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EXISTENCE OF $\psi-$ BOUNDED SOLUTIONS FOR NONHOMOGENEOUS LINEAR DIFFERENTIAL EQUATIONS

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ABSTRACT . In this article we present a necessary and sufficient condition for the existence of $\psi-$ bounded solution on $\mathbb{R}$ of the nonhomogeneous linear dif - ferential equation $x' = A(t)x + f(t)$. We associate that with the condition of the concept $\psi-$ dichotomy on $\mathbb{R}$ of the homogeneous linear differential equation

$$x' = A(t)x.$$

1 . INTRODUCTION

The existence of $\psi-$ bounded and $\psi-$ stable solutions on $\mathbb{R}_+$ for systems of or - dinary differential equations has been studied by many authors ; see for exam - ple Akinyele [1] , Avramescu [2] , Constantin [4] , Diamandescu [5 , 6 , 7] . Denote by $\mathbb{R}^d$ the $d-$ dimensional Euclidean space . Elements in this space are denoted by $x = (x_1, x_2, ..., x_d)^T$ and their norm by $\| x \| = \max \{| x_1 |, | x_2 |, ..., | x_d |\}$. For real $d \times d$ matrices , we define norm $| A | = \sup_{\| x \| \leqslant 1} \| Ax \|$. Let $\mathbb{R}_+ = [0, \infty)$, $\mathbb{R}_- = (-\infty, 0]$, $J = \mathbb{R}_-, \mathbb{R}_+$ or $\mathbb{R}$ and $\psi i : J \to (0, \infty), i = 1, 2, ..., d$ be con - tinuous functions . Set

$$\psi = \mathrm{diag}[\psi 1, \psi 2, ..., \psi d].$$

Definition 1 . 1 . A function $f : J \to \mathbb{R}^d$ is said to be

- $\psi-$ bounded on J if $\psi(t)f(t)$ is bounded on J.
- $\psi-$ integrable on J if $f(t)$ is measurable and $\psi(t)f(t)$ is Lebesgue integrable on J.
- $\psi-$ integrally bounded on J if $f(t)$ is measurable and the Lebesgue integrals $\int_t^{t+1} \| \psi(u)f(u) \| du$ are uniformly bounded for any $t, t+1 \in J$.

In $\mathbb{R}^d$, consider the following equations

$$x' = A(t)x + f(t), \tag{1.1}$$
$$x' = A(t)x. \tag{1.2}$$

where $A(t)$ is continuous matrix on $J, f(t)$ is a continuous function on J. Let $Y(t)$ be fundamental matrix of (1 . 2) with $Y(0) = I_d$, the identity $d \times d$ matrix . The

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 $d \times d$ matrices P_1, P_2 is said to be the pair of the supplementary projections if

$$P_1^2 = P_1, P_2^2 = P_2, P_1 + P_2 = I_d.$$

Definition 1 . 2 . The equation (1 . 2) is said to have a $\psi-$ exponential dichotomy
on J if there exist positive constants K, L, α, β and a pair of the supplementary projections P_1, P_2 such that

$$| \psi(t)Y(t)P_1Y^{-1}(s)\psi^{-1}(s) | \leqslant Ke^{-\alpha(t-s)} \quad \text{for} s \leqslant t, s, t \in J, \tag{1.3}$$

$$| \psi(t)Y(t)P_2Y^{-1}(s)\psi^{-1}(s) | \leqslant Le^{\beta(t-s)} \quad \text{for} t \leqslant s, s, t \in J. \tag{1.4}$$

The equation (1 . 2) is said to have a $\psi-$ ordinary dichotomy on J if (1 . 3) , (1 . 4) hold

$$\text{with} \alpha = \beta = 0.$$

We say that (1 . 2) has a $\psi-$ bounded grow if for some fixed $h > 0$ there exists a constant $C \geqslant 1$ such that every solution $x(t)$ of (1 . 2) is satisfied

$$\| \psi(t)x(t) \| \quad \leqslant C \| \psi(s)x(s) \| \text{ for} s \leqslant t \leqslant s + h, s, t \in J. \tag{1.5}$$

Remark 1 . 3 . It is easy to see that if (1 . 2) has a $\psi-$ exponential dichotomy on $\mathbb{R}_+$ and on $\mathbb{R}_-$ with a pair of the supplementary projections P_1, P_2 then (1 . 2) has a $\psi-$ exponential dichotomy on $\mathbb{R}$ with the pair of the supplementary projections

$$P_1, P_2.$$

Theorem 1 . 4 ([3 , 5 , 7]) **.** *(a) The equation (1 . 1) has at least one $\psi-$ bounded s o lution on $\mathbb{R}_+$ for every $\psi-$ integrable function f on $\mathbb{R}_+$ if and only if (1 . 2) has a $\psi-$ o rdinary dichotomy on $\mathbb{R}_+$. (b) The equation (1 . 1) has at least one $\psi-$ bounded s o lution on $\mathbb{R}_+$ for every $\psi-$ inte - grally bounded function f on $\mathbb{R}_+$ if and only if (1 . 2) has a $\psi-$ exponential dichotomy on $\mathbb{R}_+$. (c) Suppose that (1 . 2) has a $\psi-$ bounded grow on $\mathbb{R}_+$. Then , (1 . 1) has at least one $\psi-$ bounded s o lution on $\mathbb{R}_+$ for every $\psi-$ bounded function f on $\mathbb{R}_+$ if and only if (1 . 2) has a $\psi-$ exponential dichotomy on $\mathbb{R}_+$.*

