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HOPF BIFURCATION IN THE IS - LM BUSINESS CYCLE MODEL WITH TIME DELAY

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ABSTRACT . The distinction between investment decisions and implementation leads us to formulate a new I S - LM business cycle model . It is shown that the dynamics depends crucially on the t ime delay parameter - the gestation t ime period of investment . The Hopf bifurcation theorem is used to predict the occurrence of a limit cycle bifurcation for the t ime delay parameter . Our analysis shows that the limit cycle behavior is independent of the assumption of nonlinearity of the investment function . An example is given to verify the theoretical results .

1 . INTRODUCTION

The Hopf bifurcation theorem [8] as a tool for establishing the existence of closed orbits in dynamical systems seems to have been originally introduced to economics by Torre [1 0] , who studied the st andard IS - LM model

$$\dot{Y} = \alpha(I(Y,r) - S(Y,r))$$
$$\dot{r} = \beta(L(Y,r) - \bar{M})$$

with Y as the gross product , I as the investment , S as the saving , L as the demand for money , and $\bar{M}$as the constant money supply . Here α and β are respectively the adjustment coefficients in the markets of goods and money . Other applications of the Hopf bifurcation theorem can be found in , Benhabib and Nishimura [2] , Medio [9] , Krawiec and Szydlowski [7] , and Asada and Yoshida [1] . In the two - dimensional case the use of bifurcation theory actually provides no new insight into known models . The real domains of bifurcation theory are dynamical systems of dimension greater than or equal to three because the Poincare - Bendixson theorem cannot be applied anymore . Gabisch and Lorenz considered an augmented IS - LM business cycle model [4 , p . 1 68] ,

$$\dot{Y} = \alpha(I(Y,K,r) - S(Y,r))$$
$$\dot{r} = \beta(L(Y,r) - \bar{M}) \quad (1.1)$$
$$\dot{K} = I(Y,K,r) - \delta K$$

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with K as the capital sto ck and δ as the depreciation rate of the capital sto ck . The model seems to be one of the simplest complete business cycle models in the Keynesian tradition . A similar model has been studied by Boldrin [3] .

In the Kalecki business cycle model [5] , Kalecki assumed that the saved part of profit is invested and the capital growth is due to past investment decisions . There is a gestation period or a time delay , after which capital equipment is available for production . Similar time lag has been introduced and discussed in [1] and [7] . In
this paper , Kalecki ' s idea is introduced into the IS - LM model (1 . 1) to formulate a generalized IS - LM business cycle model as follow

$$\begin{aligned} \dot{Y} &= \alpha(I(Y,K,r) - S(Y,r)) \\ \dot{r} &= \beta(L(Y,r) - \bar{M}) \\ \dot{K} &= I(Y(t-T),K,r) - \delta K \end{aligned} \tag{1.2}$$

with T as the time delay parameter .

Investment depends on income at the time investment decisions are make and on capital sto ck at the t ime investment is finished . The latter is a consequence of the fact that at time $t-T$ there are some investments which will be finished between
$t-T$ and t. We assume that capital sto ck produced in this period is taken into consideration when new investments are planned . The Hopf bifurcation theorem is applied to predict the occurrence of a limit cycle bifurcation for the time delay parameter . The crucial role in the creation of the limit cycle is Kalecki ' s time delay parameter , rather than the assumption of the S - shaped investment function . An example is given to show that a Hopf bifurcation can o ccur in the present IS - LM model – a linear system with time delay .

2 . L INEAR I S - LM MODEL

Assume that the investment function I, the saving function S, and the demand for money L depend linearly on their arguments , that is

$$\begin{aligned} I &= \eta Y - \delta_1 K - \beta 1^r \\ S &= l_1 Y + \beta 2^r \\ L &= l_2 Y - \beta 3^r \end{aligned}$$

with $\eta, \delta_1, l_1, l_2, \beta 1, \beta 2, \beta 3$ positive constants . Now system (1 . 2) becomes

$$\begin{aligned} \dot{Y} &= \alpha((\eta - l_1)Y - (\beta 1 + \beta 2)r - \delta_1 K) \\ \dot{r} &= \beta(l_2 Y - \beta 3^r - \bar{M}) \\ \dot{K} &= \eta Y(t-T) - \beta 1^r - (\delta + \delta_1)K \end{aligned} \tag{2.1}$$

