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A MINIMAX INEQUALITY FOR A CLASS OF FUNCTIONALS AND APPLICATIONS TO THE EXISTENCE OF SOLUTIONS

FOR TWO - POINT BOUNDARY - VALUE PROBLEMS

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Abstract . In this paper , we establish an equivalent statement to minimax inequality for a special class of functionals . As an application , we prove the existence of three solutions to the Dirichlet problem

$$u''_{-}(x) + m(x)u(x) = \lambda f(x, u(x)), \quad x \in (a, b),$$
$$u(a) = u(b) = 0,$$

where $\lambda > 0, f : [a, b] \times \mathbb{R} \to \mathbb{R}$ is a continuous function which changes sign on $[a, b] \times \mathbb{R}$ and $m(x) \in C([a, b])$ is a positive function .

1 . Introduction

Given two G $\hat{a}$ teaux differentiable functionals Φ and T on a real Banach space X, the minimax inequality

$$\sup_{\lambda \geq 0} \inf_{u \in X} {}^{(\Phi(u)} + \lambda(\rho - T(u))) < \inf_{u \in X} \sup_{\lambda \geq 0} (\Phi(u) + \lambda(\rho - T(u))), \quad \rho \in \mathbb{R}, \tag{1.1}$$

plays a fundamental role for establishing the existence of at least three critical points for the functional $\Phi(u) - \lambda T(u)$.

In this work some conditions that imply the minimax inequality (1 . 1) are pointed out and equivalent formulations are proved .

In this paper , our approach is based on a three critical - point theorem proved in [8] (Theorem 2 . 1) which stated below for the reader ' s convenience . Also we state a technical lemma that enables us to apply the theorem .

Lemma 2 . 2 below establishes an equivalent statement of minimax inequality (1 . 1) for a special class of functionals , while its consequences (Lemmas 2 . 5 and 2 . 7) guarantee some conditions so that minimax inequality holds .

Finally , we apply Theorem 2 . 1 to elliptic equations , by using an immediate con - sequence of Lemma 2 . 2 : We consider the boundary - value problem

$$-u''(x) + m(x)u_u^{(x)}{}_{(a)} == \lambda_u^{f(x,}{}_{(b)=0,}^{u\,(x)),} \quad x \in (a, b), \quad (1.2)$$

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where $\lambda > 0, f : [a,b] \times \mathbb{R} \to \mathbb{R}$ is a continuous function which changes sign on $[a,b] \times \mathbb{R}, m$ is a continuous , positive function and we establish some conditions on f so that problem (1 . 2) admits at least three weak solutions .

We say that u is a weak solution to (1 . 2) if $u \in W_0^{1,2}([a,b])$ and

$$\int_a^b u'(x)v'(x)dx + \int_a^b m(x)u(x)v(x)dx - \lambda \int_a^b f(x,u(x))v(x)dx = 0$$
$$\text{forevery} v \in W_0^{1,2}([a,b]).$$

By arguments similar to those in problem (1 . 2) , we will have the existence of at least three weak solutions for the problem

$$-u''(x) + m(x)u(^{x)=}_{u\ (a)=}\lambda h_u^{1(x)}{}^{h_2\ (u(x)),}_{(b)=}{}_{0}, \quad x \in (a,b) \tag{1.3}$$

where $h_1 \in C([a,b])$ is a function which changes sign on $[a,b]$ and $h_2 \in C(\mathbb{R})$ is a positive function . The existence of at least three weak solutions is also proved for the problem

$$-u''(x) + m(x_u^{)u(x)}{}_{(a)=} = u^{\lambda f(u(x)),}_{(b)\ =0,} \quad x \in (a,b) \tag{1.4}$$

where $f : \mathbb{R} \to \mathbb{R}$ is a continuous function which changes sign on $\mathbb{R}$.

Conditions that guarantee the existence of multiple solutions to differential equa - tions are of interest because physical processes described by differential equations can exhibit more that one solution . For example , certain chemical reactions in tubu - lar reactors can be mathematically described by a nonlinear , two - point boundary value problem with the interest in seeing if multiple steady - states to the problem exist . For a recent treatment of chemical reactor theory and multiple solutions see [2 , section 7] and references therein .