Theorem 1 . 5 ([7]) **.** *Suppose that (1 . 1) has a $\psi-$ exponential dichotomy on $\mathbb{R}_+$ and , $P_1 \neq 0, P_2 \neq 0$. If $\lim_{t\to\infty} \| \psi(t)f(t) \| = 0$ then every $\psi-$ bounded s o lution $x(t)$ of (1 . 1) is such that $\lim_{t\to\infty} \| \psi(t)x(t) \| = 0$.*

2 . Preliminaries

Lemma 2 . 1 . *(a) Let (1 . 2) has a $\psi-$ exponential dichotomy on $\mathbb{R}_+$ with a pair of the supplementary projections P_1, P_2. If $Q1, Q2$ is a pair of the supplementary projections such that $ImP_1 = ImQ1$, then (1 . 2) als o has a $\psi-$ exponential dichotomy on $\mathbb{R}_+$ with the pair of the supplementary projections $Q1, Q2$. (b) Let (1 . 2) have a $\psi-$ exponential dichotomy on $\mathbb{R}_-$ with a pair of the supplemen - tary projections P_1, P_2. If $Q1, Q2$ is a pair of supplementary projections such that $ImP_2 = ImQ2$, then (1 . 2) als o has a $\psi-$ exponential dichotomy on $\mathbb{R}_-$ with the pair of th e supplementary projections $Q1, Q2$.*

Proof . First , we prove in the case of $J = \mathbb{R}_+$. Note that (1 . 2) has a $\psi-$ exponential dichotomy on $\mathbb{R}_+$ with the pair of the supplementary projections P_1, P_2 if only if following st atements are satisfied :

$$\| \psi(t)Y(t)P_1\xi \| \quad \leqslant K' e^{-\alpha(t-s)} \| \psi(s)Y(s)\xi \| \quad \text{for all } \xi \in \mathbb{R}^d \text{ and } t \geqslant s \geqslant 0, \quad (2.1)$$

$$\| \psi(t)Y(t)P_2\xi \| \leqslant L' e^{\beta(t-s)} \| \psi(s)Y(s)\xi \| \quad \text{for all } \xi \in \mathbb{R}^d \text{ and } s \geqslant t \geqslant 0. \quad (2.2)$$

 In fact , if (1 . 3) and (1 . 4) are true , we have for any vector $y \in \mathbb{R}^d$

$$\| \psi(t)Y(t)P_1Y^{-1}(s)\psi^{-1}(s)y \| \quad \leqslant Ke^{-\alpha(t-s)} \| y \| \quad \text{for} t \geqslant s \geqslant 0,$$
$$\| \psi(t)Y(t)P_2Y^{-1}(s)\psi^{-1}(s)y \| \leqslant Le^{\beta(t-s)} \| y \| \quad \text{for} s \geqslant t \geqslant 0.$$

Choose $y = \psi(s)Y(s)\xi$, we obtain (2 . 1) , (2 . 2) . Conversely , suppose that inequalities
(2 . 1) , (2 . 2) are true . For any vector $y \in \mathbb{R}^d$, putting $\xi = Y^{-1}(s)\psi^{-1}(s)y$ we get (1 . 3) , (1 . 4) .

Now prove the lemma . It follows from $KerP_2 = ImP_1 = ImQ1 = KerQ2$ that $P_2Q1 = 0$. Hence $P_1Q1 = P_1Q1 + P_2Q1 = Q1$. Similarly $Q1^P1 = P_1$. Then

$$P_1 - Q1 = P_1^2 - P_1Q1 = P_1(P_2 - Q2), \tag{2.3}$$
$$P_1 - Q1 = -Q1^P2 = P_1P_2 - Q1^P2 = (P_1 - Q1)P_2. \tag{2.4}$$

For each $u \in \mathbb{R}^d$, put $\xi = (P_1 - Q1)u$. The relation (2 . 3) implies that $\xi \in ImP_1$, then $P_1\xi = \xi$. Result from (2 . 1) , for $s = 0$ that

$$\| \psi(t)Y(t)[P_1 - Q1]u \| \leqslant K'e^{-\alpha t} \| \psi(0)[P_1 - Q1]u \|, t \geqslant 0. \tag{2.5}$$

