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OPTIMAL CONTROL OF AN EPIDEMIC THROUGH EDUCATIONAL CAMPAIGNS

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Abstract . In this work we study the best strategy for educational campaigns during the outbreak of an epidemic . Assuming that the epidemic is described by the simplified SIR model and that the total t ime of the campaign is limited due to budget , we consider two possible scenarios . In the first scenario we have a campaign oriented to decrease the infection rate by stimulating susceptibles to have a protective behavior . In the second scenario we have a campaign oriented to increase the removal rate by stimulating the infected to remove themselves from the infected class . The optimality is taken to be to minimize the total number of infected by the end of the epidemic outbreak . The tech - nical tool used to determine the optimal strategy is the Pontryagin Maximum Principle .

1 . Introduction

In this work we study the best strategy for educational campaigns during the outbreak of an epidemic . We assume that the epidemic is described by the simplified SIR model [1 6] and also assume that the total time of the campaign is budget limited . Optimality is measured minimizing the total number of infected at the end of the optimal outbreak . If we cannot make a campaign during all the epidemic time , what is the optimal way of using the time we have ? How many campaigns should we make ? What should be their intensities ? When should they start ? The difficult point is , of course , how to model the effect of the campaign on the spread of the epidemic . Here we face two problems : first , the model must be intuitively plausible and second , it must be mathematically tractable .

With respect to the first requirement we will model the campaign effects by reducing the rate at which the disease is contracted from an average individual *during* the campaign (called shortly infection rate) . We justify this with an example : suppose during a flu outbreak one starts a campaign orienting susceptibles to avoid contracting the virus (assuming some protective behavior , e . g . , washing hands , avoiding close environments , etc .) . The effect of the campaign will be that the probability of a susceptible contracting the virus will decrease . The same reasoning applied to a campaign oriented to the infected (e . g . stimulating quarantine) will be modelled increasing the rate at which an average individual leaves the infective rate

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(called shortly removal rate) . With respect to the second requirement we assume , for mathematical simplicity , that this reduction (increase) is bounded below (above) and the campaigns cost are linear on the controls . With those hypotheses the problem renders itself to analytical treatment and we can prove the main facts about the optimal campaign . The theorems of section 4 reduce the dimension of the optimal problem allowing a complete numerical study of the problem .

Application of control theory to epidemics is a very large field . A comprehensive survey of control theory applied to epidemiology was performed by Wickwire [1 7] . Many different models with different objective functions have been proposed (see [8 , 9 , 1 2] and more recently [3 , 1 8]) . A major difficulty in applying control theoretic methods to practical epidemiology problems is the commonly made assumption that one has total knowledge of the state of the epidemics [7] .

2 . Statement of the problem

We denote by $S(t), I(t), R(t)$ the number of susceptible , infectives and removed in a closed population of size N at time t. We assume the controlled dynamics

$$\begin{aligned} \dot{S} &= -u_1 SI, \\ \dot{I} &= u_1 SI - u_2 I, \\ \dot{R} &= u_2 I, \end{aligned} \tag{2.1}$$

The above models assume a mass - action type interaction (for more realistic inter - actions see [4]) . We let positive constants β and γ denote the infection and removal rates respectively without the influence of an education campaign . Our controls are $u_1(t), u_2(t)$ with $u_1(t) \in [\beta_m, \beta]$ and $u_2(t) \in [\gamma, \gamma M]$ with $0 < \beta_m$. Observe that $u_1(t)$ and $u_2(t)$ regulate the goals and efforts of two types of campaigns . For example , if $u_2(t) = \gamma$ for all t we are controlling only the infection rate . In this case $u_1(t) = \beta$ will correspond to not having a campaign affecting the susceptibles and $u_1(t) = \beta_m$ will correspond to the maximum effort that can be made . The reciprocal case will be if $u_1(t) = \beta$ for all t. In this scenario we will be controlling only the removal rate γ. The above considerations motivate the introduction of the following cost constraints .

$$J_1 = \int_0^{t^*} [(\beta - u_1(t)) + (u_2(t) - \gamma)] I(t) dt, \tag{2.2}$$

$$J_2 = \int_0^{t^*} (\beta - u_1(t)) + (u_2(t) - \gamma) dt, \tag{2.3}$$

In both cases the cost is linear in the controls u_1 and u_2. In the first case the cost of the campaign is supposed to be proportional to the number of infected (if one assumes that the number of infected is proportional to the number of regions where the disease o ccurs and therefore , to the number of regions to be covered by the campaign , higher the number of infected , higher the costs) . The second case assumes that the cost is independent of the number of infected .

