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FIXED POINT THEOREM AND ITS APPLICATION TO PERTURBED INTEGRAL EQUATIONS IN MODULAR FUNCTION SPACES

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Abstract . In this paper , we present a modular version of Krasnoselskii ' s fixed point theorem . Then this result is applied to the existence of solutions to perturbed integral equations in modular function spaces .

1 . Introduction

Using the same argument as in [1] , we present a modular version of Krasnoselskii ' s fixed point theorem , result that is well known in Banach spaces . The modular ρ considered here is convex , satisfies the Fatou property , and satisfies the Δ_2- condition . We are interested in the existence of a fixed point for the application $S : B \to B$; where B is a convex , closed , and bounded subset of $X_\rho; S = T + U$ with $T : B \to B$ that satisfies a contraction type hypothesis (see [1]) ; and $U : B \to B$ is $rho - hyphen$ completely continuous .

Since ρ satisfies the Δ_2- condition ,U being $\rho-$ completely continuous is equivalent
to the condition U, $\| \cdot \|_\rho$ – completely continuous , where $\| \cdot \|_\rho$ is the Luxemburg norm .
On the other hand if T is $\rho-$ contraction , then T is not necessarily $\| \cdot \| \rho^{-\text{contraction}}$ (see counterexample in [5 , page 945 , Ex . 2 . 1 5]) .

We apply our main theorem to the study of solutions to the perturbed integral equation

$$u(t) = \exp(-t)f0 + \int_0^t \exp(s-t)(T+h)u(s)ds \tag{1.1}$$

in the modular space $C^\varphi = C([0,b], L^\varphi)$, where L^φ is the Musielak - Orlicz space , $f0$ is a fixed element in L^φ. Some hypotheses on the operators T and h are stated below . Also , we present an example of this class of equations .

For more details about modular spaces , we refer the reader to the books edited by Musielak [9] and by Kozlowski [6] . Now recall some definitions .

Let X be an arbitrary vector space over $K(K = \mathbb{R}$ or $K = \mathbb{C})$.
(a) A functional $\rho : X \to [0, +\infty]$ is called modular if

$$\text{(i)} \quad \rho(x) = 0 \text{implies} x = 0.$$

(i i) $\rho(-x) = \rho(x)$ for all x in X in the case of X being real .$\rho(e^{it}x) = \rho(x)$ for any real t in the case of X being complex .

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(ii i) $\rho(\alpha x+\beta y)\leq\rho(x)+\rho(y)$ for $\alpha,\beta\geq 0$ and $\alpha+\beta=1$.

If in place of (i ii) there holds

(i ii ') $\rho(\alpha x+\beta y)\leq\alpha\rho(x)+\beta\rho(y)$ for $\alpha,\beta\geq 0$ and $\alpha+\beta=1$,

then the modular ρ is called convex . (b) If ρ is a modular in X, then the set $X_\rho=\{x\in X:\rho(\lambda x)\to 0$ as $\lambda\to 0\}$ is called a modular space . (c) (i) If ρ is a modular in X, then $\mid x\mid_\rho=\inf\{u>0,\rho(\frac{x}{u})\leq u\}$ is a $F-$ norm .

(ii) If ρ is a convex modular , then $\| x\|\rho = Inf\{u > 0,\rho(\frac{x}{u}) \leq 1\}$ is called the

Luxemburg norm .

Let X_ρ be a modular space . (a) A sequence $(x_n)_n\in\mathbb{N}$ in X_ρ is said to be (i)$\rho-$ convergent to x, denoted by $x_n\rho\to x$, if $\rho(x_n-x)\to 0$ as $n\to+\infty$. (ii)$rho-hyphen$ Cauchy if $\rho(x_n-x_m)\to 0$ as $n,m\to+\infty$. (b)X_ρ is $\rho-$ complete if any $\rho-$ Cauchy sequence is $\rho-$ convergent .

(c) A subset B of X_ρ is said to be $rho-hyphen$ closed if for any sequence $(x_n)_n\in\mathbb{N}\subset B$, such

that $x_n\rho\to x$, then $x\in B$. Here $_____^{\rho}_{B}$ denotes the closure of B in the sense of ρ.

We say that the subset A of X_ρ is $\rho-$ bounded if :

$\sup_{x,y\in A}\rho(x-y)<+\infty$, and let the $rho-hyphen$ diameter of A, denoted by $\delta_\rho(A)$, to be

$$\delta_\rho(A)=\sup_{x,y\in A}^{\rho(x}-y).$$

Recall also that ρ has the Fatou property if $\rho(x-y)\leq\lim\inf\rho(x_n-yn)$, whenever

$$x_n\rho\to x\text{and}yn\rho\to y.$$

We say that ρ satisfies the Δ_2- condition if : $\rho(2x_n)\to 0$ as $n\to+\infty$ whenever $\rho(x_n)\to 0$ as $n\to+\infty$, for any sequence $(x_n)_n\in\mathbb{N}$ in X_ρ.

