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EIGENVALUES AND EIGENVECTORS FOR THE QUATERNION MATRICES OF DEGREE TWO

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Abstract

In this paper we give a computation method , in a particular case , for eigenvalues and eigenvectors of the quaternion matrices of degree two with elements in the generalized quaternion division algebra $\mathbb{H}(\alpha,\beta)$. It is known (see [1]) that every quaternion matrix has at least one charac - teristic root , but there is not yet giving a computing method . By using [4] we give such a computing method for eigenvalues and eigenvectors of the quaternion matrices of degree two with elements in the generalized quaternion division algebra $\mathbb{H}(\alpha,\beta)$.

Let $\mathbb{H}(\alpha,\beta)$ be the generalized quaternion division algebra over the comu - tative field K with $charK \neq 2$.

Definition 1 *Let $A \in \mathcal{M}_n(\mathbb{H}(\alpha,\beta))$ and $\lambda \in \mathbb{H}(\alpha,\beta)$. The quaternion λ is called an **eigenvalue** of th e matrix A(or a **characteristic root**) , if there exists a matrix $x \in \mathcal{M}_{n\times 1}(\mathbb{H}(\alpha,\beta))$, $x \neq 0$, such that $Ax = x\lambda$. The matrix x is called the **eigenvector** of the matrix A.*

Proposition 1 *Two s imilar matrices have the same characteristic roots .*

Proof . Let $A \sim B$, i . e . there exists an invertible matrix $T \in \mathcal{M}_n(\mathbb{H}(\alpha,\beta))$ such that $B = TAT^{-1}$. Let $\lambda \in \mathbb{H}(\alpha,\beta)$ be an eigenvalue for the matrix A, then we find the matrix $x \in \mathcal{M}_{n\times 1}(\mathbb{H}(\alpha,\beta))$ such that $Ax = x\lambda, x \neq 0$. Let

$$y = Tx.\text{Then}By = TAT - 1_y = TAx = Tx\lambda = y\lambda.\square$$

Proposition 2 *Let $A \in \mathcal{M}_n(\mathbb{H}(\alpha,\beta))$ and le t $\lambda \in \mathbb{H}(\alpha,\beta)$ be an e igenvalue of the matrix A. If $\rho \in \mathbb{H}(\alpha,\beta)$, $\rho \neq 0$, then $\rho^{-1}\lambda\rho$ is als o an e igenvalue of the matrix A.*

Key Words : quaternion matrices ; generalized quaternion algebra .

Proof . From $Ax = x\lambda$, we get $A(x\rho) = x\lambda\rho = (x\rho)\rho^{-1}\lambda\rho$.□ **Remark 1** From the Proposition 2 , we see that , if the vector corresponding
to the eigenvalue λ is x, then $x\rho$ is the eigenvector corresponding to the cha -

racteristicroot$\rho^{-1}\lambda\rho$.

Proposition 3 ([1]) *Let K be an arbitrary field , not necessarily commutative , with $charK \neq 2$. If $A = (a_{ij})_{\overline{i,j=}}1, n \in \mathcal{M}_n(K)$, then we have a triangular invertiblefor al l $i >^{matrix} j + 1, i^T_{,} j^{such} \in \{1that_{,2,\ldots,} C^{=}_{n\}}$.□$T^{-1}AT$, $C = (c_{ij})_{\overline{i,j=}}1, n$, where $c_{ij} = 0$,*
Let $\mathbb{H}$ be the real quaternion algebra and let f be the polynomial of degree n :

$$f(X) = a_0 X a_1 X ... X a_n + g(X),$$

where $a_0, a_1, ..., a_n \in \mathbb{H}, a_i \neq 0$ for every $i = \text{-}1_{,n}$ and $g(X)$ is a finite sum of monomials of the form $b_0 X b_1 X ... X b_m$, where m⋖∼∼⋊⋋⋉⋊≈⊓≊⊃⋖n.

