = (\begin{smallmatrix}\ln \\ -2\end{smallmatrix} 1(\begin{smallmatrix}t \\ t\end{smallmatrix} +^{+} 1^{1})_{-2}^{+} c_1 + c_2)$$

It is easy to see that these solutions are $\Psi-$ bounded on $\mathbb{R}_+$ if and only if $c_2 = 0$. In

$$\parallel \Psi(t) x(t) \parallel \quad = 0. \qquad \text{thiscase}, \lim_{t\to\infty}$$

Note that the asymptotic properties of the components of the solutions are not the same . This is obtained by using a matrix Ψ rather than a scalar .

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