2 . Main result

Theorem 2 . 1 . *Let ρ be a convex modular that satisfies the Δ_2- condition ,X_ρ be a*

$\rho-$ complete modular space and B be a convex ,$\rho-$ clos ed ,$\rho-$ bounded subset of X_ρ. As - sume that U and T are two applications from B into B such that U is $\rho-$ completely continuous and there exist real numbers $k>0$, and $c>\max(1,k)$ that satisfy $\rho(c(Tx-Ty))\leq k\rho(x-y)$ for any x,y in B. And $T(B)+U(B)\subset B$. Then the operator $S=T+U$ has a fixed point .

Remark 2 . 2 . Since an operator $rho-hyphen$ Lipschitz is not necessarily $\| . \| \rho^{-\text{Lipschitz}}$(see

counterexample in [5 , page 945 , Ex . 2 . 1 5]) , then the result above gives a modular version of Krasnoselskii ' s fixed point theorem .

We need the following lemma for proving Theorem 2 . 1 .

Lemma 2 . 3 . *Let ρ be a convex modular and X_ρ be a modular space . If a subset B*

of X_ρ is $\rho-$ bounded then B is $\| . \| \rho^{-bounded.}$

Proof . Suppose that B is not $\| . \| \rho^{-\text{bounded.}}$ So there exist sequences $(x_n)_n\in\mathbb{N}$ and $(yn)_{n\in}\mathbb{N}$ in B such that $\| x_n-yn\| \rho\to+\infty$ as $n\to+\infty$. Hence for any $A>1$ there exists $N\in\mathbb{N}$, such that if $n>N$, then $\| x_n-yn\| \rho>A$ i . e . $\| \frac{x_n-yn}{A}\|_\rho>1$ whenever $n>N$. This implies $\rho(\frac{x_n-yn}{A})\geq\| \frac{x_n-yn}{A}\|_\rho>1$ (see [9 , p . 8]) . Hence ,

$1<\rho(\frac{x_n-yn}{A})\leq\frac{1}{A}\rho(x_n-yn)$ whenever $n>N$. So $A<\rho(x_n-yn)$ for any $n>N$. This shows that B is not $rho-hyphen$ bounded . Hence , the lemma is established . □

Proof of Theorem 2 . 1 . Firstly , we show that the operator $I - T$ is a bij ection from B into $U(B)$(where I is the identity function) . Let x in B, and consider the following sequence defined by $yn+1 = Ty_n + Ux$, with $y0$ a fixed element in B. Then the
sequence $(yn)_{n\in}\mathbb{N}$ is Cauchy . Indeed ,

$$\begin{aligned}\rho(yn + m - yn) &= \rho(Ty_{m+n-}1 - Ty_{n-}1)\\ &= \rho(\frac{1}{c}(c(Ty_{n+m-}1 - Ty_{n-}1)))\\ &\leq \frac{k}{c}\rho(ym + n - 1 - yn - 1),\end{aligned}$$

by induction , we have

$$\rho(ym + n - yn) \leq (\frac{k}{c})^n \rho(ym - y0)$$

and by hypothesis , B is $\rho-$ bounded , then we have $\rho(ym - y0) \leq \delta_\rho(B) < \infty$ for any

$$m \in \mathbb{N}, \text{whichimplies}$$
$$\rho(ym + n - yn) \leq (\frac{k}{c})^n \delta_\rho(B),$$

and by hypothesis $c > \max(1, k)$ we have $(\frac{k}{c})^n \to 0$ as $n \to +\infty$. Therefore , $\rho(ym + n - yn) \to 0$ as $n, m \to +\infty$. Which implies that the sequence $(yn)_n \in \mathbb{N}$ is $rho - hyphen$
Cauchy . Since X_ρ is $rho - hyphen$ complete , B is closed and T is continuous then the sequence $(yn)_n \in \mathbb{N}$ is convergent to an element $y \in B$ and $y = Ty + Ux$. Indeed ,

$$\begin{aligned}\rho(\frac{y - Ty - U(x)}{2}) &= \rho(\frac{y - yn + yn - Ty - U(x)}{2})\\ &= \rho(\frac{y - yn + Ty_{n-}1 - Ty}{2})\\ &\leq \rho(y - yn) + \rho(Ty_{n-}1 - Ty),\end{aligned}$$

which implies that $y - Ty = U(x)$.

Then it follows that for any $x \in B$, there exists $y \in B$ such that $(I - T)y = Ux$. Therefore , we get that $(I - T)(B) \subset U(B)$(Indeed , if we suppose that there exists $y \in B$ such that $y - Tyelement - slashU(B)$ i . e . , for any $x \in B$, we have $y - Ty \neq U(x)$ which is absurd) , and $I - T$ is a surj ective operator from B into $U(B)$.

Let $y1, y2$ in B such that $(I - T)y1 = (I - T)y2$, then $y1 - y2 = Ty1 - Ty2$; therefore ,
$\rho(y1 - y2) \leq \frac{k}{c}\rho(y1 - y2)$, and since $c > \max(1, k)$ it follows that $\rho(y1 - y2) = 0$ and $y1 = y2$. Which shows that $I - T$ is inj ective operator . Therefore , $I - T$ is a bij ection operator from B into $U(B)$.