In [2] , it is shown that , if the polynomial f has a single term of degree n, then the equation $f(x) = 0$ has exactly n solutions in $\mathbb{H}$.
Proposition 4 ([1]) *Let $A \in \mathcal{M}_n(\mathbb{H})$, then the matrix A has an e igenvalue* .□
In the next , let $\mathbb{H}(\alpha, \beta)$ be the generalized quaternion division algebra over
the commutative field K with $charK \neq 2$. It is known that $\mathbb{H}(\alpha, \beta)$ is an
algebra of degree two , then every element $x \in \mathbb{H}(\alpha, \beta)$ satisfies a relation of the form :

$$x^2 + t(x)x + n(x) = 0,$$

where $t(x), n(x) \in K$ are the **t race** and the **norm** of the element x.
If $\{1, e_1, e_2, e_3\}$ is a basis in $\mathbb{H}(\alpha, \beta)$ and $x \in \mathbb{H}(\alpha, \beta)$, then , for $x = a + be_1 + ce_2 + de_3$, the element $\bar{x} = a - be_1 - ce_2 - de_3$ is called the **conjugate** of the element x and we have the relations :

$$x + \bar{x} = t(x) \quad \text{and} x\bar{x} = n(x)$$

Proposition 5 ([4]) *Let* $a, b \in \mathbb{H}(\alpha,\beta), a \neq 0, b \neq 0$. *Then the linear equation*

$$ax = xb \tag{5.1.}$$

has nonzero s o lutions , $x \in \mathbb{H}(\alpha,\beta)$, *if and only if :*

$$t(a) = t(b) \quad and n(a - a_0) = n(b - b_0), \quad (5.2.)$$
$$where a = a_0 + a_1e_1 + a_2e_2 + a_3e_3, b = b_0 + b_1e_1 + b_2e_2 + b_3e_3.\Box$$

Proposition 6 ([4]) *i) If* $a = a_0+a_1e_1+a_2e_2+a_3e_3, b = b_0+b_1e_1+b_2e_2+b_3e_3 \in \mathbb{H}(\alpha,\beta)$ *with* $b \neq \bar{a}_, a, bslash - element K$, *then the s o lutio ns of the equation (5 . 1 .) , with* $t(a) = t(b)$ *and* $n(a - a_0) = n(b - b_0)$, *are found in* $\mathcal{A}(a, b)$(*the alge bra generated b y th e e lements* a *and* b) *and have th e form :*

$$x = \lambda_1(a - a_0 + b - b_0) + \lambda_2(n(a - a_0) - (a - a_0)(b - b_0)), \tag{6.1.}$$

where $\lambda_1, \lambda_2 \in K$ *are arbitrary . ii) If* $b = \bar{a}_,$ *th en the general s o lution of the equation (5 . 1 .) is* $x = x_1e_1 + x_2e_2 + x_3e_3$, *where* $x_1, x_2, x_3 \in K$ *and th ey satisfy the identity :*

$$\alpha a_1x_1 + \beta a_2x_2 + \alpha\beta a_3x_3 = 0.\Box \tag{6.2.}$$

Proposition 7 ([4]) *Let* $a \in \mathbb{H}(\alpha,\beta), a element - slash K$. *If there exists* $r \in K$ *such that* $n(a) = r^2$, *then* $a = \bar{q}rq^{-1}$, *where* $q = r + \bar{a}, q^{-1} = lin\bar{e} - q_{n(q)}$.
Proof . By hypothesis we have $a(r + \bar{a}) = ar + a\bar{a} = ar + n(a) = = ar + r^2 = (a + r)r$. From $\bar{q} = r + a$ it results $\bar{q}r = aq.\Box$
Proposition 8 ([4]) *Let* $a \in \mathbb{H}(\alpha,\beta)$ *with* $a element - slash K$, *if there exist* $r, s \in K$ *with*
the properties $n(a) = r^4, n(r^2 + \bar{a}) = s^2$, *then th e quadratic equation* $x^2 = a$
has two s o lutions of the form : $x = \pm\frac{r(r^2+a)}{s}$.
Proof . By *Proposition 7 ,* it results that a has the form
$a = \bar{q}r^2q^{-1}$, where $q = r^2 + \bar{a}_.$ Because $q^{-1} = lin\bar{e} - q_{n(q)}$, we obtain