In recent years , many authors have studied multiple solutions from several points of view and with different approaches and we refer to [1 , 3 , 4 , 7] and the references therein for more details , for instance , in their interesting paper [3] , the authors studied problem

$$u^{u''}_{(0)}{}^{+}\lambda^{f\ (u)}_{=u(1)} =_= 0^{;}_{0}, \tag{1.5}$$

(in the case independent of λ) by using a multiple fixed - point theorem to obtain three symmetric positive solutions under growth conditions on f.

Also , in [4] , the author proves multiplicity results for the problem (1 . 5) which for each $\lambda \in [0,+\infty[$, admits at least three solutions in $W_0^{1,2}([0,1])$ when f is a continuous function .

In particular , in [1] we obtained the existence of an interval $\Lambda \subseteq [0,+\infty[$ and a positive real number q such that , such that for each $\lambda \in \Lambda$ problem

$$\Delta_p u + \lambda f(x_u, u_=) = 0 a_{\text{on}}(x)^{|u|^{p-2}u}_{\partial}{}_{\Omega}, \quad \text{in}\Omega, \tag{1.6}$$

where $\Delta_p u = \text{div}\ (|\ \nabla u\ |^{p-2}\ \nabla u)$ is the $p-$ Laplacian operator , $\Omega \subset \mathbb{R}^N (N \geq 2)$ is non -

empty bounded open set with smooth boundary $\partial\Omega, p > N, \lambda > 0, f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a continuous function and positive weight function $a(x) \in C(\overline{\quad\quad}\Omega)$, admits at least three weak solutions whose norms in $W_0^{1,p}(\Omega)$ are less than q.

 For additional approaches to the existence of multiple solutions to boundary - value problems , see [2 , 5 , 6] and references therein .

2 . Main results

First , we recall the three critical point theorem by Ricceri [8] when choosing

$$h(\lambda) = \lambda\rho.$$

Theorem 2 . 1 . *Let X be a s eparable and reflexive real Banach space ; $\Phi : X \to \mathbb{R}$ a continuously G â teaux differentiable and s equentially weakly lower s emicontinuous functional whose G â t eaux derivative admits a continuous inverse on X^*; $\Psi : X \to \mathbb{R}$ a continuously G â t eaux differentiable functional whose G â teaux derivative is com - pact . Assume that*

$$\| u \lim \|_{\to +\infty} (\Phi(u) + \lambda\Psi(u)) = +\infty$$

for al l $\lambda \in [0, +\infty[$, and that th ere exists $\rho \in \mathbb{R}$ such that

$$\sup_{\lambda \geq 0} \inf_{u \in X}^{(\Phi(u)} +\lambda\Psi(u) + \lambda\rho) < \inf_{u \in X} \sup_{\lambda \geq 0} (\Phi(u) + \lambda\Psi(u) + \lambda\rho).$$

Then , there exists an open interval $\Lambda \subseteq [0, +\infty[$ and a positive real number q such that , for each $\lambda \in \Lambda$, the equation

$$\Phi'(u) + \lambda\Psi'(u) = 0$$

has at least three s o lutions in X whose norms are less than q.

Here and in the sequel , X will denote the Sobolev space $W_0^{1,2}([a, b])$ with the norm

$$\| u \| := (\int_a^b | u'(x) |^2 \, dx)1/2,$$

$f : [a, b] \times \mathbb{R} \to \mathbb{R}$ is a continuous function and $g : [a, b] \times \mathbb{R} \to \mathbb{R}$ is defined by

$$g(x, t) = \int_0^t f(x, \xi) d\xi$$

for each $(x, t) \in [a, b] \times \mathbb{R}$. Now , we define

$$\| u \| * := (\int_a^b (| u'(x) |^2 + m(x) \mid u(x) |^2) dx)1/2.$$

So the Poincar é ' s inequality and the positivity of the function $m(x) \in C([a, b])$, there exist positive suitable constants c_1 and c_2 such that

$$c_1 \| u \| \leq \| u \|_* \leq c_2 \| u \| \tag{2.1}$$

(i . e . , the above norms are equivalent) . We now introduce two positive special functionals on the Sobolev space X : For $u \in X$, let

$$\Phi(u) := \frac{\| u \|_*^2}{2},$$

$$T(u) := \int_a^b g(x, u(x)) dx$$

 Let $\rho, r \in \mathbb{R}, w \in X$ be such that $0 < \rho < T(w)$ and $0 < r < \Phi(w)$. We put

$$\beta 1(\rho, w) := \rho \frac{\Phi(w)}{T(w)}, \tag{2.2}$$

$$\beta 2(r, w) := r \frac{T(w)}{\Phi(w)}, \tag{2.3}$$

$$\beta 3(\rho, w) := \frac{1}{c_1} (\frac{b-a}{2} \beta 1(\rho, w))^{1/2}, \tag{2.4}$$