The characteristic equation of equation (2 . 1) has the form

$$\det \begin{pmatrix} \alpha(\eta - l_1) - \lambda & -\alpha(\beta 1 + \beta 2) & -\alpha\delta_1 \\ \beta l_2 & -\beta\beta 3 - \lambda & 0 \\ \eta e^{-\lambda T} & -\beta 1 & -(\delta + \delta_1) - \lambda \end{pmatrix} = 0$$

that is

$$\lambda^3 + A\lambda^2 + B\lambda + C + D\lambda e^{-\lambda T} + E e^{-\lambda T} = 0 \tag{2.2}$$

 where

$$A = \delta + \delta_1 + \beta\beta 3 - \alpha(\eta - l_1),$$

$$B = (\delta + \delta_1)(\beta\beta 3 - \alpha(\eta - l_1)) + \alpha\beta l_2(\beta 1 + \beta 2) - \alpha\beta\beta 3(\eta - l_1),$$

$$C = -\alpha\beta\beta 1^l 2^\delta 1 - (\delta + \delta_1)\alpha\beta(\beta 3(\eta - l_1) - l_2(\beta 1 + \beta 2)),$$

$$D = \alpha\eta\delta_1, \quad E = \alpha\beta\beta 3\eta\delta_1$$

Generally speaking , transcendental equation (2 . 2) cannot be solved analytically and has indefinite number of roots . In essence , we have two main tools besides direct numerical integration ; firstly , the linear stability analysis , especially in the case of small time delay , and secondly , the Hopf bifurcation theorem . In the following sections , we discuss both approaches .

3 . Linear stability analysis

For small time delay T, the method of linear stability analysis is much convenient to find the bifurcation point . To this end , let $e^{-\lambda T} \approx 1 - \lambda T$, then the eigenvalue equation (2 . 2) becomes

$$\lambda^3 + (A - DT)\lambda^2 + (B + D - ET)\lambda + C + E = 0 \tag{3.1}$$

By the Hopf bifurcation theorem and the Routh - Huwitz criteria , a Hopf bifurcation o ccurs at a value $T = T_0$ where [4 , pp 1 66] ,

$$A - DT_0 > 0, \quad B + D - ET_0 > 0, \quad C + E > 0 \tag{3.2}$$

and

$$(A - DT_0)(B + D - ET_0) = C + E \tag{3.3}$$

Let

$$g(\lambda, T) = \lambda^3 + (A - DT)\lambda^2 + (B + D - ET)\lambda + C + E$$

Evaluating g at $T = T_0$ yields

$$g(\lambda, T_0) = \lambda^3 + s\lambda^2 + k^2\lambda + k^2 s$$

where $s = A - DT_0, k^2 = B + D - ET_0$. The eigenvalues of (3 . 1) at T_0 are

$$\lambda_0(T_0) = -s = -(A - DT_0)$$

$$\lambda_{1,2}(T_0) = \pm ik = \pm i(B + D - ET_0)\frac{1}{2}$$

where i is the imaginary unit . Differentiating implicitly $g(\lambda(T), T)$ yields

$$\frac{d\lambda}{dT} = -\frac{\partial g}{\partial T}\Big/\frac{\partial g}{\partial \lambda} = -\frac{-D\lambda^2 - E\lambda}{3\lambda^2 + 2(A - DT)\lambda + B + D - ET}$$

Evaluating the required derivatives of g at T_0, we have

$$\frac{d\lambda_1(T_0)}{dT} = -\frac{(Dk^2 - Eki)(-3k^2 + B + D - ET_0 - 2k(A - DT_0)i)}{P^2 + R^2} \tag{3.4}$$

where $P = -3k^2 + B + D - ET_0, R = 2(A - DT_0)k$. The real part of (3 . 4) is

$$\mathrm{Re}(\frac{d\lambda_1(T_0)}{dT}) = -\frac{Dk^2(-3k^2 + B + D - ET_0) - 2Ek^2(A - DT_0)}{P^2 + R^2}$$

and Re $(\frac{d\lambda_1(T_0)}{dT}) > 0$ is equivalent to

$$-D(B + D - ET_0) < Ek^2(A - DT_0) \tag{3.5}$$

 Noting that D and E are positive , inequality (3 . 5) holds if the following conditions are fulfilled

$$A - DT_0 > 0 \quad \text{and} \quad B + D - ET_0 > 0$$

So inequality (3 . 2) is sufficient to have positive slope of the real part of the eigen - value $\lambda_1(T)$. This fact guarantees the bifurcation to a limit cycle for $T = T_0$ according to the Hopf bifurcation theorem .

