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A SEMILINEAR ELLIPTIC PROBLEM INVOLVING NONLINEAR BOUNDARY CONDITION AND SIGN - CHANGING POTENTIAL

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ABSTRACT . In this paper , we study the multiplicity of nontrivial nonnegative solutions for a semilinear elliptic equation involving nonlinear boundary con - dition and sign - changing potential . With the help of the Nehari manifold , we prove that the semilinear elliptic equation :

$$
\begin{aligned}
-\Delta u + u &= \lambda f(x) \mid u \mid^{q-2} u \quad \text{in}\Omega, \\
\frac{\partial u}{\partial \nu} &= g(x) \mid u \mid^{p-2} u \quad \text{on}\partial\Omega,
\end{aligned}
$$

has at least two nontrivial nonnegative solutions for λ is sufficiently small .

1 . INTRODUCTION

In this paper , we consider the multiplicity of nontrivial nonnegative solutions for the following semilinear elliptic equation

$$
\begin{aligned}
-\Delta u + u &= \lambda f(x) \mid u \mid^{q-2} u \quad \text{in}\Omega, \\
\frac{\partial u}{\partial \nu} &= g(x) \mid u \mid^{p-2} u \quad \text{on}\partial\Omega,
\end{aligned} \tag{1.1}
$$

where $1 < q < 2 < p < \frac{2(N-1)}{N-2}, \lambda > 0, \Omega$ is a bounded domain in $\mathbb{R}^N$ with smooth boundary , $\frac{\partial}{\partial \nu}$ is the outer normal derivative and $f, g : \bar{\Omega} \to \mathbb{R}$ are continuous functions which change sign in $\bar{\Omega}$. Associated with (1 . 1) , we consider the energy

functional$J_\lambda$in$H^1(\Omega)$,

$$
J_\lambda(u) = \frac{1}{2} \parallel u \parallel_{H^1}^2 - \frac{\lambda}{q} \int_\Omega f \mid u \mid^q dx - \frac{1}{p} \int_{\partial\Omega} g \mid u \mid^p ds.
$$

where ds is the measure on the boundary and $\parallel u \parallel 2H_1 = \int_\Omega \mid \nabla u \mid^2 + u^2 dx$. It is well known that J_λ is of C^1 in $H^1(\Omega)$ and the solutions of equation (1 . 1) are the critical points of the energy functional J_λ.

The fact that the number of solutions of equation (1 . 1) is affected by the non - linear boundary conditions has been the fo cus of a great deal of research in recent years . Garcia - Azorero , Peral and Rossi [1 0] have investigated (1 . 1) when $f \equiv g \equiv 1$.

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They found that there exist positive numbers Λ_1, Λ_2 with $\Lambda_1 \leq \Lambda_2$ such that equa - tion (1 . 1) admits at least two positive solutions for $\lambda \in (0, \Lambda_1)$ and no positive solu - tion exists for $\lambda > \Lambda_2$. Also see Chipot - Chlebik - Fila - Shafrir [4] , Chipot - Shafrir - Fila [5] , Flores - del Pino [8] , Hu [1 1] , Pierrotti - Terracini [1 4] and Terraccini [1 6] where problems similar to equation (1 . 1) have been studied .

The purpose of this paper is to consider the multiplicity of nontrivial nonnegative solutions of equation (1 . 1) with sign - changing potential . We prove that equation (1 . 1) has at least two nontrivial nonnegative solutions for λ is sufficiently small .

Theorem 1 . 1 . *There exists* $\lambda_0 > 0$ *such that for* $\lambda \in (0, \lambda_0)$, *equation* (1 . 1) *has at least two nontrivial nonnegative s o lutions .*