Clearly , $\beta 1(\rho, w), \beta 2(r, w)$ and $\beta 3(\rho, w)$ are positive . Now , we put

$$\delta_1 := \inf\{\frac{(b-a)^{1/2}}{2c_1} \parallel u \parallel_* \in \mathbb{R}^+; T(u) \geq \rho\},$$

$$\delta_2 := \inf \{\tfrac{(b-a)^{1/2}}{2c_1} \parallel u \parallel_* \in \mathbb{R}^+, \quad \text{such that}$$

$$(b-a)(x,t) \in [a,b] \times [\max_{\|u\|} -\frac{(b-a)^{1/2}}{2c_1} *, \frac{(b-a)^{1/2}}{2c_1} \parallel u \parallel_*] g(x,t) \geq \rho\}$$

and

$$\delta_\rho := \delta_1 - \delta_2. \tag{2.5}$$

Clearly , $\delta_1 \geq \delta_2$. Taking into account that for every $u \in X$,

$$\max_{x \in [a,b]} \mid u(x) \mid \leq \frac{(b-a)^{1/2}}{2} \parallel u \parallel$$

and (2 . 1) , we have

$$\max_{x \in [a,b]} \mid u(x) \mid \leq \frac{(b-a)^{1/2}}{2c_1} \parallel u \parallel_*$$

for each $u \in X$. So that

$$T(u) = \int_a^b g(x, u(x)) dx \leq (b-a) \max g(x,t)$$

$$\text{where}(x,t) \in [a,b] \times [-\frac{(b-a)^{1/2}}{2c_1} \parallel u \parallel *, \quad \frac{(b-a)^{1/2}}{2c_1} \parallel u \parallel_*]. \text{Namely}$$

$$T(u) \leq (b-a) \max g(x,t),$$

where$(x,t) \in \quad [a,b] \times [-\frac{(b-a)^{1/2}}{2c_1} \parallel u \parallel * \quad , \quad \frac{(b-a)^{1/2}}{2c_1} \parallel u \parallel *]; \therefore, \{\frac{(b-a)^{1/2}}{2c_1} \parallel u \parallel * \quad \in$

$\mathbb{R}^+; T(u) \geq \rho\}$ is a subset of

$$\{\frac{(b-a)^{1/2}}{2c_1} \parallel u \parallel_* \in \mathbb{R}^+ \text{suchthat}$$

$$(b-a)(x,t) \in [a,b] \times [\max_{\|u\|} -\frac{(b-a)^{1/2}}{2c_1} *, \frac{(b-a)^{1/2}}{2c_1} \parallel u \parallel *] g(x,t) \geq \rho\}.$$

So , we have $\delta_1 \geq \delta_2$ and $\delta_\rho \geq 0$.

Our main results depend on the following lemma :

Lemma 2 . 2 . *Assume that there exist* $\rho \in \mathbb{R}, w \in X$ *such that*

$$\text{(i)} \quad 0 < \rho < T(w),$$

(i i) $(b-a) \max_{(x,t)\in[a,b]\times[-\beta3(\rho,w)+\delta_\rho,} \beta3(\rho,w)-\delta_\rho] g(x,t) \quad < \rho,$ *where* $\beta3(\rho,w)$ *is given by* (2 . 4) *and* δ_ρ *by* (2 . 5) .