$$a = r^2\bar{q}q^{-1} = r^2\bar{q}^{lin\bar{e}-q}_{n(q)} = r^{2lin\bar{e}-q^2}_{s\phantom{2lin\bar{e}-q}2} = (\frac{r}{s}\bar{q})^2, \therefore$$

$$x_1 = \frac{r}{s}\bar{q}, x_2 = -\frac{r}{s}\bar{q}$$

are the claimed solutions .$\Box$

Proposition 9 ([4]) *Let $a, b, c \in \mathbb{H}(\alpha,\beta)$ such that ab and $b^2 - c$ do not be long to K . If ab and $b^2 - c$ satisfy th e conditions in Proposition 8 , then the equations $xax = b$ and $x^2 + bx + xb + c = 0$ have s o lutions .*

Proof. $xax = b \iff (ax)^2 = ab \text{and} x^2 + bx + xb + c = 0 \iff (x + b)^2 = b^2 - c.\square$

Proposition 10 ([4]) *If $b, c \in \mathbb{H}(\alpha,\beta) \setminus \{K\}$ satisfy the conditions $bc = cb$, $\frac{b^2}{4} - c \neq 0$ and there exists $r \in K$ such that $n(\frac{b^2}{4} - c) = r^4$ and $n(r^2 + \frac{\bar{b}^2}{4} - \bar{c}) = s^2, s \neq 0,$ then th e equation*

$$x^2 + bx + c = 0 \tag{10.1}$$

has s o lutio ns in $\mathbb{H}(\alpha,\beta)$.

Proof . Let $x_0 \in \mathbb{H}(\alpha,\beta)$ be a solution of the equation (1 0 . 1 .) . Because $0_x^2 = t(x_0)x_0 - n(x_0)\text{cedilla} - \text{s i } x_0^2 + bx_0 + c = 0$, it results that $t(x_0)x_0 - n(x_0) + bx_0 + c = 0$,

$$\therefore (t(x_0) + b)x_0 = c + n(x_0).$$

Because $t(x_0) + b \neq 0, t(x_0), n(x_0) \in K, 1 \in \mathcal{A}(b,c)$, we have

$$t(x_0) + b\text{cedilla} - \text{sic} + n(x_0) \in \mathcal{A}(b,c).$$

Therefore $x_0 \in \mathcal{A}(b,c)$. Because $bc = cb$, we obtain that $\mathcal{A}(b,c)$ is commuta - t ive , therefore x_0 commutes with every element of $\mathcal{A}(b,c)$. Then the equation (1 0 . 1 .) can be written :

$$(x + \frac{b}{2})2 - \frac{b^2}{4} + c = 0$$

and we use *Proposition* 8.$\square$

We consider now the case $n = 2$, hence we take $A = (a_{ij})_{i,j=\overline{}1,2} \in \mathbb{H}(\alpha,\beta)$. **Case I** . Let $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in \mathbb{H}(\alpha,\beta)$ with $a_{21} \neq 0$. Let $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \neq 0$ be the eigenvector corresponding to the eigenvalue λ of the matrix A. We suppose that $x_2 \neq 0$. Then the vector $xx_2^{-1} = \begin{pmatrix} x_1x_2^{-1} \\ 1 \end{pmatrix}$ is the eigenvector corresponding to the eigenvalue $x_2\lambda x_2^{-1}$ for the matrix A. Therefore we have got an eigenvector of the form $x = \begin{pmatrix} x_1 \\ 1 \end{pmatrix}$. Then the relation $Ax = x\lambda$ is equivalent to the next system :

$$\{ \quad a^{x_1}_{11_{a}2} 1^{x}1 +_+ a_{12^2_{a}22} =_= x_1 \overline{\lambda\lambda} \quad (*)$$