Our goal will be to find the optimal control strategies that minimize the total number of infected over the course of the epidemic outbreak (equivalently , that maximize the total number of susceptibles) . In this work , the end of the epidemic outbreak will be defined as a (very large) time instant t^* for which $I(t^*) < 1$(see remark (2 . 1) about the existence of t^*). In other words , t^* is the first time such that

$I(t^*) < 1$. This is a technicality in order to avoid dealing with a infinite horizon control problem . Since in the simplified SIR model the only way to enter in the removed class is from the infected class , the total number of infected at the end of the epidemics is given by $\lim_{t\to\infty} R(t)$. However ,

$$\dot{R} = \gamma I,$$

and since we always assume $R(0) = 0$, we obtain that the total number of infected is given by

$$\lim_{t\to\infty} R(t) = \int_0^\infty \gamma I(t)dt.$$

Remark 2 . 1 . We make some remarks that are important for what follows .

(1) Since $S(t) + I(t) + R(t) = N$ we will ignore the last equation of (2 . 1) .

(2) The set $M = \{I \geq 0, S \geq 0, S + I \leq N\}$ is an invariant set for system (2 . 1) . (3) In the simplified SIR model we always have that $\lim_{t\to\infty} I(t) = 0$(see e . g .

[6]).

Since we are working only on M and the controls u_1 and u_2 are bounded and positive , we will always have that $\lim_{t\to\infty} I(t) = 0$ for any control . This establishes the existence of t^*.

The constant cost constraints J_1 and J_2 can be imposed introducing a new variable w to our system . We obtain the two control systems

$$\begin{aligned} \dot{S} &= -u_1 SI, \\ \dot{I} = u_1 SI - u_2 I, \quad Y_1 &= \int_0^{t^*} I(t)dt \\ \dot{w} = ({}^(\beta - u_1(t)) + (u_2(t) - \gamma))I(t), \quad w(0) &= 0, \quad w(t^*) = C. \end{aligned} \tag{2.4}$$

with cost J_1, and the system

$$\begin{aligned} \dot{S} &= -u_1 SI, \\ \dot{I} = u_1 SI - u_2 I, \quad Y_2 &= \int_0^{t^*} u_2(t)I(t)dt \\ \dot{w} = (\beta - u_1(t)) + (u_2(t) - \gamma), \quad w(0) &= 0, \quad w(t^*) = C. \end{aligned} \tag{2.5}$$

with cost J_2. In both systems we are imposing $J_1 = J_2 = C$, where C is a constant .

Remark 2 . 2 . The constant C is the value of the total amount of campaign effort . We will assume henceforth that C is such that the controls can not be at the maximum effort level during the whole time period .

The problems will be referred as problem C 1 and C 2 respectively . The goal is to find the optimal controls to (2 . 4) that minimize Y_1 and the optimal controls to (2 . 5) that minimize Y_2. We will refer to the first problem as problem C 1 and to the second

problem as problem $C2$. As it will turn out , problem C 1 is trivial . We will assume that the admissible controls u_1 and u_2 are measurable lo cally bounded functions . Since u_1 and u_2 appear linearly in our control problems , an optimal control will in

general be a combination of bang - bang controls and singular controls (see [1 4 , 1 1]) .