Then , there exists $\rho \in \mathbb{R}$ *such that*

$$\sup_{\lambda \geq 0} \inf_{u \in X} (\Phi(u) + \lambda(\rho - T(u))) < \inf_{u \in X} \sup_{\lambda \geq 0} (\Phi(u) + \lambda(\rho - T(u))).$$

Proof . From (i i) , we obtain

$$\beta 3(\rho, w) - \delta_\rho element - slash\{l \in \mathbb{R}^+ : (b-a)(x,t) \in^{\max} [a,b] \times [-l,l] g(x,t) \geq \rho\}.$$

Moreover

$$\inf\{l \in \mathbb{R}^+; (b-a)(x,t) \in^{\max} [a,b] \times [-l,l] g(x,t) \geq \rho\} \geq \beta 3(\rho, w) - \delta_\rho;$$

in fact , arguing by contradiction , we assume that there is $l_1 \in \mathbb{R}^+$ such that

$$(b-a)(x,t) \in [\max_{a,b] \times [-} l_1, l_1] g(x,t) \geq \rho$$

and

$$l_1 < \beta 3(\rho, w) - \delta_\rho,$$

so

$$(b-a)(x,t) \in [a,b] \times [-\beta 3(\max \rho, w) + \delta_\rho, \beta 3(\rho, w) - \delta_\rho] g(x,t) \geq (b-a)(x,t) \in [\max_{a,b] \times [} -l_1, l_1] g(x,t) \geq \rho.$$

This is a contradiction . So

$$\inf\{l \in \mathbb{R}^+; (b-a)(x,t) \in^{\max} [a,b] \times [-l,l] g(x,t) \geq \rho\} > \beta 3(\rho, w) - \delta_\rho.$$

Therefore ,

$$\begin{aligned} &\inf\{\frac{(b-a)^{1/2}}{2c_1} \parallel u \parallel * \in \mathbb{R}^+ : \\ &(b-a)(x,t) \in [a,b] \times [\max_{\|u\|} -\frac{(b-a)^{1/2}}{2c_1} *, \frac{(b-a)^{1/2}}{2c_1} \parallel u \parallel_*] g(x,t) \geq \rho\} \\ &> \beta 3(\rho, w) - \delta_\rho; \end{aligned}$$

namely $\beta 3(\rho, w) < \delta_1$. So , we have

$$\inf\{\frac{\parallel u \parallel_*^2}{2} \in \mathbb{R}^+; T(u) \geq \rho\} > \beta 1(\rho, w),$$

or equivalently

$$\inf\{\Phi(u); \quad u \in T^{-1}([\rho, +\infty[)\} > \rho \frac{\Phi(w)}{T(w)},$$

and , taking in to account that (i) holds , one has

$$\frac{\inf\{\Phi(u); \quad u \in T^{-1}([\rho, +\infty[)\}}{\rho} > \frac{\Phi(w) - \inf\{\Phi(u); \quad u \in T^{-1}([\rho, +\infty[)\}}{T(w) - \rho}.$$

Now , there exists $\lambda \in \mathbb{R}$ such that

$$\lambda > \frac{\Phi(w) - \inf\{\Phi(u); \quad u \in T^{-1}([\rho, +\infty[)\}}{T(w) - \rho}$$

and

$$\lambda < \inf_{-}^{\{\Phi(} u); \quad u \in T^{-1}([\rho, +\infty[)\}.$$

$$\rho$$

or equivalently

$$\inf\{\Phi(u); \quad u \in T^{-1}([\rho, +\infty[)\} > \Phi(w) + \lambda(\rho - T(w))$$

 and

$$\lambda\rho < \inf\{\Phi(u); \quad u \in T^{-1}([\rho, +\infty[)\}.$$

Therefore , thanks to the $0 < \rho < T(w)$, we obtain

$$\inf_{u\in X}^{(\Phi(u)} +\lambda(\rho - T(u))) < \inf\{\Phi(u); \quad u \in T^{-1}([\rho, +\infty[)\}. \tag{2.6}$$

On other hand ,

$$\inf_{u\in X}^{(\Phi(u)} +\lambda(\rho - T(u))) \le (\Phi(0) + \lambda(\rho - T(0))) = \lambda\rho. \tag{2.7}$$

So , with (2 . 6) and (2 . 7) , one has

$$\sup_{\lambda\ge 0} {}^{\inf}_{u} \in X (\Phi(u) + \lambda(\rho - T(u))) < \inf\{\Phi(u); \quad u \in T^{-1}([\rho, +\infty[)\}.$$

Therefore , thanks to the

$$\inf_{u\in X} \sup_{\lambda\ge 0}(\Phi(u) + \lambda(\rho - T(u))) = \inf\{\Phi(u); \quad u \in T^{-1}([\rho, +\infty[)\},$$

we have

$$\sup_{\lambda\ge 0} {}^{\inf}_{u} \in X (\Phi(u) + \lambda(\rho - T(u))) < \inf_{u\in X} \sup_{\lambda\ge 0}(\Phi(u) + \lambda(\rho - T(u))).$$