Among the other interesting problems which are similar of equation (1 . 1) , Ambro - setti - Brezis - Cerami [3] have investigated the equation

$$-\Delta u = \lambda \,|_u\, u\,|^{q-}_{=}\,{}^{2}_{0}{}^{u} +_{\mathrm{on}}\, |\, u\,|^{p-}\, \partial^{2u}_{\Omega,} \quad \mathrm{in}\Omega, \tag{1.2}$$

where $1 < q < 2 < p \leq \frac{2N}{N-2}$. They proved that there exists $\lambda_0 > 0$ such that (1 . 2) admits at least two positive solutions for $\lambda \in (0, \lambda_0)$, has a positive solution for $\lambda = \lambda_0$, and no positive solution for $\lambda > \lambda_0$. Actually , Adimurthi - Pacella - Yadava [1] , Damascelli - Grossi - Pacella [6] , Ouyang - Shi [1 3] and Tang [1 7] proved that there exists $\lambda_0 > 0$ such that equation (1 . 2) in the unit ball $B^N(0;1)$ has exactly two positive solutions for $\lambda \in (0, \lambda_0)$, has exactly one positive solution for $\lambda = \lambda_0$ and no positive solution exists for $\lambda > \lambda_0$. Generalizations of the result of equation (1 . 2) were done by Ambrosetti - Azorero - Peral [2] , de Figueiredo - Gossez - Ubilla [9] and Wu [1 8] .

This paper is organized as follows . In section 2 , we give some notation and pre - liminaries . In section 3 , we prove that (1 . 1) has at least two nontrivial nonnegative solutions for λ is sufficiently small .

2 . Notation and Preliminaries

Throughout this section , we denote by S_p, C_p the best Sobolev embedding and trace constant for the operators $H^1(\Omega) \to L^p(\Omega), H^1(\Omega) \to L^p(\partial\Omega)$, respectively . Now , we consider the Nehari minimization problem : For $\lambda > 0$,

$$\alpha_\lambda = \inf\{J_\lambda(u) : u \in \mathbf{M}_\lambda\},$$

$$\text{where}\mathbf{M}_\lambda = \{u \in H^1(\Omega) \setminus \{0\} : \langle J'_\lambda(u), u\rangle = 0\}.\text{Define}$$

$$\psi\lambda(u) = \langle J'_\lambda(u), u\rangle = \| u \|^2_{H^1} - \lambda \int_\Omega f \mid u \mid^q dx - \int_{\partial\Omega} g \mid u \mid^p ds.$$

$$\text{Thenfor}u \in \mathbf{M}_\lambda,$$

$$\langle \psi'_\lambda(u), u\rangle = 2 \| u \|^2_{H^1} - \lambda q \int_\Omega f \mid u \mid^q dx - p \int_{\partial\Omega} g \mid u \mid^p ds.$$

Similarly to the method used in Tarantello [1 5] , we split $\mathbf{M}_\lambda$ into three parts :

$$\mathbf{M}^+_\lambda = \{u \in \mathbf{M}_\lambda : \langle \psi'_\lambda(u), u\rangle > 0\},$$
$$\mathbf{M}^0_\lambda = \{u \in \mathbf{M}_\lambda : \langle \psi'_\lambda(u), u\rangle = 0\},$$
$$\mathbf{M}^-_\lambda = \{u \in \mathbf{M}_\lambda : \langle \psi'_\lambda(u), u\rangle < 0\}.$$

Then , we have the following results .

Lemma 2 . 1 . *There exists* $\lambda_1 > 0$ *such that for ea ch* $\lambda \in (0, \lambda_1)$ *we have* $\mathbf{M}_\lambda^0 = \phi$. *Proof .* We consider the following two cases .
Case (I) : $u \in \mathbf{M}_\lambda$ and $\int_{\partial\Omega} g \mid u \mid^p ds \leq 0$. We have

$$\lambda \int_\Omega f \mid u \mid^q dx = \ \parallel u \parallel 2H_1 - \int_{\partial\Omega} g \mid u \mid^p ds.$$

Thus ,

$$\begin{aligned}\langle \psi_\lambda'(u), u\rangle &= 2 \parallel u \parallel_{H\ 1}^2 - \lambda q \int_\Omega f \mid u \mid^q dx - p \int_{\partial\Omega} g \mid u \mid^p ds \\ &= (2-q) \parallel u \parallel 2H_1 + (q-p) \int_{\partial\Omega} g \mid u \mid^p ds > 0 \\ &\qquad \text{andso} u \in \mathbf{M}_\lambda^+.\end{aligned}$$