We replace λ from the second equation in the first one and we get : $a_{11}x_1+a_{12} = x_1(a_{21}x_1+a_{22})$, hence $x_1a_{21}x_1 + x_1a_{22} - a_{11}x_1 - a_{12} = 0$. We multi - ply this last relation to the left side with a_{21}. It results $a_{21}x_1a_{21}x_1 + a_{21}x_1a_{22} - a_{21}a_{11}x_1 - a_{21}a_{12} = 0$. We denote $a_{21}x_1 = t$ and we obtain

$$t^2 + ta_{22} - a_{21}a_{11}a^{-}2^1_1t - a_{21}a_{12} = 0. \qquad (**)$$

If $a_{22} = -a_{21}a_{11}a^{-}2^1_1 = b$, we denote $c = -a_{21}a_{12}$, and if $, b^2 - celement - slashK$ and there exist $r, s \in K$ with the properties $n(b^2 - c) = r^4$ and $n(r^2 + \overline{}_{b^2-c}) \quad = s^2$, then we may use the *Proposition 8* getting $(t + b)^2 = b^2 + a_{21}a_{12}$, therefore :

$$t = \pm\frac{r}{s}(r^2 + b^2 - c) - b.$$

It results that $a_{21}x_1 == \pm\frac{r}{s}(r^2 + b^2 - c) - b$ hence

$$a_{21}x_1 = \pm\frac{r}{s}(r^2 + a_{21}2_{a_1 1}a^{-}2^1_1 + a_{21}a_{12}) + a_{21}a_{11}a^{-}2^1_1.\text{Therefore}$$

$$x_1 = \pm\frac{r}{s}(r^2a^{-}2^1_1 + 2_{a_1 1}a^{-}2^1_1 + a_{12}) + a_{11}a^{-}2^1_1,$$

and , for the eigenvalue λ, we have the expression :

$$\lambda = \pm\frac{r}{s}(r^2 + 2^2_{a}2 + a_{21}a_{12}),$$

$$\because a_{22} = -a_{21}a_{11}a^{-}2^1_1 \text{and} a_{21}2_{a_1 1}a^{-}2^1_1 = a_{21}a_{11}a_{11}a^{-}2^1_1 = -a_{22}a_{21}a_{11}a^{-}2^1_1 = $$
$$= 2^2_{a}2.$$

Case II . If $a_{22} \quad \neq \quad -a_{21}a_{11}a^{-}2^1_1, a_{21} \quad \neq 0$, then the equation $(**)$ is writ - ten $(t + a_{22})^2 - 2^2_{a}2 - a_{22}t - a_{21}a_{11}a^{-}2^1_1t - a_{21}a_{12} = 0$. Equivalently , we get :

$$(t + a_{22})^2 - (a_{22} + a_{21}a_{11}a^{-}2^1_1)(t + a_{22}) + a_{21}a_{11}a^{-}2^1_1a_{22} - a_{21}a_{12} = 0.\text{Denoting}$$

$-(a_{22} + a_{21}a_{11}a^{-}2^1_1) = b, a_{21}a_{11}a^{-}2^1_1a_{22} - a_{21}a_{12} = c$ and $t + a_{22} = v$, we obtain the equation :

$$v^2 + bv + c = 0. \qquad (***)$$

If $b, c \in \mathbb{H}(\alpha, \beta) \setminus \{K\}, bc = cb, \frac{b^2}{4} - c \neq 0$ and there exists $r \in K$ such that $n(\frac{b^2}{4} - c) = r^4$ and $n(r^2 + \frac{\bar{b}^2}{4} - \bar{c}) \quad = s^2, s \neq 0$, we may use *Proposition 1 0* and we obtain the solutions . If these conditions are not satisfied , we can say
only that the solutions of the equation $(***)$ are in the algebra generated by

$$b \text{and} c.$$

Case III . If $a_{21} = 0$, and $a_{12} \neq 0$, then the vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is the eigenvec - tor for the eigenvalue $\lambda = a_{11}$. If $a_{21} = 0$ and $a_{12} = 0$, we have $a_{22} = \lambda$ and then the system $(*)$ is equivalent to the equation $a_{11}x_1 = a_{22}x_1$ and it s nonzero solutions are given by *Proposition 6*. If we have $t(a_{11}) = t(a_{22})$ and $n(a\prime_{11}) = n(a'2_2)$, where $a\prime_{11} = a_{11} - t(a_{11})$ and $a'2_2 = a_{22} - t(a_{22})$, then the solutions have the form (6 . 1 .) for $a_{11} \neq \bar{a}_{22}$ or have the form (6 . 2 .)

$$\text{for} a_{11} = \bar{a}_{22.}$$

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