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EXISTENCE OF SOLUTIONS FOR A SECOND ORDER ABSTRACT FUNCTIONAL DIFFERENTIAL EQUATION WITH STATE - DEPENDENT DELAY

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Abstract . In this paper we study the existence of mild solutions for abstract partial functional differential equation with state - dependent delay .

1 . Introduction

In this note we study the existence of mild solutions for a second order abstract Cauchy problem with state dependent delay described in the form

$$x''(t) = Ax(t) + f(t, x_{\rho(t,x_t)}), \quad t \in I = [0,a], x_0 = \varphi \in \mathcal{B}, \tag{1.1}$$

$$x'(0) = \zeta 0 \in X, \tag{1.2}$$

where A is the infinitesimal generator of a strongly continuous cosine function of bounded linear operator $(C(t))t \in \mathbb{R}$ defined on a Banach space $(X, \| \cdot \|)$; the function $x_s : (-\infty, 0] \to X, \quad x_s(\theta) = x(s+\theta)$, belongs to some abstract phase space $\mathcal{B}$ described axiomatically and $f : I \times \mathcal{B} \to X, \rho : I \times \mathcal{B} \to (-\infty, a]$ are appropriate functions .

Functional differential equations with st ate - dependent delay appear frequently in applications as model of equations and for this reason the study of this type of equations has received great attention in the last years . The literature devoted to this subject is concerned fundamentally with first order functional differential equations for which the state belong to some finite dimensional space , see among another works , [1 , 2 , 3 , 4 , 5 , 8 , 1 0 , 1 1 , 1 2 , 1 3 , 1 9 , 24 , 23] . The problem of the existence of solutions for first order partial functional differential equations with state - dependent delay have been treated in the literature recently in [1 4 , 1 5 , 1 6] .
To the best of our knowledge , the existence of solutions for second order abstract partial functional differential equations with state - dependent delay is an untreated topic in the literature and this fact is the main motivation of the present work .

2 . Preliminaries

In this section , we review some basic concepts , notations and properties needed to establish our results . Throughout this paper , A is the infinitesimal generator of

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a strongly continuous cosine family $(C(t))t \in \mathbb{R}$ of bounded linear operators on the Banach space $(X, \| \cdot \|)$. We denote by $(S(t))t \in \mathbb{R}$ the associated sine function which is defined by $S(t)x = \int_0^t C(s)xds$, for $x \in X$, and $t \in \mathbb{R}$. In the sequel , N and $\tilde{N}$ are positive constants such that $\| C(t) \| \leq N$ and $\| S(t) \| \leq \tilde{N}$, for every $t \in I$.

In this paper , $[D(A)]$ represents the domain of A endowed with the graph norm given by $\| x \| A = \| x \| + \| Ax \|, x \in D(A)$, while E stands for the space formed by the vectors $x \in X$ for which $C(\cdot)x$ is of class C^1 on $\mathbb{R}$. We know from Kisi ń sky [1 8] , that E endowed with the norm

$$\| x \| E = \| x \| + \sup_{0 \leq t \leq 1} \| AS(t)x \|, \quad x \in E, \tag{2.1}$$

is a Banach space . The operator - valued function

$$\mathcal{H}(t) = \begin{bmatrix} C(t) & S(t) \\ AS(t) & C(t) \end{bmatrix}$$

is a strongly continuous group of bounded linear operators on the space $E \times X$ generated by the operator $\mathcal{A} = \begin{bmatrix} 0 & I \\ A & 0 \end{bmatrix}$ defined on $D(A) \times E$. It follows from this that $AS(t) : E \to X$ is a bounded linear operator and that $AS(t)x \to 0$, as $t \to 0$, for each $x \in E$. Furthermore , if $x : [0, \infty) \to X$ is lo cally integrable , then $y(t) = \int_0^t S(t-s)x(s)ds$ defines an $E-$ valued continuous function . This assertion is a consequence of the fact that

$$\int_0^t \mathcal{H}(t-s) \begin{bmatrix} 0 \\ x(s) \end{bmatrix} ds = \quad [\textstyle\int_0^t S(t-s)x(s)ds \quad \displaystyle\int_0^t C(t-s)x(s)ds]$$

defines an $E \times X-$ valued continuous function . In addition , it follows from the definition of the norm in E that a function $u : I \to E$ is continuous if , and only if , is continuous with respect to the norm in X and the set of functions $\{AS(t)u : t \in [\ 0\ ,\ 1\ ]\ \}$ is an equicontinuous subset of $C(I, X)$.