Case (II) : $u \in \mathbf{M}_\lambda$ and $\int_{\partial\Omega} g \mid u \mid^p ds > 0$. Suppose that $\mathbf{M}_\lambda^0 \neq \phi$ for all $\lambda > 0$. If $u \in \mathbf{M}_{\lambda,}^0$ then we have

$$\begin{aligned}0 = \langle \psi_\lambda'(u), u\rangle &= 2 \parallel u \parallel_{H\ 1}^2 - \lambda q \int_\Omega f \mid u \mid^q dx - p \int_{\partial\Omega} g \mid u \mid^p ds \\ &= (2-q) \parallel u \parallel 2H_1 - (p-q) \int_{\partial\Omega} g \mid u \mid^p ds.\end{aligned}$$

Thus ,

$$\parallel u \parallel_{H\ 1}^2 = p\frac{-q}{2-q} \int_{\partial\Omega} g \mid u \mid^p ds \tag{2.1}$$

and

$$\lambda \int_\Omega f \mid u \mid^q dx = \parallel u \parallel_{H\ 1}^2 - \int_{\partial\Omega} g \mid u \mid^p ds = \frac{p-2}{2-q} \int_{\partial\Omega} g \mid u \mid^p ds. \tag{2.2}$$

Moreover ,

 where $K(p,q) = (^{line-minus-q2}_{p-q})^{(p-1)/(p-2)}(^{minus-line-p2}_{2-q})$.
Then $I_\lambda(u) = 0$ for all $u \in \mathbf{M}_\lambda^0$. Indeed ,
from (2 . 1) and (2 . 2) it follows that for $u \in \mathbf{M}_\lambda^0$ we have

$$I_\lambda(u) = K(p,q)\Big(\frac{\| u \|_{H^1}^{2(p-1)}}{\int_{\partial\Omega} g \mid u \mid^p ds}\Big) 1/(p-1) - \lambda \int_\Omega f \mid u \mid^q dx$$

 Proof . (i) Case (I) : $\int_{\partial\Omega} g \mid u \mid^p ds \leq 0$. We have

$$\lambda \int_{\Omega} f \mid u \mid^q dx = \quad \| u \| 2H_1 - \int_{\partial\Omega} g \mid u \mid^p ds > 0.$$

Case (II) : $\int_{\partial\Omega} g \mid u \mid^p ds > 0$. We have

$$\| u \| 2H_1 - \lambda \int_{\Omega} f \mid u \mid^q dx - \int_{\partial\Omega} g \mid u \mid^p ds = 0$$

and

$$\| u \|_{H\ 1}^2 > p \frac{-q}{2-q} \int_{\partial\Omega} g \mid u \mid^p ds.$$

Thus ,

$$\lambda \int_{\Omega} f \mid u \mid^q dx = \quad \| u \| 2H_1 - \int_{\partial\Omega} g \mid u \mid^p ds > \frac{p-2}{2-q} \int_{\partial\Omega} g \mid u \mid^p ds > 0.$$

(ii) Since

$$(2-q) \| u \| 2H_1 - (p-q) \int_{\partial\Omega} g \mid u \mid^p ds = \langle \psi'_\lambda(u), u \rangle < 0.$$

It follows that $\int_{\partial\Omega} g \mid u \mid^p ds > 0$. This completes the proof . □ For each $u \in \mathbf{M}_\lambda^-$, we write

$$t_{\max} = \Big(\frac{(2-q) \| u \| 2H_1}{(p-q) \int_{\partial\Omega} g \mid u \mid^p ds} \Big) 1/(p-2) < 1.$$

Then we have the following lemma .