By (2 . 4) we conclude

$$K'e^{-\alpha t} \| \psi(0)[P_1 - Q1]u \| \quad = \leqslant K^{K',e^{-\alpha t}\| \psi(0)\,[P1 \quad -Q1]P2^u \quad \|}_{|\psi(0)| \quad |P_1 - Q1|e-\alpha t_{\|} P_2u\|}, \quad t \geqslant 0. \tag{2.6}$$

Applying (2 . 2) , for $t = 0$, we get

$$\begin{aligned} \| P_2u \| = & \quad \| \psi^{-1}(0)\psi(0)P_2u \| \\ & \leqslant | \psi^{-1}(0) ||| \psi(0)P_2u \| \\ \leqslant L'e^{-\beta s} | \psi^{-1}(0) ||| \psi(s)Y(s)u \|, & \quad \text{for} s \geqslant 0. \end{aligned} \tag{2.7}$$

The relations (2 . 5) – (2 . 7) imply

$$\| \psi(t)Y(t)[P_1 - Q1]u \| \leqslant \leqslant K_1^{K'L'_{|\psi(0)|}}{}_{e^{\beta(t-s)}} \||_{\psi}^{\psi^{-1}(0)|}\ {}^{|P_1}_{(s)Y(s)u} \| -, Q^1_{\text{for}} \mid e-{}^{\alpha t}t^{e\ -\beta t_{\|}}_{,s\geqslant\ 0.} \psi(s)Y(s)u \| \tag{2.8}$$

On the other hand , by (2 . 2) we get

$$\| \psi(t)Y(t)P_2u \| \leqslant L'e^{\beta(t-s)} \| \psi(sY(s))u \|, \quad \text{for} 0 \leqslant t \leqslant s. \tag{2.9}$$

It follows from $Q2 = P_2 + P_1 - Q1, (2.8)$ and (2 . 9) that

$$\begin{aligned} \| \psi(t)Y(t)Q2^u \| \leqslant \quad \| \psi(t)Y(t)P_2u \| \quad & + \| \psi(t)Y(t)[P_1 - Q1]u \| \\ & \leqslant (L' + K_1)e^{\beta(t-s)} \| \psi(s)Y(s)u \| \\ \leqslant L_2e^{\beta(t-s)} \| \psi(s)Y(s)u \|, & \quad \text{for} 0 \leqslant t \leqslant s. \end{aligned} \tag{2.10}$$

Similarly , for $u \in \mathbb{R}^d$, we have

$$\| \psi(t)Y(t)Q1^u \| \quad \leqslant K_2e^{-\alpha(t-s)} \| \psi(s)Y(s)u \|, \quad \text{for} 0 \leqslant s \leqslant t. \tag{2.11}$$

Then from this inequality , (2 . 1 0) and the preceding note it follows that (1 . 2) has a $\psi-$ exponential dichotomy on $\mathbb{R}_+$ with the pair of the supplementary projections $Q1, Q2$. In the case of $J = \mathbb{R}_-$, the proof is similar . $\square$

Remark 2 . 2 . (a) Suppose that (1 . 2) has a $\psi-$ exponential dichotomy on $\mathbb{R}_+$ with
a pair of supplementary projections P_1, P_2. The set $P_1\mathbb{R}^d$ is the subspace of $\mathbb{R}^d$ consisting of the values $x(0)$ of all $\psi-$ bounded solutions $x(t)$ on $\mathbb{R}_+$ of (1 . 2) . In fact ,
denote by X_1 this subspace , if $v \in P_1\mathbb{R}^d$ then $v \in X_1$ by virtue of (2 . 1) . Conversely if $u \in X_1$, we have to show that $P_2u = 0$. Suppose otherwise that $P_2u \neq 0$, by (2 . 1) , (2 . 2) we have $\| \psi(t)Y(t)P_1u \|$ is bounded and the limit of $\| \psi(t)Y(t)P_2u \|$ is ∞, as t tend to ∞. Denote y the solution of (1.2), $y(0) = u$. The relation $\psi(t)y(t) - \psi(t)Y(t)P_1u = \psi(t)Y(t)P_2u$ follows that y is non $\psi-$ bounded on $\mathbb{R}_+$, which is a contradiction .
(b) Similarly if (1 . 2) has a $\psi-$ exponential dichotomy on $\mathbb{R}_-$ with a pair of supple -
mentary projections P_1, P_2 then the set $P_2\mathbb{R}^d$ is the subspace of $\mathbb{R}^d$ consisting of the values $x(0)$ of all $\psi-$ bounded solutions $x(t)$ on $\mathbb{R}_-$ of (1 . 2) .

(c) Suppose that (1 . 2) has a $\psi-$ exponential dichotomy on $\mathbb{R}$, then (1 . 2) has no nontrivial $\psi-$ bounded solution on $\mathbb{R}$. In fact if $x(t)$ is the $\psi-$ bounded solution of (1 . 2) on $\mathbb{R}$ then it is $\psi-$ bounded on $\mathbb{R}_+$ and on $\mathbb{R}_-$. Because equation (1 . 2) has a $\psi-$ exponential dichotomy on $\mathbb{R}_+$, and on $\mathbb{R}_-$ with a pair of supplementary projections P_1, P_2, by preceding notice we have $P_2x(0) = 0$ and $P_1x(0) = 0$. Hence $x(0) = 0$, then $x(t)$ is the trivial solution of (1 . 2) .