The existence of solutions for the second - order abstract Cauchy problem

$$x''(t) = Ax(t) + h(t), \quad t \in I, \tag{2.2}$$

$$x(0) = w, \quad x'(0) = z, \tag{2.3}$$

where $h : I \to X$ is an integrable function , is studied in [22] . Similarly , the existence of solutions of semi - linear second - order abstract Cauchy problems has been treated in [2 1] . We only mention here that the function $x(\cdot)$ given by

$$x(t) = C(t)w + S(t)z + \int_0^t S(t-s)h(s)ds, \quad t \in I, \tag{2.4}$$

is called a mild solution of (2 . 2) - (2 . 3) , and that when $w \in E$ the function $x(\cdot)$ is of class C^1 on I and

$$x'(t) = AS(t)w + C(t)z + \int_0^t C(t-s)h(s)ds, \quad t \in I. \tag{2.5}$$

For additional details on the cosine function theory , we refer the reader to [6 , 22 , 2 1] .

In this work we will employ an axiomatic definition for the phase space $\mathcal{B}$ which is similar at those introduced in [1 7] . Specifically , $\mathcal{B}$ will be a linear space of functions mapping $(-\infty, 0]$ into X endowed with a seminorm $\| \cdot \| \mathcal{B}$ and satisfying the following

asumptions :

(A 1) If $x : (-\infty, b] \to X, b > 0$, is continuous on $[0, b]$ and $x_0 \in \mathcal{B}$, then for every $t \in [0, b]$ the following conditions hold :

(a) x_t is in $\mathcal{B}$.

(b) $\| x(t) \| \leq H \| x_t \| \mathcal{B}$.

(c) $\| x_t \| \mathcal{B} \leq M(t) \| x_0 \| \mathcal{B} + K(t) \sup\{\| x(s) \|: 0 \leq s \leq t\}$,

where $H > 0$ is a constant ; $K, M : [0, \infty) \to [1, \infty)$, K is continuous , M is lo cally bounded and H, K, M are independent of $x(\cdot)$.

(A 2) For the function x in (A 1), x_t is a $\mathcal{B}-$ valued continuous function on $[0, b]$.

(B 1) The space $\mathcal{B}$ is complete .

Example 2 . 1 (The phase space $\mathbf{C_r} \times \mathbf{L^p}$(**g** ; **X**)) **.** Let $g : (-\infty, -r) \to \mathbb{R}$ be a positive Lebesgue integrable function and assume that there exists a non - negative and lo cally bounded function γ on $(-\infty, 0]$ such that $g(\xi + \theta) \leq \gamma(\xi) g(\theta)$, for all $\xi \leq 0$ and $\theta \in (-\infty, -r) \setminus N_\xi$, where $N_\xi \subseteq (-\infty, -r)$ is a set with Lebesgue measure zero . The space $C_r \times L^p(g; X)$ consists of all classes of functions $\varphi : (-\infty, 0] \to X$ such that φ is continuous on $[-r, 0]$, Lebesgue - measurable and $g \| \varphi \|^p$ is Lebesgue integrable on $(-\infty, -r)$. The seminorm in $C_r \times L^p(g : X)$ is defined by

$$\| \varphi \| \mathcal{B} := \sup\{\| \varphi(\theta) \| \quad : -r \leq \theta \leq 0\} + (integraldisplay - minus^{-r}_{\infty} g(\theta) \| \varphi(\theta) \| p_{d\theta}) 1/p.$$

Assume that $g(\cdot)$ verifies the conditions (g - 5) , (g - 6) and (g - 7) in the nomenclature of [1 7] . In this case , $\mathcal{B} = C_r \times L^p(g; X)$ verifies assumptions (A 1) , (A 2) , (B 1) see [1 7 , Theorem 1 . 3 . 8] for details . Moreover , when $r = 0$ and $p = 2$ we have that

$$H = 1, M(t) = \gamma(-t)^{1/2} \text{and} K(t) = 1 + (\int_{-t}^{0} g(\theta) d\theta)^{1/2} \text{for} t \geq 0.$$