Lemma 2 . 4 . *Let* $p^* = line - p_{p-q}$ *and* $\lambda_2 = \binom{minus-line-p2}{p-q} \binom{line-minus-q2}{p-q} \frac{2-q}{p-2} C_p^{\frac{p(2-q)}{2-p}} S_p^{-q} \| f \|_{Lp^*}^{-1}$. *Then for each* $u \in \mathbf{M}_\lambda^-$ *and* $\lambda \in (0, \lambda_2)$, *we have*

(i) *if* $\int_{\Omega} f \mid u \mid^q dx \leq 0$, *then* $J_\lambda(u) = \sup_{t \geq 0} J_\lambda(tu) > 0$;

(i i) *if* $\int_{\Omega} f \mid u \mid^q dx > 0$, *then th ere is a unique* $0 < t^+ = t^+(u) < t_{\max}$ *such that*

$$t^+ u \in \mathbf{M}_\lambda^+ and$$
$$J_\lambda(t^+ u) = 0 \leq t^{\mathrm{i}} \leq_{t_{\max}}^{\mathrm{nf}} J_\lambda(tu), J_\lambda(u) = \sup_{\geq t t_{\max}} J_\lambda(tu).$$

Proof . Fix $u \in \mathbf{M}_\lambda^-$. Let

$$h(t) = t^{2-q} \| u \|_{H\ 1}^2 - t^{p-q} \int_{\partial\Omega} g \mid u \mid^p ds \quad \mathrm{for} t \geq 0.$$

We have $h(0) = 0$, $h(t) \to -\infty$ as $t \to \infty$, $h(t)$ achieves its maximum at $t_{\max}$, increasing for $t \in [0, t_{\max})$ and decreasing for $t \in (t_{\max}, \infty)$. Moreover ,

$$h(t_{\max})$$

$$= \Big(\frac{(2-q) \| u \| 2H_1}{(p-q) \int_{\partial\Omega} g \mid u \mid^p ds} \Big) \frac{2-q}{p-2} \| u \| 2H_1 - \Big(\frac{(2-q) \| u \|_{H^1}^2}{(p-q) \int_{\partial\Omega} g \mid u \mid^p ds} \Big) \frac{p-q}{p-2} \int_{\partial\Omega} g \mid u \mid^p ds$$

$$= \quad \| u \| qH_1 [(\tfrac{2}{p} {}^{-q}_{-q}) \frac{2-q}{p-2} - (\tfrac{2}{p} {}^{-q}_{-q}) \frac{p-q}{p-2}] \Big(\frac{\| u \|_{H^1}^p}{\int_{\partial\Omega} g \mid u \mid^p ds} \Big) \frac{2-q}{p-2}$$

 (i) : $\int_\Omega f \mid u \mid^q dx \leq 0$. There is a unique $t^- > t_{\max}$ such that $h(t^-) = \lambda \int_\Omega f \mid u \mid^q dx$

$$\text{and} h'(t^-) < 0. \text{Now,}$$

$$(2-q) \parallel t^- u \parallel 2H_1 - (p-q) \int_{\partial\Omega} \mid t^- u \mid^p ds$$

$$= (t^-)^{1+q}[(2-q)(t^-)^{1-q} \parallel u \parallel 2H_1 - (p-q)(t^-)^{p-q-1} \int_{\partial\Omega} g \mid u \mid^p ds]$$

$$= (t^-)^{1+q} h'(t^-) < 0,$$

and

$$\langle J'_\lambda(t^- u), t^- u\rangle$$

$$= (t^-)^2 \parallel u \parallel_{H\ 1}^2 - (t^-)^q \lambda \int_\Omega f \mid u \mid^q dx - (t^-)^p \int_{\partial\Omega} g \mid u \mid^p ds$$

$$= (t^-)^q [h(t^-) - \lambda \int_\Omega f \mid u \mid^q dx] = 0.$$

Thus , $t^- u \in \mathbf{M}_\lambda^-$ or $t^- = 1$. Since for $t > t_{\max}$, we have

$$(2-q) \parallel tu \parallel_{H\ 1}^2 - (p-q) \int_{\partial\Omega} g \mid tu \mid^p ds < 0,$$

$$\frac{d^2}{dt^2} J_\lambda(tu) < 0,$$

$$\frac{d}{dt} J_\lambda(tu) = t \parallel u \parallel 2H_1 - \lambda t^{q-1} \int_\Omega f \mid u \mid^q dx - t^{p-1} \int_{\partial\Omega} g \mid u \mid^p ds = 0 \quad \text{for} t = t^-.$$

$$\text{Thus, } J_\lambda(u) = \sup_{t \geq 0} J_\lambda(tu). \text{Moreover,}$$

$$J_\lambda(u) \geq J_\lambda(tu) \geq \frac{t^2}{2} \parallel u \parallel_{H\ 1}^2 - \frac{t^p}{p} \int_{\partial\Omega} g \mid u \mid^p ds \quad \text{forall} t \geq 0.$$