Lemma 2 . 3 ([8]) **.** *Let $h(t)$ be a non - negative , locally integrable such that*
$\int_t^{t+1} h(s)ds \leqslant c,$ *for al l $t \in \mathbb{R}$*
If $\theta > 0$ then , for all $t \in \mathbb{R}$,

$$\int_t^{\infty} e^{-\theta(s-t)}h(s)ds \leqslant c[1 - e^{-\theta}]^{-1}, \tag{2.12}$$

$$integraldisplay - minus_{\infty}^{t} e^{-\theta(t-s)}h(s)ds \leqslant c[1 - e^{-\theta}]^{-1}. \tag{2.13}$$

Proof . We prove (2 . 1 2) , the proof of (2 . 1 3) is similar .

$$\begin{aligned}\int_{t+m}^{t+m+1} e^{-\theta(s-t)}h(s)ds &\leqslant \int_{t+m}^{t+m+1} e^{-\theta(t+m)}e^{\theta t}h(s)ds \\ &= \int_{t+m}^{t+m+1} e^{-\theta m}h(s)ds \leqslant ce^{-\theta m}\end{aligned}$$

implies that

$$\int_t^{\infty} e^{-\theta(s-t)}h(s)ds = \sum_{\infty}^{m=0} \int_{t+m}^{t+m+1} e^{-\theta(s-t)}h(s)ds \leqslant c\sum_{m\ \infty}^{=0} e^{-\theta m} = c[1 - e^{-\theta}]^{-1}$$

□

Lemma 2 . 4 . *Equation (1 . 1) has at least one $\psi-$ bounded s o lution on $\mathbb{R}$ for every $\psi-$ integrally bounded function f on $\mathbb{R}$ if and only if the fo l lowing three conditions are satisfied :*

(1) *Equation (1 . 1) has at least one s o lution on $\mathbb{R}, \psi-$ bounded on $\mathbb{R}_+$ for every $\psi-$ integrally bounded function f on $\mathbb{R}_+$*

(2) *Equation (1 . 1) has at least one s o lution on $\mathbb{R}, \psi-$ bounded on $\mathbb{R}_-$ for every $\psi-$ integrally bounded function f on $\mathbb{R}_-$.*

(3) *Every s o lution of (1 . 2) is th e sum of two s o lution of (1 . 2) , one of that is $\psi-$ bounded on $\mathbb{R}_+$, another is $\psi-$ bounded on $\mathbb{R}_-$.*

Proo f − period Suppose the three conditions are satisfied we have to prove that (1 . 1) has at least one $\psi-$ bounded solution on $\mathbb{R}$ for every $\psi-$ integrally bounded function f on $\mathbb{R}$. Every $\psi-$ integrally bounded function f on $\mathbb{R}$ is $\psi-$ integrally bounded function f on $\mathbb{R}_+$ and on $\mathbb{R}_-$. Then for each $\psi-$ integrally bounded function f on $\mathbb{R}$ exists the solution $y1$ and $y2$ of (1 . 1) , which is defined on $\mathbb{R}$ and corresponding $\psi-$ bounded on

$\mathbb{R}_+$ and on $\mathbb{R}_-$. Denote by $x(t)$ the solution of (1 . 2) such that $x(0) = y2(0) - y1(0)$. By 3 , we get $x(t) = x_1(t) + x_2(t)$, here x_1, x_2 are two solutions of (1 . 2) , that are corresponding $\psi-$ bounded solution on $\mathbb{R}_+$ and $\mathbb{R}_-$. Set $z_1 = y1 + x_1, z_2 = y2 - x_2$. Hence z_1 and z_2 are the solutions of (1 . 1) corresponding $\psi-$ bounded solution on $\mathbb{R}_+$ and on $\mathbb{R}_-$. Further , $z_2(0) = y2(0) - x_2(0) = y1(0) + x_1(0) = z_1(0)$, then $z_1 = z_2$. Consequently z_1 is a $\psi-$ bounded solution on $\mathbb{R}$ of (1 . 1) .