Secondly , we show that $(I - T)^{-1}$ is continuous . Let $(x_n)_n \in \mathbb{N} \subset U(B)$ be a convergent sequence to $x \in U(B)$, and consider the sequence defined by $z_n =$
$(I - T)^{-1}(x_n)$, then $(z_n)_n \in \mathbb{N}$ is $\rho-$ Cauchy . Indeed ,

$$\begin{aligned}z_{n+m} - z_n &= z_{m+n} - Tz_{m+n} + Tz_{m+n} - Tz_n + Tz_n - z_n\\ &= x_{m+n} + Tz_{m+n} - Tz_n - x_n\\ &= x_{m+n} - x_n + Tz_{m+n} - Tz_n;\end{aligned}$$

therefore , if we take α such that $\frac{1}{\alpha} + \frac{1}{c} = 1$, then

$$\begin{aligned}\rho(z_{m+n} - z_n) &= \rho(\frac{1}{c}(c(Tz_{m+n} - Tz_n)) + \frac{1}{\alpha}\alpha(x_{m+n} - x_n)) \\ &\leq \frac{k}{c}\rho(z_{m+n} - z_n) + \frac{1}{\alpha}\rho(\alpha(x_{m+n} - x_n)).\end{aligned}$$

 Then ,

$$\rho(z_{m+n} - z_n) \leq \frac{c}{c-k}\frac{1}{\alpha}\rho(\alpha(x_{m+n} - x_n)).$$

And since $\rho(x_{m+n} - x_n) \to 0$ as $m, n \to +\infty$, then by the Δ_2- condition $\rho(\alpha(x_{m+n} - x_n)) \to 0$ as $m, n \to +\infty$. Therefore , $\rho(z_{m+n} - z_n) \to 0$ as $m, n \to +\infty$, and by hypothesis X_ρ is $rho-hyphen$ complete , then the sequence $(z_n)_n \in \mathbb{N}$ is convergent to an element $z \in B$. On the other hand , $x_n = z_n - T(z_n)$ is convergent to $x = z - T(z)$. Indeed ,

$$\rho(\frac{z_n - T(z_n) - z + T(z)}{2}) \leq \rho(z_n - z) + \rho(T(z_n) - T(z)).$$

Since $\rho(z_n - z) \to 0$ as $n \to +\infty$ and T is continuous , $\rho(\frac{z_n - T(z_n) - z + T(z)}{2}) \to 0$ as $n \to +\infty$, and by Δ_2- condition we have $\rho(z_n - T(z_n) - (z - T(z)) \to 0$ as $n \to +\infty$. Therefore , $(I - T)^{-1}(x_n)$ converges to $(I - T)^{-1}(x)$, which implies that $(I - T)^{-1}$ is continuous .

Finally , we consider the function f defined by

$$f(x) = (I - T)^{-1}U(x).$$

Since U is $rho-hyphen$ completely continuous and $(I-T)^{-1}$ is hyphen $-$ rho continuous , it follows by
the Δ_2- condition that U is $\| \cdot \|_\rho -$ completely continuous and $(I-T)^{-1}$ is $\| . \| \rho^-$ continuous . Which implies that f is $\| . \| \rho^{-\text{completely}}$ continuous from B into B. By the Δ_2- condition , B is $\| . \|_\rho -$ closed . Then , using Lemma 2 . 3 and Schauder ' s fixed
point theorem , f has a fixed point . Let x_0 be such that $f(x_0) = x_0$, then we have $x_0 = f(x_0) = (I-T)^{-1}U(x_0)$ which implies that $x_0 = (T+U)(x_0)$. Therefore , S has a fixed point , which completes the proof . □

The next section presents an application of Theorem 2 . 1 . We study the existence of solutions in the modular space $C^\varphi = C([0, b], L^\varphi)$. For details about the spaces C^φ and L^φ, we refer the reader to [1] and to books edited by Musielak [9] and Kozlowski [6] .

3 . Perturbed integral equations

In this section , we study the existence of solutions to perturbed integral equations on the Musielak - Orlicz space L^φ. For this , we begin by setting the functional framework of this integral equation .

F − u nctional framework . Let L^φ be the Musielak - Orlicz space . Then both the modular ρ and its associated F - norm satisfy the Fatou property . Hence forth , we assume that ρ is convex and satisfies the Δ_2- condition (the $F-$ norm becomes the Luxemburg norm [4]) . Therefore , we have

$$\| x_n - x \| \rho \to 0 \iff \rho(x_n - x) \to 0$$

as $n \to +\infty$ on L^φ. This implies that the topologies generated by $\| . \|_\rho$ and ρ are equivalent . Note that , under such conditions on $\rho, (L^\varphi(\Omega), \| . \|_\rho)$ is a Banach space ,

$$\text{where}\Omega = [0, b].$$

We denote by $C^\varphi = C([0, b], L^\varphi)$ the space of all $rho-hyphen$ continuous functions from $[0, b]$ to L^φ, endowed with the modular ρa defined by $\rho a(u) = \sup_{t\in[0,b]} \exp(-@)\rho(u(t))$,
where $a \geq 0$. On the space C^φ one can consider the three topologies associated with the modular ρa (see [9] and [2]) , the Luxemburg norm $\| . \|_{\rho a}$, and the norm

$$| \, . \, |_{\infty} \text{ definedby } | \, u \, |_{\infty} = \sup_{t \in [0,b]} \| \, u(t) \, \|_{\rho} \, .$$