By routine computations , $g(t) = \frac{t^2}{2} \parallel u \parallel 2H_1 - \frac{t^p}{p} \int_{\partial\Omega} g \mid u \mid^p ds$ achieves its maximum at

$$t_0 = (\parallel u \parallel 2H_1 / \int_{\partial\Omega} g \mid u \mid^p ds)^{1/(p-2)}. \text{Thus,}$$

$$J_\lambda(u) \geq p \frac{-2}{2p} \left(\frac{\parallel u \parallel pH_1}{\int_{\partial\Omega} g \mid u \mid^p ds} \right) \frac{2}{p-2} > 0.$$

(ii) : $\int_\Omega f \mid u \mid^q dx > 0$. By (2 . 6) and

$$h(0) = 0 < \lambda \int_\Omega f \mid u \mid^q dx \leq \lambda \parallel f \parallel Lp^* S_p^q \parallel u \parallel_{H^1}^q$$

 We have $t^{+}u \in \mathbf{M}_{\lambda}^{+}$, $t^{-}u \in \mathbf{M}_{\lambda}^{-}$, and $J_{\lambda}(t^{-}u) \geq J_{\lambda}(tu) \geq J_{\lambda}(t^{+}u)$ for each $t \in [t^{+}, t^{-}]$ and $J_{\lambda}(t^{+}u) \leq J_{\lambda}(tu)$ for each $t \in [0, t^{+}]$. Thus , $t^{-} = 1$ and

$$J_{\lambda}(u) = \sup_{t \geq 0} J_{\lambda}(tu), J_{\lambda}(t^{+}u) = 0 \leq t^{\mathrm{i}} \leq^{\mathrm{nf}}_{t_{\max}} J_{\lambda}(tu).$$

This completes the proof . □ Next , we establish the existence of nontrivial nonnegative solutions for the equa - tion

$$-\Delta u + u_u =_= \lambda_0 f(x_{\mathrm{on}}) \mid u \mid^{q-} \partial^{2_u}_{\Omega,} \quad \mathrm{in}\Omega, \tag{2.7}$$

Associated with equation (2 . 7) , we consider the energy functional

$$K_{\lambda}(u) = \frac{1}{2} \parallel u \parallel^2_{H\ 1} - \frac{\lambda}{q} \int_{\Omega} f \mid u \mid^q dx$$

and the minimization problem

$$\beta\lambda = \inf\{K_{\lambda}(u) : u \in \mathbf{N}_{\lambda}\},$$

where $\mathbf{N}_{\lambda} = \{u \in H_0^1(\Omega) \setminus \{0\} : \langle K'_{\lambda}(u), u\rangle = 0\}$. Then we have the following result .

Theorem 2 . 5 . *Suppose that* $\lambda > 0$. *Then equation* (2 . 7) *has a nontrivial nonneg - ative s o lution* v_{λ} *with* $K_{\lambda}(v_{\lambda}) = \beta\lambda < 0$.

Proof . First , we need to show that K_{λ} is bounded below on $\mathbf{N}_{\lambda}$ and $\beta\lambda < 0$. Then

for$u \in \mathbf{N}_{\lambda}$,

$$\parallel u \parallel 2H_1 = \lambda \int_{\Omega} f \mid u \mid^q dx \leq \lambda \parallel f \parallel Lq^* S_p^{-\frac{q}{2}} \parallel u \parallel^q_{H^1} .$$

where $p^* = p\frac{p}{-q}$. This implies

$$\parallel u \parallel H^1 \leq (\lambda \parallel f \parallel Lp^* S_p^{-\frac{q}{2}}) \frac{1}{2-q}. \tag{2.8}$$