Conversely , now if (1 . 1) has at least one $\psi-$ bounded solution on $\mathbb{R}$ for every $\psi-$ integrally bounded function f on $\mathbb{R}$ we have to prove three condition are satisfied . The conditions 1 , 2 are satisfied since every $\psi-$ integrally bounded function f on $\mathbb{R}_+$, or $\mathbb{R}_-$ is the restriction of a $\psi-$ integrally bounded function f on $\mathbb{R}$. We prove that the condition 3 is satisfied . Set

$$h(t) = \begin{cases} 0 & \text{for } \mid t \mid \geqslant 1 \\ 1 & \text{for} t = 0 \\ \text{linear} & \text{for} t \in [-1, 0], t \in [0, 1] \end{cases}$$

Fix a solution $x(t)$ of (1 . 2) . Then $h(t)x(t)$ is a $\psi-$ integrally bounded function on $\mathbb{R}$. Set $y(t) = x(t) \int_0^t h(s)ds$, we have

$$y'(t) = A(t)x(t) \int_0^t h(s)ds + h(t)x(t) = A(t)y(t) + h(t)x(t).$$

By hypothesis , the equation

$$y'(t) = A(t)y(t) + h(t)x(t)$$

has a solution $\widetilde{y}(t)$, which is $\psi-$ bounded on $\mathbb{R}$. Set $x_1(t) = \widetilde{y}(t) - y(t) + \frac{1}{2}x(t)$ and $x_2(t) = \widetilde{y}(t) + y(t) + \frac{1}{2}x(t)$. It follows from $\int_{-1}^0 h(t)dt = \int_0^1 h(t)dt = \frac{1}{2}$ that $x_1(t) = \widetilde{y}(t)$ for $t \geqslant 1; x_2(t) = \widetilde{y}(t)$ for $t \leqslant -1$. Then x_1, x_2 are the corresponding $\psi-$ bounded solutions on $\mathbb{R}_+, \mathbb{R}_-$ of (1 . 2) . Consequently the solution $x(t)$ of (1 . 2) is the sum of two solutions $x_1(t)$ and $x_2(t)$ of (1 . 2) , those solutions satisfy the

condition 3 . The lemma is proved . □

3 . Main results

Theorem 3 . 1 . *Equation (1 . 1) has at least one $\psi-$ bounded s o lution on $\mathbb{R}_-$ for every $\psi-$ integrally bounded function f on $\mathbb{R}_-$ if and only if (1 . 2) has a $\psi-$ exponential dichotomy on $\mathbb{R}_-$.*

Proof . This Theorem can be shown as in [3 , Theorem 3 . 3] . We give the main steps

of the proof as follows . In the proof of " if part " : Suppose that $\int_{t-1}^t \parallel \psi(s)f(s) \parallel ds \leqslant c$

 for $t \leqslant 0$. By using Lemma 2 . 3 we get

$$\begin{aligned} \| integraldisplay - minus_{\infty}^{t} \psi(t)Y(t)P_1Y^{-1}(s)ds \| &\leqslant minus - integraldisplay_{\infty}^{t} \mid \psi(t)Y(t)P_1Y^{-1}(s)\psi^{-1}(s) \| \\ &\leqslant integraldisplay - minus_{\infty}^{t} e^{-\alpha(t-s)} \| \psi(s)f(s) \| \, ds \leqslant \end{aligned}$$

and

$$\begin{aligned} \| \int_t^0 \psi(t)Y(t)P_2Y^{-1}(s)f(s)ds \| &\leqslant \int_t^0 e^{-\beta(s-t)} \| \psi(s)f(s) \| \, ds \\ &\leqslant \int_t^{\infty} e^{-\beta(s-t)} \| \psi(s)f(s) \| \, ds \leqslant c(1-e^{-\beta})^{-1}. \end{aligned}$$

It follows that the function

$$\widetilde{x}(t) = minus - integraldisplay_{\infty}^{t} \psi(t)Y(t)P_1Y^{-1}(s)f(s)ds - \int_t^0 \psi(t)Y(t)P_2Y^{-1}(s)f(s)ds$$

is bounded on $\mathbb{R}$. Hence the function

$$\begin{aligned} x(t) &= \psi^{-1}(t)\widetilde{x}(t) \\ = integraldisplay - minus_{\infty}^{t} \psi(t)Y(t)P_1Y^{-1}(s)f(s)ds &- \int_t^0 \psi(t)Y(t)P_2Y^{-1}(s)f(s)ds \end{aligned}$$

is $\psi-$ bounded on $\mathbb{R}_-$. On the other hand

$$\begin{aligned} x'(t) = A(t)(integraldisplay - minus_{\infty}^{t} Y(t)P_1Y^{-1}(s)f(s)ds - \int_t^0 Y(t)P_2Y^{-1}(s)f(s)ds) \\ +Y(t)P_1Y^{-1}(t)f(t) + Y(t)P_2Y^{-1}(t)f(t) \\ = A(t)x(t) + f(t), \end{aligned}$$

it implies that $x(t)$ is a solution of (1 . 1) . In the proof of " only if part " : The set

$$\widetilde{C}_{\psi} = \{x : \mathbb{R}_- \to \mathbb{R}^d : x$$

is $\psi-$ bounded and continuous on $\mathbb{R}_-\}$. It is a Banach space with the norm $\| x \|_{C-e_{\psi}} =$ $\sup t \leqslant 0 \| \psi(t)x(t) \|$. The first step : we show that (1 . 1) has a unique $\psi-$ bounded solution $x(t)$ with $x(0) \in \widetilde{X}_1 = P_1\mathbb{R}^d$ for each $f \in \widetilde{C}_{\psi}$ and $\| x \| C - e_{\psi} \leqslant$ $r \| f \|_{e-C_{\psi}}$, here r
is a positive constant independent of f.