3 . Optimality Problem C 1

Problem C 1 is such that all the differential equations involved are multiplied by the positive function $I(t)$. This motivates the introduction of a new parameter s defined by

$$s(t) = \int_0^t I(t)dt.$$

Observing that $\frac{d}{dt} = I\frac{d}{ds}$ we obtain for the objective functional that

$$Y_1 = \int_0^{t^*} Idt = \int_0^{s^*} ds = s^*,$$

where $s^* = \int_0^{t^*} I(t)dt$. Therefore , problem C 1 write as the minimum time problem

$$\begin{aligned} S' &= -u_1 S, \\ I' &= u_1 S - u_2 \\ w' = (\beta - u_1) + (u_2 - \gamma), &\quad w(0) = 0, w(s^*) = C, \end{aligned} \tag{3.1}$$

where $\prime = \frac{d}{ds}$, and $s^* = \int_0^{t^*} I(t)dt$. Now we see that in order for the variable $w(s)$ achieve the value C in the smallest possible time s, it suffices that the derivative w' be the largest possible , therefore it suffices that $u_1(s) = \beta_m$ and $u_2(s) = \gamma M$.

4 . Optimality Problem C 2

Our main tool for the study of the optimality of system (2 . 5) will be the Pon - tryagin Maximum Principle (PMP) [1 , 1 4] . Let pS, pI and pw denote the adjoint variables to S, I, and w respectively . The Hamiltonian for problem $C2$ is

$$H = pS(-u_1 SI) + pI(u_1 SI - u_2 I) + pw[(\beta - u_1(t)) + (u_2(t) - \gamma)] - u_2(t)I.$$

That we write as

$$H = g + u_1\phi 1 + u_2\phi 2, \tag{4.1}$$

where

$$g \equiv pw(\beta - \gamma), \phi 1 \equiv SI(pI - pS) - pw, \phi 2 \equiv -I(pI + 1) + pw.$$

The adjoint variables satisfy Hamilton ' s equations

$$\dot{p}S = -\frac{\partial \mathcal{H}}{\partial S}, \quad \dot{p}I = -\frac{\partial \mathcal{H}}{\partial I}, \quad \dot{p}w = -\frac{\partial \mathcal{H}}{\partial w}, \tag{4.2}$$

that are given by

$$\begin{aligned} \dot{p}S &= u_1 I(ps - pI), \\ \dot{p}I &= u_1 S(ps - pI) + u_2(pI + 1), \\ \dot{p}w &= 0. \end{aligned} \tag{4.3}$$

By the PMP , the optimal controls $u_1(t), u_2(t)$ are the ones that maximize $\mathcal{H}$(we are ignoring abnormal controls see [1]) . PMP implies that at optimal traj ectories the following transversality conditions will hold [1 4]

$$pS(0) = pI(0) = 0 \quad \text{and} \quad pS(t^*) = pI(t^*) = 0. \tag{4.4}$$

This is implied by the boundary conditions to be satisfied by w. The derivatives of the functions $\phi 1$ and $\phi 2$ along the flow of hamiltonian dynamical system induced by (4 . 1) can be computed using (2 . 5) and (4 . 3) . We obtain

$$\dot{\phi}1 = u_2 IS(pS + 1), \tag{4.5}$$

$$\dot{\phi}2 = -u_1 IS(pS + 1). \tag{4.6}$$

From where it follows that

$$u_1\dot{\phi}1 + u_2\dot{\phi}2 = 0.$$

Remark 4 . 1 . The existence of the optimal control for problem C_2 is given by an application of Filipov ' s theorem [1 , 1 5] : We observe that the vector field X defined by (2 . 5) is bounded in M and complete (M is compact) . Also the controls are bounded and for each fixed allowed pair (u_1, u_2) the set

$$\bar{X}(u_1, u_2) = \{SI \begin{pmatrix} u_1 \\ -u_1 \\ 0 \end{pmatrix} + I \begin{pmatrix} 0 \\ u_2 \\ 0 \end{pmatrix}, \quad \text{for} S, I \in M\}$$

is convex , which implies that the set $X(u_1, u_2) = \{X \text{ for } S, I \in M\}$ is convex . To apply directly Filipov ' s theorem it remains to establish the compact support of the vector fields . But this is not necessary by the boundness and completeness of the vector fields (see discussion in [1] and [5]) .

