EXISTENCE OF $\Psi$–BOUNDED SOLUTIONS FOR A SYSTEM OF DIFFERENTIAL EQUATIONS

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Abstract. In this article, we present a necessary and sufficient condition for the existence of solutions to the linear nonhomogeneous system $x' = A(t)x + f(t)$. Under the condition stated, for every Lebesgue $\Psi$–integrable function $f$, there is at least one $\Psi$–bounded solution on the interval $(0, +\infty)$.

1. INTRODUCTION

We give a necessary and sufficient condition for the nonhomogeneous system

$$x' = A(t)x + f(t)$$

(1.1)
to have at least one $\Psi$–bounded solution for every Lebesgue $\Psi$–integrable function $f$, on the interval $\mathbb{R}_+ = [0, +\infty)$. Here $\Psi$ is a continuous matrix function, instead of a scalar function, which allows a mixed asymptotic behavior of the components of the solution.

The problem of $\Psi$–boundedness of the solutions for systems of ordinary differential equations has been studied by many authors; see for example Akinyele [1], Constantin [3], Avramescu [2], Hallam [5], and Morchalo [6]. In these papers, the function $\Psi$ is a scalar continuous function: increasing, differentiable, and bounded in $[1]$; nondecreasing with $\Psi(t) \geq 1$ on $\mathbb{R}_+$ in $[3]$.

Let $\mathbb{R}^d$ be the Euclidean $d$–space. Elements in this space are denoted by $x = (x_1, x_2, ..., x_d)^T$ and their norm by $\|x\| = \max \{|x_1|, |x_2|, ..., |x_d|\}$. For $d \times d$ real matrices, we define the norm $\|A\| = \sup_{\|x\| \leq 1} \|Ax\|$.

Let $\Psi_i : \mathbb{R}_+ \rightarrow (0, \infty), i = 1, 2, ..., d$, be continuous functions, and let

$$\Psi = \text{diag} [\Psi_1, \Psi_2, ..., \Psi_d].$$

Then the matrix $\Psi(t)$ is invertible for each $t \geq 0$.

Definition. A function $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}^d$ is said to be $\Psi$–bounded on $\mathbb{R}_+$ if $\Psi(t)\varphi(t)$ is bounded on $\mathbb{R}_+$.

Definition. A function $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}^d$ is said to be Lebesgue $\Psi$–integrable on $\mathbb{R}_+$ if $\varphi(t)$ is measurable and $\Psi(t)\varphi(t)$ is Lebesgue integrable on $\mathbb{R}_+$.

By a solution of (1.1), we mean an absolutely continuous function satisfying the system for almost all $t \geq 0$.

2000 Mathematics Subject Classification. 34D55, 34C11.

Key words and phrases. $\Psi$–bounded, Lebesgue $\Psi$–integrable function.

Let $A$ be a continuous $d \times d$ real matrix and the associated linear differential system be

$$y' = A(t)y.$$  \hfill (1.2)

Also let $Y$ be the fundamental matrix of (1.2) with $Y(0) = I_d$, the identity $d \times d$ matrix.

Let $X_1$ denote the subspace of $\mathbb{R}^d$ consisting of all vectors which are values of $\Psi$-bounded solutions of (1.2) at $t = 0$. Let $X_2$ be an arbitrary closed subspace of $\mathbb{R}^d$, supplementary to $X_1$. Let $P_1, P_2$ denote the corresponding projections of $\mathbb{R}^d$ onto $X_1, X_2$.

### 2. The Main Results

In this section, we give the main results of this Note.

**Theorem 2.1.** If $A$ is a continuous $d \times d$ real matrix, then (1.1) has at least one $\Psi$-bounded solution on $\mathbb{R}_+$ for every Lebesgue $\Psi$-integrable function $f$ on $\mathbb{R}_+$ if and only if there is a positive constant $K$ such that

$$\| \Psi(t_1)Y(t_1)P_1 Y^{-1}(s_1)\Psi^{-1}(s_1) \| \leq Ke^{Kt} \text{ for } 0 \leq st \leq t_s.$$  \hfill (2.1)