Hence ,

$$K_{\lambda}(u) = \frac{1}{2} \parallel u \parallel H^1 - \frac{\lambda}{q} \int_{\Omega} f \mid u \mid^q dx$$

$$= (\frac{1}{2} - \frac{1}{q}^{)} \parallel u \parallel^2_{H\ 1}$$

$$\leq (\frac{1}{2} - \frac{1}{q}^{)} (\lambda \parallel f \parallel Lp^* S_p^{-\frac{q}{2}}) \frac{1}{2-q}$$

for all $u \in \mathbf{N}_{\lambda}$ and $\beta\lambda < 0$. Let $\{v_n\}$ be a minimizing sequence for K_{λ} on $\mathbf{N}_{\lambda}$. Then by (2 . 8) and the compact imbedding theorem , there exist a subsequence $\{v_n\}$ and v_{λ} in $H_0^1(\Omega)$ such that

$$v_n \rightharpoonup v_{\lambda} \quad \mathrm{weaklyin} H_0^1(\Omega)$$

and

$$v_n \to v_{\lambda} \quad \mathrm{stronglyin} L^q(\Omega). \tag{2.9}$$

First , we claim that $\int_\Omega f \mid v_\lambda \mid^q dx > 0$. If not ,

$$K_\lambda(v_n) \geq \frac{1}{2} \parallel v_\lambda \parallel 2H_1 - \frac{\lambda}{q} \int_\Omega f \mid v_\lambda \mid^q dx + o(1) \geq \frac{1}{2} \parallel v_\lambda \parallel^2_{H\ 1} + o(1),$$

this contradicts $K_\lambda(v_n) \to \beta\lambda(\Omega) < 0$ as $n \to \infty$. Thus , $\int_\Omega f \mid v_\lambda \mid^q dx > 0$. In particular , v_λ $equivalence - negationslash 0$. Now , we prove that $v_n \to v_\lambda$ strongly in $H_0^1(\Omega)$. Suppose otherwise , then $\| v_\lambda \| H^1 < \liminf_{n\to\infty} \| v_n \| H^1$ and so

$$\| v_\lambda \|_{H\ 1}^2 - \lambda \int_\Omega f \mid v_\lambda \mid^q dx < \lim_{n\to\infty} \inf(\| v_n \|_{H\ 1}^2 - \lambda \int_\Omega f \mid v_n \mid^q dx) = 0.$$

Since $\int_\Omega f \mid v_\lambda \mid^q dx > 0$, there is a unique $t_0 \neq 1$ such that $t_0 v_\lambda \in \mathbf{N}_\lambda$. Thus ,

$$t_0 v_n \rightharpoonup t_0 v_\lambda \quad \text{weaklyin} H_0^1(\Omega).$$

Moreover ,

$$K_\lambda(t_0 v_\lambda) < K_\lambda(v_\lambda) < \lim_{n\to\infty} K_\lambda(v_n) = \beta\lambda,$$

which is a contradiction . Hence $v_n \to v_\lambda$ strongly in $H_0^1(\Omega)$. This implies $v_\lambda \in \mathbf{N}_\lambda$ and

$$K_\lambda(v_n) \to K_\lambda(v_\lambda) = \beta\lambda \quad \text{as} n \to \infty.$$

Since $K_\lambda(v_\lambda) = K_\lambda(\| v_\lambda \|)$ and $\| v_\lambda \| \in \mathbf{N}_\lambda$, without loss of generality , we may
assume that v_λ is a nontrivial nonnegative solution of equation (2.7). ☐ Then we have the following results .

$$\text{(i)} \quad \alpha_\lambda \leq \lambda_\alpha^+ \leq \beta\lambda < 0; \qquad \textbf{Lemma} 2.6.$$