□

Remark 2 . 3 . Note that $\sup_{\lambda\ge 0}\inf_{u\in X}(\Phi(u)+\lambda(\rho-T(u)))$ is well defined , because $\lambda \to \inf_{u\in X}(\Phi(u) + \lambda(\rho - T(u)))$ is upper semicontinuous in $[0, +\infty[$ and tends to

$$-\infty \text{as} \lambda \to +\infty.$$

Remark 2 . 4 . If $\beta 3(\rho, w) - \delta_\rho \le 0$ in Lemma 2 . 2 , ; then then the lemma still holds .
Because , $\beta 3(\rho, w) \le \delta_1 - \delta_2 \le \delta_1$, and by arguing as in the proof of Lemma 2 . 2 , the results holds .
If instead of condition (ii) in Lemma 2 . 2 , we put

$$(b - a)(x, t) \in [a, b] \times [\max_{-\beta 3(\rho,w)}, \beta 3(\rho, w)]g(x, t) < \rho,$$

then the result holds , because

$$\begin{aligned}(b - a)(x, t) \in [a, b] \times [-\beta 3(\max \rho, w) + \delta_\rho, \beta 3(\rho, w) - \delta_\rho]g(x, t) \\ \le (b - a)(x, t) \in [a, b] \times [\max_{-\beta 3(\rho,w}), \beta 3(\rho, w)]g(x, t) < \rho.\end{aligned}$$

So , we have the following result .
Lemma 2 . 5 . *Assume that there exist* $\rho \in \mathbb{R}$, $w \in X$ *such that*

$$\text{(i)} \quad 0 < \rho < T(w),$$

(i i) $(b - a) \max_{(x,t)\in[a,b]} \times[-\beta 3(\rho, w), \beta 3(\rho, w)]g(x, t) < \rho,$ *where* $\beta 3(\rho, w)$ *is given* *by* (2 . 4) *Then , there exists* $\rho \in \mathbb{R}$ *such that*

$$\sup_{\lambda \geq 0} {}^{\inf}_{u} {}_{\in X} (\Phi(u) + \lambda(\rho - T(u))) < \inf_{u \in X} \sup_{\lambda \geq 0} (\Phi(u) + \lambda(\rho - T(u))).$$

Proposition 2 . 6 . *The fo l lowing ass ertions are equivalent :* (a) *There are* $\rho \in \mathbb{R}, w \in X$ *such that*

$$(i) 0 < \rho < T(w),$$

(ii) $(b-a) \max_{(x,t) \in [a,b]} \times [-\beta 3(\rho, w), \beta 3(\rho, w)] g(x,t) < \rho,$ *where* $\beta 3(\rho, w)$ *is given by* (2 . 4) .

 (b) *There are* $r \in \mathbb{R}, w \in X$ *such that*

$$(iii) 0 < r < \Phi(w),$$

$$(iv) \quad (b-a) \max_{(x,t)\in[a,b]} \times [-\frac{1}{c_1}\sqrt{\frac{b-a}{2}r}, \frac{1}{c_1}\sqrt{\frac{b-a}{2}r}] g(x,t) \quad < \quad \beta 2(r,w), \quad where$$

$\beta 2(r,w)$ *is given by* (2 . 3) . *Proof .* (a) $\Rightarrow$ (b) . First we note that $0 < \Phi(w)$, because if $\Phi(w) = 0$, one has $\frac{(b-a)^{1/2}}{2c_1} \parallel w \parallel * = 0$. Hence , taking into account (ii) , one has

$$T(w) \leq (b-a)(x,t) \in [a,b] \times [\max -\frac{(b-a)^{1/2}}{2c_1}{}_{\parallel w} \parallel_*, \frac{(b-a)^{1/2}}{2c_1} \parallel w \parallel *] g(x,t) = 0,$$

and that is in contradiction to (i) . We now put $\beta 1(\rho, w) = r$. We obtain $\rho = \beta 2(r,w)$ and $\beta 3(\rho,w) = \frac{1}{c_1}\sqrt{\frac{b-a}{2}r}$. Therefore , from (i) and (i i) , one has $0 < r < \Phi(w)$ and

$$(b-a)(x,t) \in [a,b] \times [\max -\frac{1}{c_1}\sqrt{\frac{b-a}{2}r}, \frac{1}{c_1}\sqrt{\frac{b-a}{2}r}] g(x,t) < \beta 2(r,w).$$