**Proof.** First, we prove the “only if” part. We define the sets: $C_\Psi = \{ x : \mathbb{R}_+ \to \mathbb{R}^d : x$ is $\Psi$-bounded and continuous on $\mathbb{R}_+ \}$, $B = \{ x : \mathbb{R}_+ \to \mathbb{R}^d : x$ is Lebesgue $\Psi$-integrable on $\mathbb{R}_+ \}$, $D = \{ x : \mathbb{R}_+ \to \mathbb{R}^d : x$ is absolutely continuous on all intervals $J \subset \mathbb{R}_+, \Psi$-bounded on $\mathbb{R}_+, x(0)$ in $X_2, x'(t) = A(t)x(t)$ in $B \}$. It is well-known that $C_\Psi$ is a real Banach space with the norm

$$\| x \|_{C_\Psi} = \sup_{t \geq 0} \| \Psi(t)x(t) \|.$$

Also, it is well-known that $B$ is a real Banach space with the norm

$$\| x \|_B = \int_0^\infty \| \Psi(t)x(t) \| \, dt.$$

The set $D$ is obviously a real linear space and

$$\| x \|_D = \sup_{t \geq 0} \| \Psi(t)x(t) \| + \| x' - A(t)x \|_B$$

is a norm on $D$.

Now, we show that $(D, \| \cdot \|_D)$ is a Banach space. Let $(x_n)_n$ be a fundamental sequence in $D$. Then, $(x_n)_n$ is a fundamental sequence in $C_\Psi$. Therefore, there exists a continuous and bounded function $x : \mathbb{R}_+ \to \mathbb{R}^d$ such that

$$\lim_{n \to \infty} \Psi(t)x_n(t) = x(t), \text{ uniformly on } \mathbb{R}_+.$$

Denote $\bar{x}(t) = \Psi^{-1}(t)x(t) \in C_\Psi$. From

$$\| x_n(t) - \bar{x}(t) \| \leq \| \Psi^{-1}(t) \| \| \Psi(t)x_n(t) - x(t) \|,$$

it follows that $\lim_{n \to \infty} x_n(t) = \bar{x}(t)$, uniformly on every compact of $\mathbb{R}_+$. Thus,

$$\bar{x}(0) \in X_2.$$
On the other hand, \((f_n(t))\), where \(f_n(t) = \Psi(t)(x'_n(t) - A(t)x_n(t))\), is a fundamental sequence in \(L\), the Banach space of all vector functions which are Lebesgue integrable on \(\mathbb{R}_+\) with the norm

\[
\| f \| = \int_0^\infty \| \Psi(t)f(t) \| \, dt.
\]

Thus, there is a function \(f\) in \(L\) such that

\[
\lim_{n \to \infty} \int_0^\infty \| f_n(t) - f(t) \| \, dt = 0.
\]

Putting \(\tilde{f}(t) = \Psi^{-1}(t)f(t)\), it follows that \(\tilde{f}(t) \in B\). For a fixed, but arbitrary, \(t \geq 0\), we have

\[
\tilde{x}(t) - \tilde{x}(0) = \lim_{n \to \infty} x_n(t) - x_n(0) = \lim_{n \to \infty} \int_0^t x'_n(s)\, ds
\]

\[
= \lim_{n \to \infty} \int_0^t \left[ (x'_n(s) - A(s)x_n(s)) + A(s)x_n(s) \right] \, ds
\]

\[
= \lim_{n \to \infty} \int_0^t \left[ \Psi^{-1}(s)[f_n(s) - f(s)] + \tilde{f}(s) + A(s)x_n(s) \right] \, ds
\]

\[
= \int_0^t \tilde{f}(s) + A(s)\tilde{x}(s) \, ds.
\]

It follows that \(\tilde{x}'(t) - A(t)\tilde{x}(t) = \tilde{f}(t) \in B\) and \(\tilde{x}(t)\) is absolutely continuous on all intervals \(J \subset \mathbb{R}_+\). Thus, \(\tilde{x}(t) \in D\). From \(\lim_{n \to \infty} \Psi(t)x_n(t) = \Psi(t)\tilde{x}(t)\), uniformly on \(\mathbb{R}_+\) and

\[
\lim_{n \to \infty} \int_0^\infty \| \Psi(t)[(x'_n(t) - A(t)x_n(t)) - (\tilde{x}'(t) - A(t)\tilde{x}(t))] \| \, dt = 0,
\]

it follows that \(\lim_{n \to \infty} \| x_n - \tilde{x} \| D = 0\). Thus, \((D, \| \cdot \| D)\) is a Banach space. Now, we define

\[T : D \to B, \quad Tx = x' - A(t)x.\]

Clearly, \(T\) is linear and bounded, with \(\| T \| \leq 1\). Let \(Tx = 0\). Then, \(x' = A(t)x, x \in D\). This shows that \(x\) is a \(\Psi\)-bounded solution of (1.2). Then, \(x(0) \in X_1 \cap X_2 = \{0\}\). Thus, \(x = 0\), such that the operator \(T\) is one-to-one.