Remark 2 . 2 . Let $\varphi \in \mathcal{B}$ and $t \leq 0$. The notation φt represents the function defined by $\varphi t(\theta) = \varphi(t + \theta)$. Consequently , if the function x in axiom (A 1) is such that $x_0 = \varphi$, then $x_t = \varphi t$. We observe that φt is well defined for every $t < 0$ since the domain of φ is $(-\infty, 0]$. We also note that , in general , $\varphi t element-slash \mathcal{B}$; consider , for example , the characteristic function $\mathcal{X}_{[\mu, 0]}, \mu < -r < 0$, in the space $C_r \times L^p(g; X)$.

Some of our results will proved using the following well know result .

Theorem 2 . 3 (Leray Schauder Alternative [7 , Theorem 6 . 5 . 4]) **.** *Let D be a convex subset of a Banach space X and assume that $0 \in D$. Let $G : D \to D$ be a completely continuous map . Then the map G has a fixed point in D or the s e t*

$$\{x \in D : x = \lambda G(x), 0 < \lambda < 1\} is unbounded.$$

The terminology and notation are those generally used in functional analysis . In particular , for Banach spaces Z, W, the notation $\mathcal{L}(Z, W)$ stands for the Banach space of bounded linear operators from Z into W and we abbreviate this notation to $\mathcal{L}(Z)$ when $Z = W$. Moreover $B_r(x, Z)$ denotes the closed ball with center at x and radius $r > 0$ in Z and , for a bounded function $x : [0, a] \to X$ and $0 \leq t \leq a$ we employ the notation $\| x \| t$ for

$$\| x \| t = \sup\{\| x(s) \|: s \in [0, t]\}. \tag{2.6}$$

This paper has four sections . In the next section we establish the existence of mild solutions for the abstract Cauchy problem (1 . 1) - (1 . 2) . In section 4 some applications are considered .

3 . Existence Results

In this section we establish the existence of mild solutions for the abstract Cauchy problem (1 . 1) - (1 . 2) . To prove our results , we assume that $\rho : I \times \mathcal{B} \to (-\infty, a]$ is a continuous function and that the following conditions are verified .

(H 1) The function $f : I \times \mathcal{B} \to X$ satisfies the following properties .

(a) The function $f(\cdot, \psi) : I \to X$ is strongly measurable for every $\psi \in \mathcal{B}$.

(b) The function $f(t, \cdot) : \mathcal{B} \to X$ is continuous for each $t \in I$.

(c) There exist an integrable function $m : I \to [0, \infty)$ and a continuous nondecreasing function $W : [0, \infty) \to (0, \infty)$ such that

$$\| f(t, \psi) \| \leq m(t) W(\| \psi \| \mathcal{B}), \quad (t, \psi) \in I \times \mathcal{B}. \tag{3.1}$$

(H 2) The function $t \to \varphi t$ is well defined and continuous from the set $\mathcal{R}(\rho^-) = \{\rho(s, \psi) : (s, \psi) \in I \times \mathcal{B}, \rho(s, \psi) \leq 0\}$ into $\mathcal{B}$ and there exists a continuous and bounded function $J^\varphi : \mathcal{R}(\rho) \to (0, \infty)$ such that $\| \varphi t \| \mathcal{B} \leq J^\varphi(t) \| \varphi \| \mathcal{B}$

$$\text{forevery} t \in \mathcal{R}(\rho).$$

Remark 3 . 1 . The condition (H 2) is frequently verify by functions continuous and bounded . In fact , if $\mathcal{B}$ verifies axiom C_2 in the nomenclature of [1 7] , then there exists L > 0 such that $\| \varphi \| \mathcal{B} \leq$ L $\sup_{\theta \leq 0} \| \varphi(\theta) \|$ for every $\varphi \in \mathcal{B}$ continuous and bounded , see [1 7 , Proposition 7 . 1 . 1] for details . Consequently ,

$$\| \varphi t \| \mathcal{B} \leq L \frac{\sup_{\theta \leq 0} \| \varphi(\theta) \|}{\| \varphi \| \mathcal{B}} \| \varphi \| \mathcal{B}$$

for every continuous and bounded function $\varphi \in \mathcal{B} \setminus \{0\}$ and every $t \leq 0$. We also observe that the space $C_r \times L^p(g; X)$ verifies axiom C_2, see [1 7 , p . 1 0] for details .