(b) $\Rightarrow$ (a) First we note that $0 < T(w)$, because if $0 \geq T(w)$, from (i ii) one has $r\frac{T(w)}{\Phi(w)} \leq 0$; namely , $\beta 2(r,w) \leq 0$. Hence , from (iv) one has

$$0 = T(0) \leq (b-a)(x,t) \in [a,b] \times [\max -\frac{1}{c_1}\sqrt{\frac{b-a}{2}r}, \frac{1}{c_1}\sqrt{\frac{b-a}{2}r}] g(x,t) < 0,$$

and this is a contradiction . We now put $\beta 2(r,w) = \rho$. We obtain $r = \beta 1(\rho,w)$ and $\frac{1}{c_1}\sqrt{\frac{b-a}{2}r} = \beta 3(\rho,w)$. Therefore , from (i ii) and (iv) , we have the conclusion . □

The following lemma is another consequence of Lemma 2 . 2 .

Lemma 2 . 7 . *Assume that there exist* $r \in \mathbb{R}$, $w \in X$ *such that*

$$\text{(i)} \quad 0 < r < \Phi(w),$$

$$\text{(i i)} \quad (b-a)\max_{(x,t)\in[a,b]\times[-\frac{1}{c_1}\sqrt{\frac{b-a}{2}r}}, \frac{1}{c_1}\sqrt{\frac{b-a}{2}r}] g(x,t) < \beta 2(r,w), \ where$$

$\beta 2(r,w)$
is given by (2 . 3) . *Then , there exists* $\rho \in \mathbb{R}$ *such that*

$$\sup_{\lambda \geq 0} \inf_{u} {}_{\in X} (\Phi(u) + \lambda(\rho - T(u))) < \inf_{u\in X} \sup_{\lambda\geq 0} (\Phi(u) + \lambda(\rho - T(u))).$$

The above lemma follows from Lemma 2 . 5 and Proposition 2 . 6 .

Finally , we are interested in ensuring the existence of at least three weak solutions for the Dirichlet problem (1 . 2) . Now , we have the following result .

Theorem 2 . 8 . *Assume that th ere exist* $\rho \in \mathbb{R}, a_1 \in L^1([a,b]), w \in X$ *and a positive*
constant γ *with* $\gamma < 2$ *such that*

$$\text{(i)} \quad 0 < \rho < \int_a^b g(x,w(x))dx,$$

$$\text{(ii)} \quad (b-a) \max_{(x,t)\in[a,b]\times[-\beta 3(\rho,w)}, \beta 3(\rho,w)] g(x,t) < \rho$$

(ii i) $g(x,t) \leq a_1(x)(1+ \mid t \mid^{\gamma})$ *almost everywhere in* $[a,b]$ *and for ea ch* $t \in \mathbb{R}$,

where $\beta 3(\rho, w)$ *is given by* (2 . 4) .

Then , there exists an open interval $\Lambda \subseteq [0,+\infty[$ *and a positive real number* q *such that , for each* $\lambda \in \Lambda$, *problem* (1 . 2) *admits at least three s o lutio ns in* X *whose norms are less than* q.

 Proof . For each $u \in$ X , we put

$$\Phi(u) = \frac{\| u \|_*^2}{2},$$
$$\Psi(u) = -\int_a^b g(x, u(x))dx.$$
$$J(u) = \Phi(u) + \lambda\Psi(u).$$

In particular , for each $u, v \in X$ one has

$$\Phi'(u)(v) = \int_a^b (u'(x)v'(x) + m(x)u(x)v(x))dx, \Psi'(u)(v) = -\int_a^b f(x, u(x))v(x)dx.$$

It is well known that the critical points of J are the weak solutions of (1 . 2) , our goal is to prove that Φ and Ψ satisfy the assumptions of Theorem 2 . 1 . Clearly , Φ is a continuously G $\hat{a}$ teaux differentiable and sequentially weakly lower semi continuous functional whose G $\hat{a}$ teaux derivative admits a continuous inverse on X^* and Ψ is a continuously G $\hat{a}$ teaux differentiable functional whose G $\hat{a}$ teaux derivative is compact .