We note that the three topologies above are equivalent in the following sense $\rho a(x_n - x) \to 0 \Leftrightarrow \quad \| x_n - x \|_{\rho a} \to 0 \Leftrightarrow | x_n - x | \infty \to 0$ as $n \to +\infty$. Indeed , let $(x_n)_n \in \mathbb{N}$ be a sequence in C^{φ} such that $| x_n - x |_{\infty} \to 0$ as $n \to +\infty$ and with $x \in C^{\varphi}$, hence for all $0 < \epsilon < 1$ there exists $N \in \mathbb{N}$ such that for any $n > N$ we have

$$\sup_{t \in [0,b]} \quad \| x_n(t) - x(t) \|_{\rho} \leq \epsilon < 1.$$

On the other hand , $\| x_n(t) - x(t) \| \rho \leq \epsilon < 1$ for all $t \in [0, b]$ implies $\rho(x_n(t) - x(t)) \leq \epsilon < 1$ for all $t \in [0, b]$. Then

$$\sup_{t \in [0,b]} \quad \exp(-@)\rho(x_n(t) - x(t)) \leq \epsilon$$

for all $n \geq N$. This implies $\rho a(x_n - x) \to 0$ as $n \to +\infty$. By the Δ_2- condition we

$$\| x_n - x \|_{\rho a} \to 0 \text{as} n \to +\infty. \qquad \text{have}$$

Conversely , by letting $u > 0$ be such that $\sup_{t \in [0,b]} \exp\ (-@)\rho(\frac{x_n(t) - x(t)}{u}) \leq 1$, we have

$$e^{-ab}\rho(\frac{x_n(t) - x(t)}{u}) \leq e^{-@}\rho(\frac{x_n(t) - x(t)}{u}) \leq 1$$

for all $t \in [0, b]$. This implies

$$e^{-ab}\rho(\frac{x_n(t) - x(t)}{u}) \leq \sup_{\in_t [0,b]} \exp(-@)\rho(\frac{x_n(t) - x(t)}{u}) \leq 1.$$

Therefore ,

$$A := \{u > 0; \sup_{\in_t [0,b]} \exp(-@)\rho(\frac{x_n(t) - x(t)}{u}) \leq 1\}$$
$$\subset B := e^{-ab}\{u > 0; \rho(\frac{x_n(t) - x(t)}{u}) \leq 1\}.$$

Hence , $\inf\ (A) \geq \inf\ (B)$, which implies

$$\| x_n - x \|_{\rho a} \geq e^{-ab} \| x_n(t) - x(t) \|_{\rho}$$

for all $t \in [0, b]$. Hence ,

$$e^{ab} \| x_n - x \| \rho a \geq \quad \sup_{t \in [0,b]} \quad \| x_n(t) - x(t) \|_{\rho} = | x_n - x |_{\infty} .$$

Therefore , $| x_n - x |_{\infty} \to 0$ as $n \to +\infty$ is equivalent to $\| x_n - x \|_{\rho a} \to 0$ as $n \to +\infty$.

To study the integral equation (1 . 1) . we set the following hypotheses :

(H 1) Let B be a convex , $rho - hyphen$ closed , hyphen – rho bounded subset of L^{φ}, and $0 \in B$.

(H 2) Let $T : B \to B$ be an application for which there exists a real number $k > 0$ such that $\rho(Tx - Ty) \leq k\rho(x - y)$ for all $x, y \in B$. Also let $h : B \to B$ be an application $\rho-$ completely continuous such that $T(B) + h(B) \subseteq B$.

(H 3) Let $f0$ be a fixed element of B.

Theorem 3 . 1 . *Under thes e hypotheses and for any* $b > 0$, *the integral equation* (1 . 1) *has a s o lution* $u \in C^{\varphi} = C([0, b], L^{\varphi})$.

When we restrict our attention to the Banach space $(L^{\varphi}, \| . \|_{\rho})$, Equation (1 . 1) can be written as

$$u'(t) + (I - (T + h))u(t) = 0.$$

 When $h \equiv 0$, Equation (1 . 1) becomes

$$u(t) = \exp(-t)f0 + \int_0^t \exp(s-t)Tu(s)ds.$$

The equation above has been studied in [1] and [3] . The proof of Theorem 3 . 1 is based on Lemma 2 . 3 and the next lemma .

Lemma 3 . 2 . *If a family $M \subset C^{\varphi}$ is equicontinuous in th e s ense of $\| \cdot \|_{\rho}$, then M is equicontinuous in the s ense of ρ.*

Proof . Recall that if $\| x \|_{\rho} < 1$, then $\rho(x) \leq \| x \|_{\rho}$ (see [9 , p . 2]) . Let $0 < \epsilon < 1$, there
exists $\delta > 0$ such that if $| t - _____ t| < \delta$ then $\| f(t) - f(_____ t) \|_{\rho} \leq \epsilon < 1$ for all $f \in M$.
Hence , $\rho(f(t) - f(_____ t)) \leq \| f(t) - f(_____ t) \| \rho \leq \epsilon$ for any $f \in M$ whenever $| t - _____ t| < \delta$. This implies that M is $rho - hyphen$ equicontinuous and the proof is complete . □

Proof of Theorem 3 . 1 . Let $a > 0$ and ρa be a modular in $D = C([0,b], B)$ defined by $\rho a(u) = \sup_{t \in [0,b]} \exp(-@)\rho(u(t))$ for $u \in D$(see [1]) .