Motivated by (2 . 4) we introduce the following concept of mild solutions for the system (1 . 1) - (1 . 2) .

Definition 3 . 2 . *A function $x : (-\infty, a] \to X$ is called a mild s o lution of the abstract Cauchy pro b lem (1 . 1) - (1 . 2) if $x_0 = \varphi, x_{\rho(s, x_s)} \in \mathcal{B}$ for every $s \in I$ and*

$$x(t) = C(t)\varphi(0) + S(t)\zeta 0 + \int_0^t S(t-s) f(s, x_{\rho(s, x_s)}) ds, \quad t \in I.$$

In the rest of this paper , M_a and K_a are the constants defined by $M_a =$

$$\sup_{t \in I} M(t) \text{and} K_a = \sup_{t \in I} K(t).$$

Lemma 3 . 3 ([1 5 , Lemma 2 . 1]) **.** *Let $x : (-\infty, a] \to X$ be a function such that*

Then

$$x_0 = \varphi and x |_[0, a] \in \mathcal{PC}.$$

$$\| x_s \| \mathcal{B} \leq (M_a + \widetilde{J}^\varphi) \| \varphi \| \mathcal{B} + K_a \sup\{\| x(\theta) \|; \theta \in [0, \max\{0, s\}]\},$$

$$s \in \mathcal{R}(\rho^-) \cup I, where \widetilde{J}^\varphi = \sup_{t \in \mathcal{R}(\rho^-)} J^\varphi(t).$$

Now , we can prove our first existence result . **Theorem 3 . 4 .** *Let conditions (H 1) , (H 2) hold and assume that* $S(t)$ *is compact for every* $t \in \mathbb{R}$. *If*

$$\widetilde{N} K_a \lim_{\xi \to \infty^+} \inf \frac{W(\xi)}{\xi} \int_0^a m(s) ds < 1,$$

then there exists a mild s o lution $u(\cdot)$ *of (1 . 1) - (1 . 2) . Moreover , if* $\varphi(0) \in E$ *then* $u \in C^1(I, X)$ *and condition (1 . 2) is verified .*

Proof . On the space $Y = \{u \in C(I, X) : u(0) = \varphi(0)\}$ endowed with the uniform convergence topology , we define the operator $\Gamma : Y \to Y$ by

$$\Gamma x(t) = C(t)\varphi(0) + S(t)\zeta 0 + \int_0^t S(t-s) f(s, \bar{x}_{\rho(s,\bar{x}_s)}) ds, \quad t \in I, \tag{3.2}$$

where $\bar{x} : (-\infty, a] \to X$ is such that $\bar{x}_0 = \varphi$ and $\bar{x} = x$ on I. From assumption (A 1) and our assumptions on φ, we infer that Γx is well defined and continuous .

Let $\bar{\varphi} : (-\infty, a] \to X$ be the extension of φ to $(-\infty, a]$ such that $\bar{\varphi}(\theta) = \varphi(0)$ on I and $\widetilde{J}^{\varphi} = \sup \{J^{\varphi}(s) : \quad s \in \mathcal{R}(\rho^-)\}$. We claim that there exists $r > 0$ such that $\Gamma(B_r(\bar{\varphi} \mid I, Y)) \subseteq B_r(\bar{\varphi} \mid I, Y)$. If this property is false , then for every $r > 0$ there exist $x^r \in B_r(\bar{\varphi} \mid I, Y)$ and $t^r \in I$ such that $r < \| \Gamma x^r(t^r) - \varphi(0) \|$. By using Lemma 3 . 3 we find that