4 . Hopf bifurcation analysis

For larger t ime delay T, the linear st ability analysis of above section is no longer effective and another approach is needed . Let $\lambda = \sigma + i\omega$ and rewrite (2 . 2) in terms of its real and imaginary parts as $\sigma^3 - 3\sigma\omega + A\sigma^2 - A\omega^2 + B\sigma + C + e^{-\sigma T}(D\sigma \cos \omega T + D\omega \sin \omega T + E \cos \omega T) = 0$ $3\sigma^2\omega - \omega^3 + 2A\sigma\omega + B\omega + e^{-\sigma T}(D\omega \cos \omega T - D\sigma \sin \omega T - E \sin \omega T) = 0$

To find the first bifurcation point , we set $\sigma = 0$. Then the above two equations reduce to

$$-A\omega^2 + C + D\omega \sin \omega T + E \cos \omega T = 0 \tag{4.1}$$

$$-\omega^3 + B\omega + D\omega \cos \omega T - E \sin \omega T = 0 \tag{4.2}$$

These two equations can be solved easily numerically . If the first bifurcation point is $(\omega_{\mathrm{bif}}, T_{\mathrm{bif}})$, then the other bifurcation points (ω, T) satisfy

$$\omega T = \omega_{\mathrm{bif}} T_{\mathrm{bif}} + 2n\pi, \quad n = 1, 2, ...$$

By squaring (4 . 1) and (4 . 2) , and then adding them , it follows that

$$\omega^6 + (A^2 - 2B)\omega^4 + (B^2 - 2AC - D^2)\omega^2 + C^2 - E^2 = 0 \tag{4.3}$$

This is a cubic equation in ω^2 and the left side is positive for large values of ω^2 and negative for $\omega = 0$ if $C^2 < E^2$. Hence , if the above condition is met , then (4 . 3) has at least one positive real root . Moreover , we have the following lemma [6] .

Lemma 4 . 1 . *Define*

$$\Delta = \frac{4}{27} 2_a^3 - \frac{1}{27} 2_{a_1} 2_a^2 + \frac{4}{27} 3_{a_1} a_3 - \frac{2}{3} a_1 a_2 a_3 + 3_a^3$$

and suppose that $a_3 > 0$. *Then necessary and sufficient conditions for the cubic equation*

$$z^3 + a_1 z^2 + a_2 z + a_3 = 0$$

to have at least one s ingle positive root for z *are (1) e ith er* $(a) a_1 < 0, a_2 \geq 0,$ *and* $2_{a_1} > 3a_2$, *or* $(b) a_2 < 0$; *and*

$$(2)\Delta < 0.$$

Denote

$$G(\lambda, T) = \lambda^3 + A\lambda^2 + B\lambda + C + D\lambda e^{-\lambda T} + Ee^{-\lambda T}$$

then

$$\frac{d\lambda}{dT} = -\frac{\partial G}{\partial T}\Big/\frac{\partial G}{\partial \lambda} = \frac{(D\lambda^2 + E\lambda)e^{-\lambda T}}{3\lambda^2 + 2A\lambda + B + (D - DT\lambda - ET)e^{-\lambda T}} \tag{4.4}$$

Evaluating the real part of this equation at $T = T_{\rm bif}$ and setting $\lambda = i\omega_{\rm bif}$ yield

$$\frac{d\sigma}{dT}|_{T=T_{\rm bif}} = Re(\frac{d\lambda}{dT})|_{T=T_{\rm bif}} = \frac{\mathrm{bif}_{\omega}^2(3\mathrm{bif}_{\omega}^4 + 2\mathrm{bif}_{\omega}^2(A^2 - 2B) + B^2 - 2AC - D^2)}{P_1^2 + Q_1^2}$$

 where

$$P_1 = -3\text{bif}_\omega^2 + B + T_{\text{bif}}(-A\text{bif}_\omega^2 + C) + D\cos\omega_{\text{bif}}T_{\text{bif}}$$
$$Q_1 = 2A\omega_{\text{bif}} + T_{\text{bif}}(-\text{bif}_\omega^3 + B\omega_{\text{bif}}) - D\sin\omega_{\text{bif}}T_{\text{bif}}$$

Let $x = \text{bif}_\omega^2$, then (4 . 3) reduces to

$$f(x) = x^3 + (A^2 - 2B)x^2 + (B^2 - 2AC - D^2)x + C^2 - E^2$$

then

$$f'(x) = 3x^2 + 2(A^2 - 2B)x + B^2 - 2AC - D^2$$

If ω_{bif} is the least positive simple root of equation (4 . 3) , unless this is a double root when we must take ω_{bif} as the next root , then

$$f'(x)|_{T=T_{\text{bif}}} > 0$$

Hence ,

$$\frac{d\sigma}{dT}|_{T=T_{\text{bif}}} = \frac{\text{bif}_\omega^2 f'(\text{b}_\omega^2\text{if})}{P_1^2 + Q_1^2} > 0$$

According to the Hopf bifurcation theorem , we come to the main result of this paper .