Thanks to (i ii) , for each $\lambda > 0$ one has

$$\| u \lim \|_{\to+\infty}(\Phi(u) + \lambda\Psi(u)) = +\infty.$$

Furthermore , thanks to Lemma 2 . 5 , from (i) and (i i) , we have

$$\sup_{\lambda\geq 0} \inf_{u\in X}^{(\Phi(u)} +\lambda\Psi(u) + \lambda\rho) < \inf_{u\in X} \sup_{\lambda\geq 0}(\Phi(u) + \lambda\Psi(u) + \lambda\rho).$$

Therefore , we can apply Theorem 2 . 1 . It follows that there exists an open interval $\Lambda \subseteq [0, +\infty[$ and a positive real number q such that , for each $\lambda \in \Lambda$, problem (1 . 2) admits at least three solutions in X whose norms are less than q. $\square$

We also have the following existence result .

Theorem 2 . 9 . *Assume that there exist* $r \in \mathbb{R}, a_2 \in L^1([a, b]), w \in X$ *and a positive constant* γ *with* $\gamma < 2$ *such that*

(i) $0 < r < \frac{\| w \|_*^2}{2}$;

(ii) $(b - a) \max_{(x,t)\in[a,b]\times[-\frac{1}{c_1}\sqrt{\frac{b-a}{2}r}}, \frac{1}{c_1}\sqrt{\frac{b-a}{2}}r]g(x, t) < \beta 2(r, w)$;

(ii i) $g(x, t) \leq a_2(x)(1+ | t |^\gamma)$ *almost everywhere in* $[a, b]$ *and for ea ch* $t \in \mathbb{R}$,

where $\beta 2(r, w)$ *is given by* (2 . 3) .

Then , there exists an open interval $\Lambda \subseteq [0, +\infty[$ *and a positive real number* q *such that , for each* $\lambda \in$ Λ, *pro b lem* (1 . 2) *admits at least three s o lutions in* X *whose norms are less than* q.

The above theorem follows from Lemma 2 . 7 and Theorem 2 . 8 .

Let $h_1 \in C([a, b])$ be a function which changes sign on $[a, b]$ and $h_2 \in C(\mathbb{R})$ be a positive function . For for $(x, t) \in [a, b] \times \mathbb{R}$, put

$$f(x, t) = h_1(x)h_2(t).$$

For for $t \in \mathbb{R}$, put

$$\alpha(t) = \int_0^t h_2(\xi) d\xi.$$

 For almost every $x \in [a,b]$, put

$$a_3(x) = \frac{a_1(x)}{h_1(x)}$$

Then , using Theorem 2 . 8 , we have the following result . **Theorem 2 . 10 .** *Assume that there exist* $\rho \in \mathbb{R}$, $a_3 \in L^1([a,b])$, $w \in X$ *and a positive constant* γ *with* $\gamma < 2$ *such that*

$$\text{(i)} \quad 0 < \rho < \int_a^b (h_1(x)\alpha(w(x)))dx;$$

$$\text{(ii)} \quad (b-a)\max_{x\in[a,b]} h_1(x) < \frac{\rho}{\alpha(\beta 3(\rho,w))};$$

(ii i) $\alpha(t) \le a_3(x)(1+ \mid t \mid^{\gamma})$ *almost everywhere in* $[a,b]$ *and for each* $t \in \mathbb{R}$, *where*

$\beta 3(\rho,w)$ *is given by* (2 . 4) .

Then , there exists an open interval $\Lambda \subseteq [0,+\infty[$ *and a positive real number* q *such that , for each* $\lambda \in \Lambda$, *problem* (1 . 3) *admits at least three s o lutio ns in X whose norms are less than* q.

Put

$$a_4(x) = \frac{a_2(x)}{h_1(x)}$$

for almost every $x \in [a,b]$. Then , by Theorem 2 . 9 , we have the following existence result .