$$r < \quad \| \Gamma x^r(t^r) - \varphi(0) \|$$

$$\leq \| C(t^r)\varphi(0) - \varphi(0) \| \quad + \| S(t)\zeta 0 \| \quad + \int_0^{t^r} \quad \| S(t^r - s) \| \| f(s, \overline{\quad} x^r_{\rho(s,(\overline{x}} \overline{\quad r)}{}_s^{)} \| ds$$

$$\leq H(N+1) \| \varphi \| \mathcal{B} + \widetilde{N} \| \zeta 0 \| + \widetilde{N} \int_0^{t^r} m(s) W(\| \text{-}xr_{\rho(s,(\overline{\quad\quad} x^r)_s)} \| \mathcal{B}) ds$$

$$\leq H(N+1) \| \varphi \| \mathcal{B} + \widetilde{N} \| \zeta 0 \| + \widetilde{N} \int_0^{t^r} m(s) W((M_a + \widetilde{J}^{\varphi}) \| \varphi \| \mathcal{B} + K_a \| \text{-}xr \|_a) ds$$

$$\leq H(N+1) \| \varphi \| \mathcal{B} + \widetilde{N} \| \zeta 0 \|$$

$$+ \widetilde{N} W((M_a + \widetilde{J}^{\varphi}) \| \varphi \| \mathcal{B} + K_a(r + \| \varphi(0) \|)) \int_0^a m(s) ds,$$

and hence

$$1 \leq \widetilde{N} K_a \lim_{\xi \to \infty} \inf \frac{W(\xi)}{\xi} \int_0^a m(s) ds,$$

which is contrary to our assumption .

Let $r > 0$ be such that $\Gamma(B_r(\bar{\varphi} \mid I, Y)) \subseteq B_r(\bar{\varphi} \mid I, Y)$. Next , we will prove that Γ is completely continuous on $B_r(\bar{\varphi} \mid I, Y)$. In the sequel , r^*, r^{**} are the numbers defined by $r^* := (M_a + \widetilde{J}^{\varphi}) \| \varphi \| \mathcal{B} + K_a(r + \quad \| \varphi(0) \|)$ and $r^{**} := W(r^*) \int_0^a m(s) ds$.

Step 1 The set $\Gamma(B_r(\bar{\varphi} \mid I, Y)(t) = \{\Gamma x(t) : x \in B_r(\bar{\varphi} \mid I, Y)\}$ is relatively compact in X for all $t \in I$.

The case $t = 0$ is obvious . Let $0 < \varepsilon < t \leq a$. Since the function $t \to S(t)$ is Lipschitz , we can select points $0 = t_1 < t_2 \cdots < t_n = t$ such that $\| S(s) - S(s') \| \leq \varepsilon$, if $s, s' \in [t_i, t_{i+1}]$ for some $i = 1, 2, ..., n-1$. If $x \in B_r(\bar{\varphi} \mid I, Y)$, from Lemma 3 . 3 follows that $\| \bar{x}_{\rho(t,\bar{x}_t)} \| \mathcal{B} \leq r^*$ and hence

$$\| \int_0^{\tau} f(s, \bar{x}_{\rho(s,\bar{x}_s)}) ds \| \leq W(r^*) \int_0^a m(s) ds = r^{**}, \quad \tau \in I. \tag{3.3}$$

 Now , from (3 . 3) we find that

$$\Gamma x(t) = C(t)\varphi(0) + S(t)\zeta 0 + \sum_{n-1}^{i=1} \int_{t_i}^{t_{i+1}} (S(s) - S(t_i)) f(t-s, \bar{x}_{\rho(t-s,\bar{x}t-s)}) ds$$
$$+ \sum_{n-1}^{i=1} S(t_i) \int_{t_i}^{t_{i+1}} f(t-s, \bar{x}_{\rho(t-s,\bar{x}t-s)}) ds$$
$$\in \{C(t)\varphi(0) + S(t)\zeta 0\} + \mathcal{C}_\varepsilon + \sum_{i=1}^{n-1} S(t_i) B_{r^{**}}(0, X).$$