4 . 1 . **Controlling the infection parameter .** In this subsection we will control only the infection parameter ; i . e . , we will assume $u_2(t) = \gamma$ for all $t \geq 0$. The pre - hamiltonian (4 . 1) is given by

$$H = \beta pw - \gamma I(pI + 1) + u_1(t)\phi 1. \tag{4.7}$$

We observe that the derivative of the switching function $\dot{\phi}1 = \gamma SI(pS + 1)$ is a continuous function and its number of zeros is determined only by the behavior of

$$pS \text{since} - \gamma IS \neq 0.$$

Lemma 4 . 2 . *If $u_2(t) = \gamma$ in th e control problem (2 . 5) then there is no open interval*

$$where \phi 1(t) = \dot{\phi}1(t) = 0.$$

Proof . Assume there exists an open interval $\mathcal{D}$, where $\phi 1(t) = \dot{\phi}1(t) = 0$ for $t \in \mathcal{D}$. The derivative of $\phi 1$ being zero implies that $pS = -1$ in $\mathcal{D}$ what implies that $\dot{p}S = 0$ and by the first equation of (4 . 3) we have that $pI = pS = -1$ in $\mathcal{D}$; but equations (4 . 3) imply that $pI = pS = -1$ for all future t($pI = pS = -1$ is an equilibrium point for the vector field (4 . 3)) what contradicts the transversality condition (4.4). □

Theorem 4 . 3 . *If $u_2(t) = \gamma$ in the control pro b lem (2 . 5) , th e optimal control $u^*1(t)$ has at most two switches .*

Proof . First we observe that when $\phi 1 = 0$ we have by (4 . 7) that

$$H - \beta pw = -\gamma I(pI + 1).$$

For latter use we multiply this equation by $-\frac{S}{\gamma}$ obtaining the equality

$$SI(pI+1) = -\frac{S}{\gamma}(H - \beta pw). \tag{4.8}$$

When $\phi 1 = 0$ we have that $SI(pI - pS) = pw$. Solving for pS and substituting back in $\dot{\phi}1$, we obtain

$$\dot{\phi}1 = \gamma SI(pS+1) = \gamma(SI(pI+1) - pw).$$

 Using (4 . 8) , we obtain that at the zeros of $\phi 1$,

$$\dot{\phi}1 = (\beta pw - H)S - pw^{\cdot} \tag{4.9}$$

From equation (4 . 9) we define the function

$$h = (\beta pw - H)S - pw.$$

By the first equation of (2 . 1) we see that S is a strictly monotone function . There - fore since pw and H are constant along the flow we have that h is a monotonic function . (4 . 9) shows that at the zeros of $\phi 1, \dot{\phi}1 = h$. Therefore , we have that
at the zeros of the C^1 function $\phi 1$, the values of its derivative $\dot{\phi}1$ is a monotonic function . Therefore $\dot{\phi}1$ can switch signs at most one time . What implies that $\phi 1$ can have at most two switches of sign (and at most three zeros). □
4 . 2 . **Controlling the removal parameter .** In this section we will assume that $u_1 = \beta$ for all times . The pre - hamiltonian is

$$H = -pw\gamma + \beta SI(pI - pS) + u_2\phi 2. \tag{4.10}$$

We observe that $\dot{\phi}2 = -\beta SI(pS + 1)$ is a continuous function . **Lemma 4 . 4 .** *If $u_1(t) = \gamma$ in th e control problem* (2 . 5) *then there is no s ingular*

$$optimalcontrolu_2(t).$$

The proof of the above lemma is similar to the proof of lemma (4 . 2) . Therefore ,
it is omitted . **Theorem 4 . 5 .** *If $u_1(t) = \beta$ in th e control pro b lem* (2 . 5) , *the optimal contro l $u_2(t)$*
has at most two switches . Proof . When $\phi 2 = 0$ we have that $pI - 1 = pw/I$. Since at the zeros of $\phi 2$ we have