Theorem 2 . 1 1 . *Assume that th ere exist* $r \in \mathbb{R}$, $a_4 \in L^1([a,b])$, $w \in X$ *and a positive constant* γ *with* $\gamma < 2$ *such that*

$$\text{(i)} \quad 0 < r < \frac{\parallel w \parallel 2_*}{2};$$

$$\text{(ii)} \quad (b-a)\max_{x\in[a,b]} h_1(x) < \frac{\beta 2(r,w)}{\alpha(\frac{1}{c_1}\sqrt{\frac{b-a}{2}r})};$$

(ii i) $\alpha(t) \le a_4(x)(1+ \mid t \mid^{\gamma})$ *almost everywhere in* $[a,b]$ *and for each* $t \in \mathbb{R}$, *where*

$\beta 2(r,w)$ *is given by* (2 . 3) .

Then , there exists an open interval $\Lambda \subseteq [0,+\infty[$ *and a positive real number* q *such that , for each* $\lambda \in \Lambda$, *pro b lem* (1 . 3) *admits at least three s o lutions in* X *whose norms are less than* q.

We now want to point out two simple consequences of Theorems 2 . 8 and 2 . 9 . Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function which changes sign on $\mathbb{R}$. For $t \in \mathbb{R}$, put $g(t) = \int_0^t f(\xi)d\xi$. So we have the following results .

Theorem 2 . 12 . *Assume that there exist* $\rho \in \mathbb{R}, w \in X$ *and two positive constants* γ *and* η *with* $\gamma < 2$ *such that*

$$\text{(i)} \quad 0 < \rho < \int_a^b g(w(x))dx;$$

$$\text{(ii)} \quad (b-a)\max_{t\in[-\beta 3(\rho,w)}, \beta 3(\rho,w)]g(t) < \rho;$$

(ii i) $g(t) \le \eta(1+ \mid t \mid^{\gamma})$ *for ea ch* $t \in \mathbb{R}$, *where* $\beta 3(\rho,w)$ *is given by* (2 . 4) .

Then , there exists an open interval $\Lambda \subseteq [0,+\infty[$ *and a positive real number* q *such that , for each* $\lambda \in \Lambda$, *pro b lem* (1 . 4) *admits at least three s o lutions in* X *whose norms are less than* q.

Theorem 2 . 13 . *Assume that the re exist* $r \in \mathbb{R}, w \in X$ *and two positive constants* γ *and* μ *with* $\gamma < 2$ *such that*

$$\text{(i)} \quad 0 < r < \frac{\| w \| 2_*}{2};$$

$$\text{((ii)} \quad (b-a) \max_{t \in [-\frac{1}{c_1}\sqrt{\frac{b-a}{2}r},} \frac{1}{c_1}\sqrt{\frac{b-a}{2}r]} g(t) < \beta 2(r, w);$$

(ii i) $g(t) \leq \mu(1+ \mid t \mid^{\gamma})$ *for ea ch* $t \in \mathbb{R}$, *where* $\beta 2(r, w)$ *is given by* (2 . 3) . *Then , there exists an open interval* $\Lambda \subseteq [0, +\infty[$ *and a positive real number* q *such that , for each* $\lambda \in \Lambda$, *pro b lem* (1 . 4) *admits at least three s o lutions in* X *whose norms are less than* q.

Example 2 . 14 . Let $\Omega = (0, 1)$ and consider the problem

$$-u'' + e^x u = u^{\lambda(e^u \, u^2 \, (3}_{(0) \;\; = \, u(1)} +_{=} u)^{),}_{0}. \quad x \in (0, 1) \tag{2.8}$$

Then , there exists an open interval $\Lambda \subseteq [0, +\infty[$ and a positive real number q such that , for each $\lambda \in \Lambda$, problem (2 . 8) admits at least three solutions in $W_0^{1,2}([0, 1])$ whose norms are less than q. In fact , by choosing $\rho = \frac{1}{4}$ and

$$w(x) = \begin{cases} x, & x \in (0, 1) \\ 0, & \text{otherwise} \end{cases}$$

so that $\beta 3(\rho, w) = \frac{1}{c_1}(\frac{e-1}{96-32e})^{1/2}$, all assumptions of Theorem 2 . 1 2 , are satisfied with $\gamma = 1, c_1$ is positive constant such that the inequality (2 . 1) hold for $m(x) = e^x$ and η sufficiently large , also with choose $r = \frac{1}{2}$ so that $\beta 2(r, w) = \frac{6-2e}{e-1}$, all assumptions of Theorem 2 . 1 3 , are satisfied with μ sufficiently large .

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