Thus ,

$$\Gamma(B_r(\bar{\varphi} \mid I, Y)(t) \subseteq \mathcal{C}_\varepsilon + \mathcal{K}_\varepsilon,$$

where $\mathcal{K}_\varepsilon$ is compact and diam $(\mathcal{C}_\varepsilon) \le \varepsilon r^{**}$, which permit us concluding that the set $\Gamma(B_r(\bar{\varphi} \mid I, Y))(t)$ is relatively compact in X since ε is arbitrary .
Step 2 The set of functions $\Gamma(B_r(\bar{\varphi} \mid I, Y))$ is equicontinuous on I.
Let $0 < \varepsilon < t < a$ and $\delta > 0$ such that $\| S(s)x - S(s')x \| < \varepsilon$, for every $s, s' \in I$ with $| s - s' | \le \delta$. For $x \in B_r(\bar{\varphi} \mid I, Y)$ and $0 < | h | < \delta$ such that $t + h \in I$ we get

$$\begin{aligned}
\| \Gamma x(t+h) - \Gamma x(t) \| \quad \le \quad & \| (C(t+h) - C(t))\varphi(0) \| + \varepsilon \| \zeta 0 \| + \widetilde{N} W(r^*) \int_t^{t+h} m(s) ds \\
& + W(r^*) \int_0^t \| (S(t+h-s) - S(t-s)) \| m(s) ds \\
\le & \| (C(t+h) - C(t))\varphi(0) \| \quad + \varepsilon \| \zeta 0 \| \quad + \widetilde{N} W(r^*) \int_t^{t+h} m(s) ds \\
& + W(r^*) \varepsilon \int_0^a m(s) ds,
\end{aligned}$$

which proves that $\Gamma(B_r(\bar{\varphi} \mid I, Y))$ is equicontinuous on I.
Proceeding as in the proof of [1 5 , Theorem 2 . 2] we can prove that Γ is continuous . Thus , Γ is completely continuous . Now , from the Schauder Fixed Point Theorem we infer the existence of a mild solution $u(\cdot)$ for (1 . 1) - (1 . 2) . The assertion concerning the regularity of $u(\cdot)$ follows directly from the properties of the space E. The proof is complete . ☐

Theorem 3 . 5 . *Let conditions (H 1) , (H 2) be satisfied . Suppose that* $S(t)$ *is compact for every* $t \in \mathbb{R}, \rho(t, \psi) \le t$ *for every* $(t, \psi) \in I \times \mathcal{B}$ *and*

$$K_a^{\widetilde{N}} \int_0^a m(s) ds < \int_C^\infty \frac{ds}{W(s)},$$

where $C = \quad (K_a N H + M_a + \widetilde{J}^\varphi) \| \varphi \| \mathcal{B} + K_a^{\widetilde{N}} \| \zeta 0 \|$ *and* $\widetilde{J}^\varphi = \sup_{t \in \mathcal{R}(\rho^-)} J^\varphi(t).$

Then there exists a mild s o lution of (1 . 1) - (1 . 2) . *If in addition ,* $\varphi(0) \in E,$ *then* $u \in C^1(I, X)$ *and condition* (1 . 2) *is verified .*

Proof . For $u \in Y = C(I, X)$ we define Γu by (3 . 2) . In order to use Theorem 2 . 3 , next we will shall *a priori* estimates for the solutions of the integral equation

 $z = \lambda\Gamma z, \lambda \in (0,1)$. If $x^\lambda = \lambda\Gamma x^\lambda, \lambda \in (0,1)$, from Lemma 3 . 3 we have that

$$\| x^\lambda(t) \| \quad \leq NH \| \varphi \| \mathcal{B} + \widetilde{N} \| \zeta 0 \| \quad + \int_0^t \widetilde{N} \| f(s, \overline{\quad} \lambda_{x\rho(s,\overline{\quad}(x^\lambda)_s)}) \| ds$$

$$\leq NH \| \varphi \| \mathcal{B} + \widetilde{N} \| \zeta 0 \|$$

$$+\widetilde{N} \int_0^t m(s) W((M_a + \widetilde{J}^\varphi) \| \varphi \| \mathcal{B} + K_a \| x^\lambda \|_{\overline{\max\{0,\rho(s,}\ (x\lambda)_s)\}})ds$$