By [1 , Prop . 2.1(3)], D is convex , $\rho a^{-\text{closed}}$ and since B is $\rho-$ bounded , then D

$$\text{is} \rho a^{-\text{bounded}}.$$

Claim : D is invariant under the operator S given by

$$Su(t) = \exp(-t)f0 + \int_0^t \exp(s-t)(T+h)u(s)ds.$$

First , we prove that Su is continuous from $[0,b]$ into $(L^{\varphi}, \| \cdot \| \rho)$. Let $t_n, t_0 \in [0,b]$ such that $t_n \to t_0$ as $n \to +\infty$. Since T and h are $rho - hyphen$ continuous , then $(T+h)u$ is $rho - hyphen$ continuous at t_0. Indeed ,

$$\begin{aligned} &\rho((T+h)u(t_n) - (T+h)u(t_0)) \\ \leq \frac{1}{2}\rho(2(Tu(t_n) - Tu(t_0)) &+ \frac{1}{2}\rho(2(hu(t_n) - hu(t_0))). \end{aligned}$$

By the Δ_2- condition , we have $\rho((T+h)u(t_n) - (T+h)u(t_0)) \to 0$ as $n \to +\infty$. Again by Δ_2- condition , $(T+h)u$ is $\| \cdot \|_{\rho} -$ continuous at t_0. Hence Su is $\| \cdot \| \rho^{-\text{continuous}}$ at

$$t_0.$$

Next , we prove that $Su(t) \in B$, for any $t \in [0,b]$. It is well known that in Banach

$$\begin{aligned} &\text{space}(L^{\varphi}, \| \cdot \|_{\rho}), \\ &\int_0^t \exp(s-t)(T+h)u(s)ds \\ \in (\int_0^t \exp(s-t)ds)_____ &\text{co}^{\|}. \|_{\rho\{(T} +h)u(s), \quad 0 \leq s \leq t\}, \end{aligned}$$

where $_____ \text{co}^{\|}. \|_{\rho}$ denotes the closure of the convex hull in the sense of $\| \cdot \| \rho$. Since $(T+h)(B) \subseteq B$, $\int_0^t \exp(s-t)(T+h)u(s)ds \in (1 - \exp(-t))_____ \text{co}^{\|}. \|_{\rho(B)}$.

But B is convex and $rho - hyphen$ closed . Thus $\overline{\text{co}}^{\|}. \|_{\rho} (B) = \overline{B}^{\|} . \| \rho \subset \overline{B}^{\rho} = B$. Therefore , $Su(t) \in$
$\exp(-t)B + (1- \exp(-t))B \subseteq B$ for all $t \in [0, b]$. Hence , D is invariant by S.

Now consider the operators : $T_1 u(t) = \exp(-t) f0 + \int_0^t \exp(s-t) Tu(s) ds$ and $h_1 u(t) = \int_0^t \exp(s-t) hu(s) ds$. Observe that $S = T_1 + h_1$. Next , we show that T_1 and h_1 satisfy the hypotheses of Theorem 2 . 1 .

(1) We note that , by the same argument in the proof of fixed point theorem (see [1]) , we show that D is invariant under h_1 and T_1 and there exists $c > \max\,(1, k_0)$ such that

$$\rho a(c(T_1 u - T_1 v)) \leq k_0 \rho a(u - v), \quad \forall u, v \in D,$$

where $1 < c \leq \frac{e^b}{e^b - 1}$, $k_0 = c\frac{k}{1+a}$ and $a \geq k$. The same techniques used in the proof of $S(D) \subset D$ are used to establish $T_1(D) + h_1(D) \subset D$: By taking the hypothesis $T(B) + h(B) \subset B$, which gives $T_1 u(t) + h_1 v(t) \in \exp\,(-t)B + (1 - \exp\,(-t))B \subset B$ for any $t \in [0, b]$ and $u, v \in D$. (2) Claim : h_1 is ρa^- completely continuous . Let $M \subset D$, then $h_1(M)$ is equicontin - uous in the sense of $\| \,.\, \|_\rho$. Indeed , let $u \in M$, we have

$$h_1 u(t) - h_1 u(\overline{\qquad} t)$$

$$= \int_0^t \exp(s - t) h u(s) ds - \int_0^t \exp(s - \overline{\qquad} t) h u(s) ds$$

$$= e^{-t} \int_0^t e^s h u(s) ds - e^{\overline{\qquad} - t} \int_0^t e^s h u(s) ds$$

$$= e^{-t} \int_0^t e^s h u(s) ds - e^{\overline{\qquad} - t} \int_0^t e^s h u(s) ds + e^{\overline{\qquad} - t} \int_0^t e^s h u(s) ds - e^{\overline{\qquad} - t} \int_0^t e^s h u(s) ds$$

$$= (e^{-t} - e^{\overline{\qquad} - t}) \int_0^t e^s h u(s) ds + e^{\overline{\qquad} - t} \overline{\qquad}_{\int^t}^{t} e^s h u(s) ds.$$