Theorem 4 . 2 . *Assume that the conditions of Lemma 4 . 1 are satisfied and* ω_{bif} *is*
the least positive root of equation (4 . 3) unless this is a double root when we must take ω_{bif} *as the next root which is s imple , th en a Hopf b ifurcation occurs as* T *passes through* T_{bif}.

A similar phenomenum appeared in the model of multiparty political system studied in [6] .

Example. When$\alpha = 3, \beta = 2, \delta = 0.1, \delta_1 = 0.5, \eta = 0.3, l_1 = 0.2, l_2 = 0.1,$

$\bar{M} = 0.5, \beta 1 = \beta 2 = \beta 3 = 0.2,$ system (2 . 1) becomes

$$\dot{Y} = 0.3Y - 0.4r - 0.5K$$
$$\dot{r} = 0.2Y - 0.4r - 0.1$$
$$\dot{K} = 0.3Y(t-T) - 0.2r - 0.6K$$

The characteristic equation (2 . 2) becomes

$$\lambda^3 + 0.7\lambda^2 + 0.18\lambda + 0.012 + 0.45\lambda e^{-\lambda T} + 0.18e^{-\lambda T} = 0$$

It is easy to verify that the conditions of Theorem 4 . 2 are fulfilled , so a limit cycle bifurcation o ccurs when the time delay parameter T passes through $T_{\text{bif}} = 0.740471$ where the relative eigenvalues are $\lambda_0 = -0.382583, \lambda_{1,2} = \pm 0.6993i$. Moreover , we can determine the approximate period of the closed orbit by

$$\tilde{T} = \frac{2\pi}{|\lambda(T_{\text{bif}})|} = \frac{2\pi}{0.6993} = 8.98496$$

which implies the period of the economical system is about 9 .

Conclusion . When we take into account the distinction between investment deci - sion and expenditure , we come to the problem of gestation lags in investment . This leads us to formulate the generalized IS - LM business cycle model with time delay . The Hopf bifurcation theorem is used to predict the o ccurrence of a bifurcation to a limit cycle for some values of the time delay parameter . Our model admits the limit cycle behavior even for a linear investment function instead of a S - shaped one . It is also shown that Kalecki ' s time delay parameter plays the crucial role of existence of limit cycle behavior . The example in section 5 verifies the analytical results .

References

[1] T . Asada , H . Yoshida ; *Stability , instability and complex behavior in macrodynamic models with policy lag* , Discrete Dynamics in Nature and Society , **5** (2001) , 281 - 295 .

[2] J . Benhabib , K . Nishimura ; *The Hopf bifurcation and the existence of closed orbit in multi - s ector models of optimal economic growth* , Journal of Economic Theory , **2 1** (1 979) , 42 1 - 444 .

[3] M . Boldrin ; *Applying bifurcation theory : Some simple results on Keynesian business cycles* , DP 8403 , University of Venice , 1 984 .

[4] G . Gabisch , H . W . Lorenz ; *Business cycle theory - A survey of methods and concepts* (Second Edition) , Springer , New York , 1 989 .

[5] M . Kalecki ; *A macrodynamic theory of business cycle* , Econometrica , **3** (1 935) , 327 - 344 .

[6] Q . J . A . Khan ; *Hopf bifurcation in multiparty political systems with time delay in switching* , Applied Mathematics Letters , **1 3** (2000) , 43 - 52 .

[7] A . Krawiec , M . Szydlowski ; *The Kaldor - Kalecki business cycle model* , Annals of Operations Research , **89** (1 999) , 89 - 100 .

[8] J . E . Marsdem , M . Mckracken ; *The Hopf b ifurcation and its application* , Springer , New York , 1 976 .

[9] A . Medio ; *Oscillations in optimal growth models* , Mimeo , University of Venice , 1 986 .

[10] V . Torre ; *Existence of limit cycles and control in complete Keynesian systems by theory of b ifurcations* , Econometrica , **45** (1 977) , 1457 - 1466 .

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