$$\leq NH \| \varphi \| \mathcal{B} + \widetilde{N} \| \zeta 0 \| + \widetilde{N} \int_0^t m(s) W((M_a + \widetilde{J}^\varphi) \| \varphi \| \mathcal{B} + K_a \| x^\lambda \| s) ds,$$

since $\rho(s, \text{-}(x^\lambda)_s) \leq s$ for all $s \in I$. Defining $\xi^\lambda(t) = (M_a + \widetilde{J}^\varphi) \| \varphi \| \mathcal{B} + K_a \| x^\lambda \| t$, we obtain

$$\xi^\lambda(t) \leq (K_a NH + M_a + \widetilde{J}^\varphi) \| \varphi \| \mathcal{B} + K_a^{\widetilde{N}} \| \zeta 0 \| + K_a^{\widetilde{N}} \int_0^t m(s) W(\xi^\lambda(s)) ds. \quad (3.4)$$

Denoting by $\beta\lambda(t)$ the right - hand side of (3 . 4) , follows that

$$\beta_\lambda'(t) \leq K_a^{\widetilde{N}} m(t) W(\beta\lambda(t))$$

and hence

$$\int_{\beta\lambda(0)=C}^{\beta\lambda(t)} \frac{ds}{W(s)} \leq K_a^{\widetilde{N}} \int_0^a m(s) ds < \int_C^\infty \frac{ds}{W(s)},$$

which implies that the set of functions $\{\beta\lambda(\cdot) : \lambda \in (0,1)\}$ is bounded in $C(I : \mathbb{R})$. This prove that $\{x^\lambda(\cdot) : \lambda \in (0,1)\}$ is also bounded in $C(I, X)$.

Arguing as in the proof of Theorem 3 . 4 we can prove that $\Gamma(\cdot)$ is completely continuous , and from Theorem 2 . 3 we conclude that there exists a mild solution $u(\cdot)$ for (1 . 1) - (1 . 2) . Finally , it is clear from the preliminaries that $u(\cdot)$ is a function in $C^1(I, X)$ which verifies (1 . 2) when $\varphi(0) \in E$. The proof is finished . $\square$

4 . Examples

In this section we consider some applications of our abstract results .

The ordinary case . If $X = \mathbb{R}^k$, our results are easily applicable . In fact , in this case the operator A is a matrix of order $n \times n$ which generates the cosine function $C(t) = \cosh\,(tA^{1/2}) = \sum_{n=1}^\infty \frac{t^{2n}}{2n!} A^n$ with associated sine function $S(t) = A^{-}\overline{\quad\quad} 2^1 \sinh\,(tA^{1/2}) = \sum_{n=1}^\infty \frac{t^{2n+2}}{(2n+1)!} A^n$. We note that the expressions $\cosh\,(tA^{1/2})$ and
$\sinh\,(t \| A \|^{1/2})$ are purely symbolic and do not assume the existence of the square roots of A. It is easy to see that $C(t), S(t), t \in \mathbb{R}$, are compact operators and that $\| C(t) \| \quad \leq \cosh\,(a \| A \|^{1/2})$ and $\| S(t) \| \quad \leq \quad \| A \|^{1/2} \sinh\,(a \| A \|^{1/2})$ for all $t \in \mathbb{R}$. The next result is a consequence of Theorems 3 . 4 and 3 . 4 .

Proposition 4 . 1 . *Assume conditions (H 1) , (H 2) . If any of the following condi - tio ns is verified ,*

(a) $K_a \| A \| 1/2 \sinh\,(a \| A \| 1/2_)$ $\liminf_{\xi \to \infty} + \frac{W(\xi)}{\xi} \int_0^a m(s) ds < 1$;