The next steps of the proof are similar to the proof of [3 , Theorem 3 . 3] , with the corresponding replacement (for example replace $t \geqslant t_0 \geqslant 0$ by $0 \geqslant t_0 \geqslant t, P_1$ by $-P_2, P_2$ by $-P_1, \infty$ by $-\infty, -\infty$ by $\infty, ...$). $\square$

Theorem 3 . 2 . *The equation* (1 . 1) *has a unique* $\psi-$ *bounded s o lution on* $\mathbb{R}$ *for ev - ery* $\psi-$ *integrally bounded function* f *on* $\mathbb{R}$ *if and only if* (1 . 2) *has a* $\psi-$ *exponential dichotomy on* $\mathbb{R}$.

Proof . First , we prove the " if " part . By Lemma 2 . 3 and in the same way as in the
proof of Theorem 3 . 1 , the function

$$x(t) = -^{t}\infty Y(t)P_1Y^{-1}(s)f(s)ds - \int_t^{\infty} Y(t)P_2Y^{-1}(s)f(s)ds$$

 is $\psi-$ bounded and continuous on $\mathbb{R}$. Moreover ,

$$x'(t) = A(t)(-^t\infty Y(t)P_1Y^{-1}(s)f(s)ds - \int_t^\infty Y(t)P_2Y^{-1}(s)f(s)ds)$$
$$+Y(t)P_1Y^{-1}(t)f(t) - Y(t)P_2Y^{-1}(t)f(t)$$
$$= A(t)x(t) + f(t),$$

it follows that $x(t)$ is a solution of (1 . 1) .

The uniqueness of the solution $x(t)$ result from (1 . 2) having no nontrivial $\psi-$ bounded solution on $\mathbb{R}$(Remark 2 . 2) . Suppose that y is a $\psi-$ bounded solution of (1 . 1) then $x-y$ is a $\psi-$ bounded solution of (1 . 2) on $\mathbb{R}$. We conclude $x=y$ since $x-y$ is the trivial solution of (1 . 2) .

We prove the " only if " par t Suppos tha (1 . 1 ha uniqu $\psi-$ bounded solution on $\mathbb{R}$ for every $\psi-$ integrally bounded function f on $\mathbb{R}$, we have to prove that (1 . 1)
has a $\psi-$ exponential dichotomy on $\mathbb{R}$. By Lemma 2 . 4 , Theorem 1 . 4 and Theorem 3 . 1 we get (1 . 2) has a $\psi-$ exponential dichotomy on $\mathbb{R}_+$ with a pair of the supplementary projections P_1, P_2 and has a $\psi-$ exponential dichotomy on $\mathbb{R}_-$. with a pair of the supplementary projections $Q1, Q2$. Remark 2 . 2 follows that $P_1\mathbb{R}^d$ is the subspace of $\mathbb{R}^d$ consisting of the values $x(0)$ of all $\psi-$ bounded solutions $x(t)$ on $\mathbb{R}_+$ of (1 . 2) and
$Q2^{\mathbb{R}^d}$ is the subspace of $\mathbb{R}^d$ consisting of the values $x(0)$ of all $\psi-$ bounded solutions $x(t)$ on $\mathbb{R}_-$ of (1 . 2) . We are going to prove that

$$\mathbb{R}^d = P_1\mathbb{R}^d \oplus Q2^{\mathbb{R}^d}_. \tag{3.1}$$

For each $u \in \mathbb{R}^d$, denote by $x = x(t)$ the solution of (1.2), $x(0) = u$. By Lemma 2 . 4 we get $x = x_1 + x_2$, where x_1, x_2 are the solutions of (1 . 2) corresponding $\psi-$ bounded on $\mathbb{R}_+, \mathbb{R}_-$. It follows from Remark 2 . 2 that $x_1(0) \in P_1\mathbb{R}^d$ and $x_2(0) \in Q2^{\mathbb{R}^d}_.$ It follows from $u = x_1(0) + x_2(0)$, that

$$\mathbb{R}^d = P_1\mathbb{R}^d + Q2^{\mathbb{R}^d}_. \tag{3.2}$$

By hypothesis (1 . 1) with $f = 0$ has unique $\psi-$ bounded solution on $\mathbb{R}$ i . e . (1 . 2) have
no nontrivial $\psi-$ bounded solution on $\mathbb{R}$. For any $v \in P_1\mathbb{R}^d \cap Q2^{\mathbb{R}^d}_,$ denote by $x(t)$ the solution of (1 . 2) such that $x(0) = v$. Then $x(t)$ is the $\psi-$ bounded solution of (1 . 2) , it implies that $x(t)$ is the trivial solution . Hence $v = 0$. Consequently