Hence ,

$$\| \, h_1 u(t) - h_1 u(\overline{\qquad} t) \, \| \, \rho \leq | \, e^{-t} - e^{\overline{\qquad} - t} \, | \, b e^b \delta_{\|} . \, \|_\rho^{(B)} + \delta_{\|} . \, \|_\rho^{(B)} | \, \overline{\qquad}_{\int^t}^{t} e^s ds \, |$$

$$\leq | \, e^{-t} - e^{\overline{\qquad} - t} \, | \, b e^b \delta \, \| \, . \, \| \, \rho^{(B)} + \delta \, \| \, . \, \| \, \rho^{(B)|e^t} - e^{-t} \, |$$

On the other hand , the functions $t \mapsto e^{-t}$ and $t \mapsto e^t$ are uniformly continuous on the compact $[0, b]$. Hence for $\epsilon > 0$, there exists $\eta 1 > 0$ such that if $| \, t - \overline{\qquad} t | < \eta 1$

then $| \, e^{-t} - e^{-t} \, | \leq \frac{\epsilon}{2 b e^b \delta \| . \| \rho^{(B)}}$, and there exists $\eta 2 > 0$ such that if $| \, t - \overline{\qquad} t | < \eta 2$ then

$$| \, e^t - e^t \, | \leq \frac{\epsilon}{2 \delta_{\|} . \, \|_\rho^{(B)}}.$$

Hence , there exists $\eta = \min\,(\eta 1, \eta 2)$ such that if $| \, t - \overline{\qquad} t | < \eta$ then $\| \, h_1 u(t) - h_1 u(\overline{\qquad} t) \, \|_\rho \leq \epsilon$ for any $u \in M$. Therefore , $h_1(M)$ is equicontinuous in the sense of $\| \,.\, \|_\rho$, and by Lemma 3.2, $h_1(M)$ is $rho - hyphen$ equicontinuous . Otherwise ,

$$h_1 u(t) = \int_0^t \exp(s - t) h u(s) ds \in (1 - \exp(-t)) \overline{\qquad} \mathrm{co}^{\|} . \, \|_{\rho\{hu(s),} \, 0 \leq s \leq t\}$$

$$\subset (1 - \exp(-t)) \overline{\qquad} \mathrm{co}^{\|} . \, \|_{\rho(h(B)).}$$

Hence $h_1(M(t)) \subset (1- \exp(-t))\overline{\text{co}}^{\|}. \| \rho(h(B))$ for all $t \in [0,b]$. But $h(B)$ is $rho-hyphen$ compact and by Δ_2- condition $h(B)$ is $\| . \| \rho$ compact , which implies that -co $\| . \| \rho(h(B))$ is com -

pact . Therefore , $h_1(M(t))$ is $\| . \| \rho$ compact for all $t \in [0,b]$, and by Ascoli ' s theorem $\overline{}^{|\cdot|_\infty}_{h_1(M)}$ is compact . Hence , by the equivalence of three topologies considered

in functional framework , $h_1(M)$ is ρa^- compact . Using the standard techniques [1 0 , proof of the Theorem 3 page 1 3] , we show that h_1 is $\| . \|_{\rho a} -$ continuous then h_1 is ρa^- continuous . Hence , h_1 is ρa^- completely continuous . It then follows from Theorem 2 . 1 that S has a fixed point which is a solution of the equation (1.1). $\square$

 3 . 1 . **Example of equation** (1 . 1) **.** In this example , we study the existence of a solution of the integral equation

$$u(t) = \exp + \int_0^t \exp(s-t) f0 + \int_0^t \exp(s-t)\left(\int_0^b \exp(-\xi) g2(s,\xi, u(\xi))d\xi\right)ds \qquad (3.1)$$

under the hypotheses st ated below . Let X_ρ be a finite dimensional vector subspace of L^φ, and ρ be a convex modular on L^φ, satisfying the Δ_2- condition . Let B be a convex , $rho-hyphen$ closed , hyphen $-$ rho bounded subset of X_ρ and $0 \in B$. Let $b > 0$ very small ,
$g1$, $g2$ be functions from $[0,b] \times [0,b] \times B$ into $B, \gamma : [0,b] \times [0,b] \times [0,b] \to \mathbb{R}^+$ and $\beta : [0,b] \times [0,b] \to \mathbb{R}^+$ be measurable functions such that :

(H 1 ') (i) $gi(t,.,x) : s \mapsto gi(t,s,x)$ where $i \in \{1,2\}$ are measurable functions on $[0,b]$ for each $x \in B$ and for almost all $t \in [0,b]$.

(i i) $gi(t,s,.) : x \mapsto gi(t,s,x)$, where $i \in \{1,2\}$, are $\rho-$ continuous on B for almostall$t, s \in [0,b]$.

(H 2 ') For any $i \in \{1,2\}, \rho(gi(t,s,x) - gi(\tau,s,x)) \le \gamma(t,\tau,s)$ for all (t,s,x) and (τ,s,x) in $[0,b] \times [0,b] \times B$ and $\lim_{t\to\tau} \int_0^b \gamma(t,\tau,s)ds = 0$ uniformly for $\tau \in [0,b]$.