(b) $\rho(t, \psi) \leq t$ *for all* $(t, \psi) \in I \times \mathcal{B}$ *and*

$$K_a \| A \|^{1/2} \sinh(a \| A \|^{1/2}) \int_0^a m(s) ds < \int_C^\infty \frac{ds}{W(s)},$$

where

$$C = (K_a \cosh(a \parallel A \parallel 1/2_{)H} + \widetilde{J}^{\varphi}) \parallel \varphi \parallel \mathcal{B} + K_a \parallel A \parallel 1/2 \sinh(a \parallel A \parallel 1/2_{)} \parallel \zeta 0 \parallel;$$

then there exists a mild s o lution of (1 . 1) - (1 . 2) . **A partial differential equation with state dependent delay .** To complete
this section , we discuss the existence of solutions for the partial differential system

$$\partial^2 u(t,\xi)$$

$$= \partial \frac{2\partial^t_{2_{u(t,}} \xi)}{\partial \xi^2} + -^t \infty a_1(s-t) u(s - \rho 1(t) \rho 2(\int_0^\pi a_2(\theta) \mid u(t,\theta) \mid^2 d\theta), \xi) ds \tag{4.1}$$

for $t \in I = [0,a], \xi \in [0,\pi]$, subject to the initial conditions

$$u(t,0) = u(t,\pi) = 0, \quad t \geq 0, \tag{4.2}$$

$$u(\tau,\xi) = \varphi(\tau,\xi), \quad \tau \leq 0, 0 \leq \xi \leq \pi. \tag{4.3}$$

To apply our abstract results , we consider the spaces $X = L^2([0,\pi]); \mathcal{B} = C_0 \times L^2(g,X)$ and the operator $Af = f''$ with domain

$$D(A) = \{x \in X : x'' \in X, x(0) = x(\pi) = 0\}.$$

It is well - known that A is the infinitesimal generator of a strongly continuous cosine function $(C(t))t \in \mathbb{R}$ on X. Furthermore , A has a discrete spectrum , the eigenval - ues are $-n^2, n \in \mathbb{N}$, with corresponding eigenvectors $z_n(\theta) = (\frac{2}{\pi})^{1/2} \sin(n\theta)$. In addition , the following properties hold :

(a) The set $\{z_n : n \in \mathbb{N}\}$ is an orthonormal basis of X.

(b) For $x \in X, C(t)x = \sum_{n=1}^\infty \cos(nt)(x,z_n)z_n$. From this expression , it follows that $S(t)x = \sum_{n=1}^\infty \frac{\sin(nt)}{n}(x,z_n)z_n$, $\| C(t) \| = \| S(t) \| \leq 1$ for all $t \in \mathbb{R}$ and that $S(t)$ is compact for every $t \in \mathbb{R}$.

(c) If Φ is the group of translations on X defined by $\Phi(t)x(\xi) = \tilde{x}(\xi+t)$, where $\tilde{x}$ is the extension of x with period 2π, then $C(t) = \frac{1}{2}(\Phi(t)+\Phi(-t))$ and $A = B^2$, where B is the generator of Φ and

$$E = \{x \in H^1(0,\pi) : x(0) = x(\pi) = 0\},$$

see [6] for details .

Assume that $\varphi \in \mathcal{B}$, the functions $a_i : \mathbb{R} \to \mathbb{R}, \rho i : [0,\infty) \to [0,\infty), i = 1,2$, are continuous , $a_2(t) \geq 0$ for all $t \geq 0$ and $L_1 = (\int_0^\infty \frac{2_{a_1}(s)}{g(s)} ds)^{1/2} < \infty$. Under these conditions , we can define the operators $f : I \times \mathcal{B} \to X, \rho : I \times \mathcal{B} \to \mathbb{R}$ by

$$f(t,\psi)(\xi) = -^0 \infty a_1(s)\psi(s,\xi)ds,$$

$$\rho(s,\psi) = s - \rho 1(s) \rho 2(\int_0^\pi a_2(\theta) \mid \psi(0,\xi) \mid^2 d\theta),$$

and transform system (4 . 1) - (4 . 3) into the abstract Cauchy problem (1 . 1) - (1 . 2) . Moreover , f is a continuous linear operator with $\| f \| \leq L_1, \rho$ is continuous and $\rho(t,\psi) \leq s$ for every $s \in [0,a]$. The next results are consequence of Theorem 3 . 5
and Remark 3 . 1 .

Proposition 4 . 2 . *Assume that φ satisfies (H 2) . Then there exists a mild s o lution of (4 . 1) – (4 . 3) .*

Corollary 4 . 3 . *If φ is continuous and bounded , th en there exists a mild s o lution of (4 . 1) – (4 . 3) .*

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