$$P_1\mathbb{R}^d \cap Q2^{\mathbb{R}^d} = 0. \tag{3.3}$$

The relations (3 . 2) and (3 . 3) imply (3 . 1) . Now , we prove the existence of a pair supplementary projections , for which (1 . 1) has a $\psi-$ exponential dichotomy on $\mathbb{R}$.
Choose the projection P of $\mathbb{R}^d$ such that $ImP = P_1\mathbb{R}^d$, $\ker P = Q2^{\mathbb{R}^d}_.$ By Lemma 2 . 1 , (1 . 2) has a $\psi-$ exponential dichotomy on $\mathbb{R}_+$, and have a $\psi-$ exponential dichotomy on $\mathbb{R}_-$ with the pair of the supplementary projections $P, I_d - P$. From Remark 1 . 3 it follows that (1 . 2) has a $\psi-$ exponential dichotomy on $\mathbb{R}$ with the pair of the supplementary projections $P, I_d - P$. The proof is complete . □

Theorem 3 . 3 . *Suppose that* (1 . 2) *has a* $\psi-$ *exponential dichotomy on* $\mathbb{R}$. *If*

$$\lim_{\to_t \pm\infty} \int_t^{t+1} \parallel \psi(s) f(s) \parallel ds = 0 \tag{3.4}$$

then the $\psi-$ *bounded s o lution of* (1 . 1) *is such that*

$$\lim_{\to_t \pm\infty} \parallel \psi(t) x(t) \parallel \quad = 0. \tag{3.5}$$

Proof . By Theorem 3 . 2 , the unique solution of (1 . 1) is

$$x(t) = integraldisplay - minus_{\infty}^{t} Y(t)P_1Y^{-1}(s)f(s)ds - \int_t^{\infty} Y(t)P_2Y^{-1}(s)f(s)ds.$$

$$\begin{aligned} \parallel \psi(t)x(t) \parallel &\leqslant integraldisplay - minus_{\infty}^{t} \parallel \psi(t)Y(t)P_1Y^{-1}(s)f(s) \parallel ds + \int_t^{\infty} \parallel \psi(t)Y(t)P_2Y^{-1}(s)f(s) \parallel ds \\ &\leqslant K integraldisplay - minus_{\infty}^{t} e^{-\alpha(t-s)} \parallel \psi(s)f(s) \parallel ds + L\int_t^{\infty} e^{-\beta(s-t)} \parallel \psi(s)f(s) \parallel ds \\ &\leqslant K_1\{integraldisplay - minus_{\infty}^{t} e^{-\alpha(t-s)} \parallel \psi(s)f(s) \parallel ds + \int_t^{\infty} e^{-\beta(s-t)} \parallel \psi(s)f(s) \parallel ds\}, \end{aligned}$$

(3 . 6)

where $K1 = \max\{K, L\}$. Denote by $\gamma = \min\{\alpha, \beta\}$. Under the hypothesis (3 . 4) , for a given $\varepsilon > 0$, there exists $T > 0$ such that

$$\int_t^{t+1} \parallel \psi(s)f(s) \parallel ds < \frac{\varepsilon}{2K_1}(1 - e^{-\gamma}) \quad \text{for } | t |> T.$$

Then from Lemma 2 . 3 and inequality (3 . 6) it follow that

$$\begin{aligned} \parallel \psi(t)x(t) \parallel &\leqslant K_{1\frac{\varepsilon}{2K}}^{\ (1}{}_{1} - e^{-\gamma})[(1 - e^{-\alpha})^{-1} + (1 - e^{-\beta})^{-1}] \\ &\leqslant K_{1\frac{\varepsilon}{2K}}^{\ (1}{}_{1} - e^{-\gamma})2(1 - e^{-\gamma})^{-1} = \varepsilon \quad \text{forall } | t |> T, \end{aligned}$$

this implies (3 . 5) . The proof is complete . □ **Corollary 3 . 4 .** *Suppose that (1 . 2) has a $\psi-$ exponential dichotomy on $\mathbb{R}$. If*

$$\lim_{\to_t \pm\infty} \parallel \psi(t)f(t) \parallel = 0 \tag{3.7}$$

then the $\psi-$ bounded s o lution of (1 . 1) is such that

$$\lim_{\to_t \pm\infty} \parallel \psi(t)x(t) \parallel \ = 0. \tag{3.8}$$

Proof . It is easy to see that (3 . 7) implies (3.4) □ Now , we consider the perturbed equation

$$x'(t) = [A(t) + B(t)]x(t) \tag{3.9}$$

where $B(t)$ is a $d \times d$ continuous matrix function on $\mathbb{R}$. We have the following result . **Theorem 3 . 5 .** *Suppose that (1 . 2) has a $\psi-$ exponential dichotomy on $\mathbb{R}$. If $\delta =$ $\sup_{t\in\mathbb{R}} \int_t^{t+1} \mid \psi(s)B(s)\psi^{-1}(s) \mid ds$ is sufficiently small , then (3 . 9) has a $\psi-$ exponential dichotomy on $\mathbb{R}$.*