(H 3 ') $\rho(g2(t,s,x) - g2(t,s,y)) \le \rho(x-y)$ for all (t,s,x) and (t,s,y) in $[0,b] \times [0,b] \times B$.

These hypotheses have been used by Martin [8] .

Now , assume that $f0$ is a fixed element of B, and that h, T are the Uryshon operators on $C([0,b],B)$ defined by :

$$(t) = \int_0^b \exp(-s) g1(t,s,u(s))ds,$$
$$[Tu](t) = \int_0^b \exp(-s) g2(t,s,u(s))ds,$$

for $t \in [0,b]$ and $u \in (C([0,b],B), \rho a)$ with $(a > 0)$.

Proposition 3 . 3 . *(1) Under the hypotheses (H 1 ') – (H 3 ') , the operator T is ρa^- Lipschitz from $C([0,b],B)$ into $C([0,b],B)$. (2) Under the hypotheses (H 1 ') – (H 2 ') , the operator h is ρa^- completely continuous*

$$fromC([0,b],B)intoC([0,b],B).$$

Proof . (1) We show that $C([0,b],B)$ is invariant by T. (i) Note that $(X_\rho, \| . \|_\rho)$ is a Banach space with finite dimension . By hypothesis (H 1 ') (i), $g2(t,.,u(.)) : s \mapsto g2(t,s,u(s))$ is measurable , and since B is $\rho-$ bounded , $g2(t,.,u(.)) : s \mapsto g2(t,s,u(s))$ is an integrable function from $[0,b]$ into $(X_\rho, \| . \|_\rho)$. Then for $u \in C([0,b],B)$, we have

$$(t) \in \int_0^b \exp(-s)ds \overline{\text{co}}^{\|}. \|_{\rho\{g2(t,\ s,u(s)), s \in [0,b]\}}$$
$$\subset (1 - \exp(-b)) \overline{\text{co}}^{\|}. \| \rho(B).$$

But B is convex and $\rho-$ closed thus $\overline{\text{co}}^{\|}. \|_{\rho(B)} = B \| . \| \rho \subset \overline{B}^{\rho} = B$. Since $0 \in B$ and $0 < 1- \exp(-b) < 1$, we have $[Tu](t) \in B$ for all $t \in [0,b]$.

 (ii) Let $u \in C([0,b],B)$ then Tu is continuous from $[0,b]$ into (B,ρ). Indeed , let
$(t_n)_n \in \mathbb{N}$ be a sequence and r in $[0,b]$ such that $t_n \to r$ as $n \to +\infty$ and we have

$$[Tu](t_n) - [Tu](r) = \int_0^b \exp(-s)(g2(t_n,s,u(s)) - g2(r,s,u(s)))ds.$$

Let $K = \{s_0, s_1, ..., s_m\}$ be a subdivision of $[0,b]$. Then $\sum_{i=0}^{m-1}(s_{i+1}-s_i)e^{-s_i}x(s_i)$ is $\| \, . \, \|_\rho -$ convergent . Thus $rho-hyphen$ converges to $\int_0^b \exp\,(-s)x(s)ds$ in X_ρ when $\mid K \mid =$

$$\sup\{\mid s_{i+1}-s_i \mid, i=0,...,m-1\} \to 0 \text{as} m \to +\infty. \text{Since}$$

$$\int_0^b \exp(-s)(g2(t,s,u(s)) - g2(\tau,s,u(s)))ds$$

$$= \lim \sum_{i=0}^{m-1}(s_{i+1}-s_i)\exp(-s_i)(g2(t,s_i,u(s_i)) - g2(\tau,s_i,u(s_i))),$$

and $\sum_{i=0}^{m-1}(s_{i+1}-s_i)\,\exp\,(-s_i) \leq \int_0^b \exp\,(-s)ds = 1 - \exp\,(-b) < 1$, then by the Fatou property we have :

$$\rho([Tu](t_n) - [Tu](r))$$

$$\leq \liminf \sum_{i=0}^{m-1}(s_{i+1}-s_i)\,\exp\,(-s_i)\rho(g2(t_n,s_i,u(s_i)) - g2(r,s_i,u(s_i)))$$

$$\leq \liminf \sum_{i=0}^{m-1}(s_{i+1}-s_i)\,\exp\,(-s_i)\gamma(t_n,r,s_i)$$

$$\leq \int_0^b \exp(-s)\gamma(t_n,r,s)ds$$

$$\leq \int_0^b \gamma(t_n,r,s)ds$$

Hence by hypothesis (H 2 ')Tu is $\rho-$ continuous at r. (2) We show that T is ρa^- Lipschitz . Let u, v in $C([0,b],B)$, we have .