Now, let \(f \in B\) and let \(x(t)\) be the \(\Psi\)-bounded solution of the system (1.1). Let \(z(t)\) be the solution of the Cauchy problem

\[z' = A(t)z + f(t), \quad z(0) = P_2x(0).\]

Then, \((x(t) - z(t))\) is a solution of (1.2) with \(P_2(x(0) - z(0)) = 0\), i.e., \(x(0) - z(0) \in X_1\). It follows that \((x(t) - z(t))\) is \(\Psi\)-bounded on \(R_+\). Thus, \(z(t)\) is \(\Psi\)-bounded on \(R_+\). It follows that \((z(t) \in D\) and \(Tz = f\). Consequently, the operator \(T\) is onto.

From a fundamental result of Banach: “If \(T\) is a bounded one-to-one linear operator from Banach space onto another, then the inverse operator \(T^{-1}\) is also bounded, we have that there is a positive constant \(K = \| T^{-1} \|^{-1}\) such that, for \(f \in B\) and for the solution \(x \in D\) of (1.1),

\[
\sup_{t \geq 0} \| \Psi(t)x(t) \| \leq K \int_0^\infty \| \Psi(t)f(t) \| \, dt.
\]
For $s \geq 0, \delta > 0, \xi \in \mathbb{R}^d$, we consider the function $f : \mathbb{R}^+ \rightarrow \mathbb{R}^d$,

$$f(t) = \begin{cases} \Psi^{-1}(t) \xi, & \text{for } s \leq t \leq s + \delta \\ 0, & \text{elsewhere.} \end{cases}$$

Then, $f \in B$ and $\| f \|_B = \delta \| \xi \|$. The corresponding solution $x \in D$ is

$$x(t) = \int_s^{s+\delta} G(t,u)du,$$

where

$$G(t,u) = \begin{cases} Y(t)P_1 Y^{-1}(u), & \text{for } 0 \leq ut \leq \delta \\ -Y(t)P_2 Y_{-1}(u), & \text{elsewhere.} \end{cases}$$

Clearly, $G$ is continuous except on the line $t = u$, where it has a jump discontinuity. Therefore,

$$\| \Psi(t)x(t) \| = \int_s^{s+\delta} \Psi(t)G(t,u)\Psi^{-1}(u)\xi du \leq K\delta \| \xi \|.$$

It follows that

$$\| \Psi(t)G(t,s)\Psi^{-1}(s)\xi \| \leq K \| \xi \|.$$

Hence,

$$| \Psi(t)G(t,s)\Psi^{-1}(s)| \leq K,$$

which is equivalent with (2.1). By continuity, (2.1) remains true also in the case $t = s$.

Now, we prove the “if” part. We consider the function

$$x(t) = \int_0^t Y(t)P_1 Y^{-1}(s)f(s)ds - \int_t^\infty Y(t)P_2 Y^{-1}(s)f(s)ds, t \geq 0,$$

where $f$ is a Lebesgue $\Psi-$ integrable function on $\mathbb{R}^+$. It is easy to see that $x(t)$ is a $\Psi-$ bounded solution on $\mathbb{R}^+$ of (1.1). The proof is now complete. □

**Remark.** By taking $\Psi(t) = I_d$ in Theorem 2.1, the conclusion in [4, Theorem 2, Chapter V] follows.

**Theorem 2.2.** Suppose that:

(1) The fundamental matrix $Y(t)$ of (1.2) satisfies the conditions:

(a) $\lim_{t \to \infty} \Psi(t)Y(t)P_1 = 0$;

(b) $| \Psi(t)Y(t)P_1 Y^{-1}(s)\Psi^{-1}(s)| \leq K, for 0 \leq s \leq t$,

$| \Psi(t)Y(t)P_2 Y^{-1}(s)\Psi^{-1}(s)| \leq K, for 0 \leq t \leq s$,

where $K$ is a positive constant and $P_1$ and $P_2$ are as in the Introduction.

(2) The function $f : \mathbb{R}^+ \rightarrow \mathbb{R}^d$ is Lebesgue $\Psi-$ integrable on $\mathbb{R}^+$. Then, every $\Psi-$ bounded solution $x(t)$ of (1.1) is such that

$$\lim_{t \to \infty} \| \Psi(t)x(t) \| = 0.$$
Proof. Let $x(t)$ be a $\Psi-$ bounded solution of (1.1). There is a positive constant $M$ such that $\| \Psi(t)x(t) \| \leq M$, for all $t \geq 0$. We consider the function

$$y(t) = x(t) - Y(t)P_1x(0) - \int_0^t Y(t)P_1Y^{-1}(s)f(s)ds + \int_t^\infty Y(t)P_2Y^{-1}(s)f(s)ds$$

for all $t \geq 0$. 
From the hypotheses, it follows that the function \( y(t) \) is a \( \Psi^- \) bounded solution of (1.2). Then, \( y(0) \in X_1 \). On the other hand, \( P_1 y(0) = 0 \). Therefore, \( y(0) = P_2 y(0) \in X_2 \). Thus, \( y(0) = 0 \) and then \( y(t) = 0 \) for \( t \geq 0 \).