$$H + \gamma pw = \beta SI(pI - pS)$$

it follows that

$$\dot{\phi}2 = H + pw(\gamma - \beta S). \tag{4.11}$$

The argument here is the same as the in proof of theorem (4 . 3) . The left hand side of (4 . 1 1) is a monotonic function . Therefore we have that at the zeros of the C^1 function $\phi 2, \dot{\phi}2$ can switch signs at most one time . Therefore $\phi 2$ can have at most two switches of sign (and at most three zeros). □
4 . 3 . **Controlling the infection and the removal parameters .** In this case we are working in a more complex case . We recall that $\dot{\phi}1 = u_2IS(pS+1)$ and $\dot{\phi}2 = -u_2 1IS(pS + 1)$. The functions $\dot{\phi}1$ and $\dot{\phi}2$ depend on the controls and are not
necessarily continuous (we are assuming that $u_1(t)$ and $u_2(t)$ are measurable lo cally bounded functions) . Therefore $\phi 1$ and $\phi 2$ are not C^1 functions and the previous reasoning does not apply in this case .
Theorem 4 . 6 . *Along the op timal s o lution the re is no time instant $\bar{t}$ for which*

$$\phi 1(^{\bar{t}}) = \phi 2(^{\bar{t}}) = 0.$$

Proof . At $\bar{t}$ we would have that $pS = pI = -1$ what contradicts the boundary conditions for w at $t = t^*$. $\square$

A corollary of this fact is that $H - g \neq 0$. As in lemma 4 . 2 , we can prove the following result .

 Theorem 4 . 7 . *There is no time inte rval for which* $\phi 1(t) = \dot{\phi} 1(t) = 0$ *and for which*

$$\phi 2(t) = \dot{\phi} 2(t) = 0.$$

Theorem 4 . 8 . *The two types of campaign , that is , the campaign for reducing* β *and the campaign for increasing* γ *are e ither time disjo int o r time nested .*

The theorem says , for example , that if you start a reducing infection rate cam - paign (RIRC) , when there is no campaign being made , then there are only two possibilities : either you start and finish a increasing removal rate campaign (IRRC) before you finish the RIRC of you wait until the RIRC is over to start the IRRC .

Proof . We recall that

$$H = g + u_1\phi 1 + u_2\phi 2.$$

Since g is constant and H is a first integral for the control system it follows that the two functions $f1 \equiv u_1\phi 1$ and $f2 \equiv u_2\phi 2$ add to a constant . The proof is a direct consequence of this fact . A campaign will start or end at a switch time , i . e . at a time where some of the functions $\phi 1$ or $\phi 2$ changes sign . Now let $\alpha \equiv H - g$. Therefore if $f1(t_1)$ is zero we have that $f2 = \alpha$ and vice - versa . Assume , by way of contradiction , that campaigns are neither disj oint neither nested . We have two cases to consider a) The number of total switches is two or b) The number of total switches is greater than two . (the case of only one switch satisfies the theorem) . If we are in case a) the only situation that does not satisfy the theorem is the one where each function has one switch and one of the campaigns (say campaign 2) starts when the other campaign (say campaign 1) is still on . In this case , since there is only one switch left , it follows that only one of the two will be turned off . As a net result we will have at least one campaign being made during all epidemic time what is ruled out by the main hypothesis of the paper : one can not make campaign for all times (see remark 2 . 2) . In case b) We have at least three zeros . Now assume , by way of contradiction , that there are two campaigns that are neither disj oint or nested . Then there is at least one switching time $\bar{t}$ for say $f2$ that is inside the $f1$ campaign interval $I = [t_1, t_2]$. Assume without lost of generality that $\bar{t}$ is a start and that there is no other switch of $f2$ in the interval $\bar{I} = [\bar{t}, t_2]$(intersection hypothesis) Now at t_1 we have $f1(t_1) = 0$ and $f2(t_1) = \alpha$. At t_2 we also have that $f1(t_2) = 0$ and $f2(t_2) = \alpha$. But this impossible , since $f2$ switches signs at $\bar{t}$ and does not switch signs in the interval $\bar{I}$. □