$$\rho([Tu](t) - [Tv](t))$$

$$\leq \liminf \sum_{i=0}^{m-1}(s_{i+1}-s_i)(\,\exp\,(-s_i))\rho(g2(t,s_i,u(s_i)) - g2(t,s_i,v(s_i)))$$

$$\leq \liminf \sum_{i=0}^{m-1}(s_{i+1}-s_i)\,\exp\,(-s_i)\rho(u(s_i) - v(s_i))$$

$$\leq \liminf \sum_{i=0}^{m-1}(s_{i+1}-s_i)\,\exp\,(as_i)\rho a(u-v).$$

$$i = 0$$

Therefore ,

$$\exp\ (-@)\rho([Tu](t) - [Tv](t)) \leq \exp\ (-@)(\int_0^b \exp\ (as)ds) \quad \rho a(u - v)$$

$$\leq \frac{e^{ba} - 1}{a} \rho a(u - v).$$

 Hence ,

$$\rho a([Tu]-[Tv]) \leq \frac{e^{ba}-1}{a}\rho a(u-v).$$

(3) Using the same argument of (1) , we show that $C([0,b],B)$ is invariant by h.
(4) Now , we claim that $h(C([0,b],B))$ is equicontinuous in the sense of ρ, and ρa^{-} compact . We have :

$$[hu](t)-[hu](\tau) = \int_0^b \exp(-s)(g1(t,s,u(s))-g1(\tau,s,u(s)))ds.$$

We easily obtain

$$\rho([hu](t)-[hu](\tau)) \leq \int_0^b \gamma(t,\tau,s)ds,$$

by using again the same argument in (1) . And since , $\lim_{t\to\tau}\int_0^b \gamma(t,\tau,s)ds = 0$ uniformly for $\tau \in [0,b]$, then $h(C([0,b],B))$ is $rho-hyphen$ equicontinuous . On the other hand , since B is $rho-hyphen$ bounded then , $h(C([0,b],B))$ is $\rho a^{-\text{bounded}}$ subset of $C([0,b],B)$. Indeed , let u,v in $C([0,b],B)$, we have

$$[hu](t)-[hv](t) = \int_0^b \exp(-s)(g1(t,s,u(s))-g1(t,s,v(s)))ds.$$

Again from (1) , we obtain

$$\begin{aligned}
&\rho([hu](t)-[hv](t)) \\
&\leq \liminf \sum_{i=0}^{m-1}(s_{i+1}-s_i)\exp(-s_i)\rho(g1(t,s_i,u(s_i))-g1(t,s_i,v(s_i))) \\
&\leq \liminf \sum_{i=0}^{m-1}(s_{i+1}-s_i)\exp(-s_i)\delta_\rho(B) \\
&\leq \Big(\int_0^b \exp(-s)ds\Big)\ \delta_\rho(B).
\end{aligned}$$

Hence ,

$$\rho a([hu]-[hv]) \leq (1-e^{-b})\delta_\rho(B) < \infty$$

Therefore , $h(C([0,b],B))$ is a $\rho a^{-\text{bounded}}$ subset of $C([0,b],B)$ and by Lemma 2 . 3 , it is $\| \cdot \|_{\rho a}$ − bounded subset of $C([0,b],B)$. On the other hand , since $(X_\rho, \| \cdot \| \rho^{)}$ is a Ba -

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nach space with finite dimensional , then for each $t \in [0,b]$ we have $h(C([0,b],B))(t)$

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is $\| \cdot \| \rho^{-\text{compact.}}$ Thus , by Ascoli ' s theorem we have $h(C([0,b],B))$ is $\| \cdot \| \rho a^{-\text{compact,}}$

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then $h(C([0,b],B))$ is ρa^- compact . Hence for any $M \subset C([0,b],B)$, we have $h(M)$ is ρa^- compact . Using the standard techniques [1 0 , Theorem 3 page 1 3] , we that h is $\| . \| \rho a^{-\text{continuous}}$ then h is $\rho a_{.}^{-\text{continuous}}$ So h is ρa^- completely continuous . $\square$

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References

[1] Taleb A . Ait , E . Hanebaly ; *A fixed point theorem and its application to integral equations in modular function spaces* , Proc . Amer . Math . Soc . 1 28 , no 2 , 41 9 - 426 (2000) .

[2] A . Hajji ; *Forme Equivalente à la Condition* Δ_2 *et Certains r é sultats de s é parations dans les Espaces Modulaires* , Math . FA / 509482 v 1 .

[3] M . A . Khamsi ; *Nonlinear Semigroups in Modular Function Spaces . Th è se d ′ é tat* , D é partement de Math é matique , R − a bat (1 994) .

[4] M . A . Khamsi ; *Uniform noncompact convexity , fixed point property in modular spaces* , Math . Jap . 40 No 3 (1 994) , 439 - 450 .

[5] M . A . Khamsi , W . M . Kozlowski , S . Reich ; *fixed point theory in modular function spaces* ,

Nonlinear Analysis , theory , methods and applications , Vol $.14, N^0 11(1990).935 - 953.$

[6] W . M . Kozlowski ; *Modular Function spaces* , Dekker New - york (1 988)

[7] M . A . Krasnoselskii ; *Topological methods in the theory of non linear integral equations* , Perg - amon press , 1 954 .

[8] R . H . Martin ; *Non linear operator and differential equations in Banach spaces* , John - Wiley .

New - york (1 976) .

[9] J . Musielak ; *Orlicz spaces and modular spaces* , L . N . vol . 1034 , S . P . (1 983) .

[10] M . Roseau ; *Equations differentielles* , Masson , Paris , 1 976 .

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