Thus, for \( t \geq 0 \) we have

\[
x(t) = Y(t)P_1 x(0) + \int_0^t Y(t)P_1 Y^{-1}(s)f(s)ds - \int_t^\infty Y(t)P_2 Y^{-1}(s)f(s)ds.
\]

Now, for a given \( \varepsilon > 0 \), there exists \( t_1 \geq 0 \) such that

\[
\int_{t_1}^\infty \| \Psi(s)f(s) \| ds < \frac{\varepsilon}{2K}, \quad \text{for } t \geq t_1.
\]

Moreover, there exists \( t_2 > t_1 \) such that, for \( t \geq t_2 \),

\[
\| \Psi(t)Y(t)P_1 \| \leq \frac{\varepsilon}{2} \| x(0) \| + \int_0^{t_1} \| Y^{-1}(s)f(s) \| ds - 1.
\]

Then, for \( t \geq t_2 \) we have

\[
\| \Psi(t)x(t) \| \leq \| \Psi(t)Y(t)P_1 \| \| x(0) \| + \int_0^{t_1} \| \Psi(t)Y(t)P_1 \| \| Y^{-1}(s)f(s) \| ds
\]

\[
+ \int_{t_1}^t \| \Psi(t)Y(t)P_1 Y^{-1}(s)\Psi^{-1}(s) \| \| \Psi(s)f(s) \| ds
\]

\[
+ \int_t^\infty \| \Psi(t)Y(t)P_2 Y^{-1}(s)\Psi^{-1}(s) \| \| \Psi(s)f(s) \| ds
\]

\[
\leq \| \Psi(t)Y(t)P_1 \| \| x(0) \| + \int_0^{t_1} \| Y^{-1}(s)f(s) \| ds
\]

\[
+ K \int_{t_1}^\infty \| \Psi(s)f(s) \| ds < \varepsilon.
\]

This shows that \( \lim t \to \infty \| \Psi(t)x(t) \| = 0 \). The proof is now complete. \( \square \)

**Remark.** Theorem 2.2 generalizes a result in Constantin [3].

Note that Theorem 2.2 is no longer true if we require that the function \( f \) be \( \Psi^- \) bounded on \( R_+ \), instead of condition (2) of the Theorem. Even if the function \( f \) is such that

\[
\lim t \to \infty \| \Psi(t)f(t) \| = 0,
\]

Theorem 2.2 does not apply. This is shown by the next example. **Example.** Consider the linear system (1.2) with \( A(t) = O_2 \). Then \( Y(t) = I_2 \) is a fundamental matrix for (1.2). Consider

\[
\Psi(t) = \begin{pmatrix} \frac{1}{t+01} & t0_41 \end{pmatrix}
\]

We have \( \Psi(t)Y(t) = \Psi(t) \), such that

\[
P_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}
\]

It follows that the first hypothesis of the Theorem is satisfied with \( K = 1 \). When we take \( f(t) = (\sqrt{t+1},(t+1)^{-2})^T \), then \( \lim_{t \to \infty} \| \Psi(t)f(t) \| = 0 \). On the other hand, the solutions of the system (1.1) are
\[ x(t) = \left( \frac{2^{t^2} + 11^{1.3/2}}{3} \right) + c2c1 \]
It follows that the solutions of the system (1.1) are unbounded on $\mathbb{R}_+$. 

**Remark.** When in the above example we consider 

$$f(t) = ((t + 1)^{-1}, (t + 1)^{-3})^T,$$

then we have 

$$\int_0^\infty \| \Psi(t)f(t) \| \, dt = 1.$$ 

On the other hand, the solutions of the system (1.1) are 

$$x(t) = \left( \frac{1}{(t + 1)^{-1} + 1 - 2c_1 + c_2} \right).$$

It is easy to see that these solutions are $\Psi_1$-bounded on $\mathbb{R}_+$ if and only if $c_2 = 0$. In this case, 

$$\lim_{t \to \infty} \| \Psi(t)x(t) \| = 0.$$ 

Note that the asymptotic properties of the components of the solutions are not the same. This is obtained by using a matrix $\Psi$ rather than a scalar.

**Acknowledgment.** The author would like to thank the anonymous referee for his/her valuable comments and suggestions.

**References**