5 . Controlling an Epidemic

In this section we study an example numerically . We will be controlling only the infection parameter . We assume that the campaign cost is independent of the number of infected , i . e . We will be considering the problem $C2$. Since the optimal campaign has at most two switches it will consist of only one campaign with maximal effort . Therefore , to determine the optimal campaign , one must only to determine the time instant when it starts . We call it the optimal start . The strategy to determine the optimal control numerically is as follows : For a fixed campaign time C we fix the susceptible and infective initial values . A grid of N starting campaign times $t_i, i = 1, ..N$ is then specified . The equations for $S(t)$ and $I(t)$(the adjoints are not used) are then integrated N times , one for each campaign starting time t_i. The total number of infected T_i by the end of the epidemic outbreak is them computed . The optimal start is the t_i that results in the smaller of all T_i.

Our goal is to understand how the optimal start depends on the campaign total time C(we will present only the results for reducing β since the results for increasing γ are equal in nature) . The model case is a severe flu epidemic described in the 4 th March 1 978 issue of the British Medical Journal . The parameters for the epidemic were determined by a best fit numerical technique in [1 3] . The values for the influenza epidemic are $N = 763, S(0) = 762, I(0) = 1, \gamma = 2.18 \times 10^{-3}$ and $\beta = 0.44036$. Time is measured in days . We plot the epidemic dynamics in figure 1 . The maximum number of infected occurs at $t = 6.49$. This instant is called (according Bailey [2]) the central epoch .

FIGURE 1 . Epidemic dynamics : The number of infected I and the number of susceptibles S. Time is measured in days .

We used a Runge - Kutta Fehlberg 7 - 8 to integrate the system of equations with tolerance 10^{-8} and step size $h = 0.01$. We will take $\beta_m \equiv 1.8 \times 10^{-3}$ what gives a reduction of 50% of the infection rate . The results obtained are valid for all ranges of reduction studied . The optimal start can be determined numerically by a simple search procedure . We partition the time interval in intervals of length 0 . 1 . Then we do the campaign (reducing the infection parameter by 50%) during time C for all starting times . In figure 2 we show the number of infected at the end of the epidemic as a function of the st arting time . Each curve represents different campaign times .

In figure 3 we show the optimal starting t ime as a function of the campaign time . We observe that as the campaign t ime increases the starting time decreases until eventually becomes zero .

Figure 4 shows that the optimal campaigns always include the central epoch . In other words , limited cost campaigns are optimal around the central epoch for non - controlled epidemics . In the figure we show in the horizontal axis the campaign duration . The two solid curves represent the time when the campaign starts (lower) and the time when the campaign finishes (upper) . The dashed curve shows the central epoch . It is always inside the campaign duration even for very small t imes .

Conclusions . In this paper we studied optimal strategies for a limited cost educa - tional campaign during the outbreak of an epidemic . Optimality was measured by the minimality of the total number of infected at the end of the outbreak . Assum - ing that the effect of the campaign was to decrease (or increase) infection (removal)

FIGURE 2 . Number of total infected at the end of epidemics as a function of the campaign starting time . Different curves represent different campaigns times C.

FIGURE 3 . Optimal starting time for different values of campaign values C.

rate we were able to show , using the Pontryagin Maximum Principle , that the op - timal campaign must consist of only one maximum effort . Numerical simulations , concerning a particular epidemic , gave us additional information about the optimal

FIGURE 4 . Relative position of the central epoch (dashed line) with respect to the optimal campaign interval .

start , i . e . the time to st art this maximum effort , in order to minimize our objective functional . Calling $\tilde{t}$ the central epoch we summarize our results in the following : If the campaign cost is proportional to the number of infected than both campaigns , to decrease infection rate and to increase removal rate must be done with maximum intensity at the start of the epidemic . If the campaign cost is independent of the number of infected and only one scenario is chosen , then 1) only one maximum effort campaign should be made , 2) all campaigns should include $\tilde{t}$. If the goals of the campaign is both to decrease infection rate *and* to increase removal rate then campaign for different scenarios must be nested or disj oint . They should never start or end at the same time .

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