Proof . By Theorem 3 . 2 it suffices to show that the equation

$$x'(t) = [A(t) + B(t)]x(t) + f(t) \tag{3.10}$$

has a unique $\psi-$ bounded solution on $\mathbb{R}$ for every $\psi-$ integrally bounded f function on $\mathbb{R}$. Denote by G_ψ the set

$$G_\psi = \{x : \mathbb{R} \to \mathbb{R}^d : x \text{ is } \psi- \text{ bounded and continuous on } \mathbb{R}\}.$$

 It is well - known that G_ψ is a real Banach space with the norm

$$\| x \| G_\psi = \sup_{t\in R} \| \psi(t)x(t) \| .$$

Consider the mapping $T : G_\psi \to G_\psi$ which is defined by

$$\begin{aligned} Tz(t) &= -^t\infty Y(t)P_1Y^{-1}(s)[B(s)z(s) + f(s)]ds \\ &- \int_t^{\infty} Y(t)P_2Y^{-1}(s)[B(s)z(s) + f(s)]ds. \end{aligned}$$

It is easy verified that $Tz \in G_\psi$. More ever if $z_1, z_2 \in G_\psi$ then

$$\begin{aligned} &\| Tz_1 - Tz_2 \| G_\psi \\ \leqslant\ & integraldisplay - minus^t_\infty \mid \psi(t)Y(t)P_1Y^{-1}(s)\psi^{-1}(s) \mid\mid \psi(s)B(s)\psi^{-1}(s) \mid\mid\mid \psi(s)z_1(s) - \psi(s)z_2(s) \| ds \\ &+ \int_t^{\infty} \mid \psi(t)Y(t)P_2Y^{-1}(s)\psi^{-1}(s) \mid\mid \psi(s)B(s)\psi^{-1}(s) \mid\mid\mid \psi(s)z_1(s) - \psi(s)z_2(s) \| ds \end{aligned}$$

By Lemma 2 . 3 , we have

$$\begin{aligned} \| Tz_1 - Tz_2 \| G_\psi &\leqslant K \| z_1 - z_2 \| G_\psi -^t \infty e^{-\alpha(t-s)} \mid \psi(s)B(s)\psi^{-1}(s) \mid ds \\ &+L \| z_1 - z_2 \| G_\psi \int_t^{\infty} e^{\beta(t-s)} \mid \psi(s)B(s)\psi^{-1}(s) \mid ds \\ &\leqslant \delta[K(1 - e^{-\alpha})^{-1} + L(1 - e^{-\beta})^{-1}] \| z_1 - z_2 \| G_\psi \end{aligned}$$

Hence , by the contraction principle , if $\delta[K(1 - e^{-\alpha})^{-1} + L(1 - e^{-\beta})^{-1}] < 1$, then the mapping T has a unique fixed point . Denoting this fixed point by z, we have

$$\begin{aligned} z(t) &= integraldisplay - minus^t_\infty Y(t)P_1Y^{-1}(s)[B(s)z(s) + f(s)]ds \\ &- \int_t^{\infty} Y(t)P_2Y^{-1}(s)[B(s)z(s) + f(s)]ds. \end{aligned}$$

It follows that $z(t)$ is a solution on $\mathbb{R}$ of (3 . 1 0) .

Now , we prove the uniqueness of this solution . Suppose that $x(t)$ is a arbitrary $\psi-$ bounded solution on $\mathbb{R}$ of (3 . 1 0) . Consider the function

$$\begin{aligned} y(t) = x(t) &- integraldisplay - minus^t_\infty Y(t)P_1Y^{-1}(s)[B(s)x(s) + f(s)]ds \\ &+ \int_t^{\infty} Y(t)P_2Y^{-1}(s)[B(s)x(s) + f(s)]ds. \end{aligned}$$

It is easy to see that $y(t)$ is a $\psi-$ bounded solution on $\mathbb{R}$ of (1 . 2) . Then from Theorem 3 . 2 follows that $y(t)$ is the trivial solution . Then

$$\begin{aligned} x(t) &= -^t\infty Y(t)P_1Y^{-1}(s)[B(s)x(s) + f(s)]ds \\ &- \int_t^{\infty} Y(t)P_2Y^{-1}(s)[B(s)x(s) + f(s)]ds. \end{aligned}$$

Hence $x(t)$ is the fixed point of mapping T. From the uniqueness of this point , it follows that $x = z$. The proof is complete . $\square$

Corollary 3 . 6 . *Suppose that* (1 . 2) *has a* $\psi-$ *exponential dichotomy on* $\mathbb{R}$. *If* $\delta =$
$\sup_{t\in\mathbb{R}} \mid \psi(t)B(t)\psi^{-1}(t) \mid$ *is sufficiently small , then* (3 . 9) *has a* $\psi-$ *exponential di - chotomy on* $\mathbb{R}$.

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