(i i) *J_λ is coe rcive and bounded below on* $\mathbf{M}_\lambda$ *for all* $\lambda \in (0, p\frac{p-2}{-q}]$. *Proof .* (i) Let v_λ be a positive solution of equation (2 . 7) such that $K(v_\lambda) = \beta\lambda$.
Since $v_\lambda \in C^2(\overline{}\Omega)$. Then we have $\int_{\partial\Omega} g \mid v_\lambda \mid^p ds = 0$ and $v_\lambda \in \mathbf{M}_\lambda^+$. This implies

$$J_\lambda(v_\lambda) = \frac{1}{2} \| v_\lambda \| 2H_1 - \frac{\lambda}{q} \int_\Omega f \mid v_\lambda \mid^q dx = \beta\lambda < 0$$
$$\text{andso} \alpha_\lambda \leq \lambda_\alpha^+ \leq \beta\lambda < 0.$$

(ii) For $u \in \mathbf{M}_\lambda$, we have $\| u \| 2H_1 = \lambda \int_\Omega f \mid u \mid^q dx + \int_{\partial\Omega} g \mid u \mid^p ds$. Then by the H ö lder and Young inequalities ,

$$J_\lambda(u) = \frac{p-2}{2p} \| u \|_{H\ 1}^2 - \lambda(^{p} \frac{-q)}{pq} \int_\Omega f \mid u \mid^q dx$$

3 . Proof of Theorem 1 . 1

First , we will use the idea of Ni - Takagi [1 2] to get the following results .

Lemma 3 . 1 . *For ea ch* $u \in \mathbf{M}_\lambda$, *there exist* $\epsilon > 0$ *and a differentiable function* $\xi : B(0;\epsilon) \subset H^1(\Omega) \to \mathbb{R}^+$ *such that* $\xi(0) = 1$, *th e function* $\xi(v)(u-v) \in \mathbf{M}_\lambda$ *and*

$$\langle \xi'(0), v\rangle = \frac{2\int_\Omega \nabla u \nabla v dx - \lambda q \int_\Omega f \mid u \mid^{q-2} uvdx - p\int_{\partial\Omega} g \mid u \mid^{p-2} uvds}{(2-q) \parallel u \parallel 2H_1 - (p-q)\int_{\partial\Omega} g \mid u \mid^p ds} \tag{3.1}$$

for al l $v \in H^1(\Omega)$. *Proof .* For $u \in \mathbf{M}_\lambda$, define a function $F : \mathbb{R} \times H^1(\Omega) \to \mathbb{R}$ by

$$F_u(\xi, w) = \langle J'_\lambda(\xi(u-w)), \xi(u-w)\rangle$$
$$= \xi^2 \int_\Omega \mid \nabla(u-w) \mid^2 + (u-w)^2 dx - \xi^q \lambda \int_\Omega f \mid u-w \mid^q dx$$
$$-\xi^p \int_{\partial\Omega} g \mid u-w \mid^p ds.$$
$$\text{Then} F_u(1,0) = \langle J'_\lambda(u), u\rangle = 0 \text{and}$$
$$\frac{d}{d\xi} F_u(1,0) = 2 \parallel u \parallel^2_{H\ 1} - \lambda q \int_{\partial\Omega} f \mid u \mid^q dx - p\int_{\partial\Omega} g \mid u \mid^p ds$$
$$= (2-q) \parallel u \parallel 2H_1 - (p-q)\int_{\partial\Omega} g \mid u \mid^p ds \neq 0.$$

According to the implicit function theorem , there exist $\epsilon > 0$ and a differentiable function $\xi : B(0;\epsilon) \subset H^1(\Omega) \to \mathbb{R}$ such that $\xi(0) = 1$,

$$\langle \xi'(0), v\rangle = \frac{2\int_\Omega \nabla u \nabla v dx - \lambda q \int_\Omega f \mid u \mid^{q-2} uvdx - p\int_{\partial\Omega} g \mid u \mid^{p-2} uvds}{(2-q) \parallel u \parallel 2H_1 - (p-q)\int_{\partial\Omega} g \mid u \mid^p ds}$$

and

$$F_u(\xi(v), v) = 0 \quad \text{forall} v \in B(0;\epsilon)$$

which is equivalent to

$$\langle J'_\lambda(\xi(v)(u-v)), \xi(v)(u-v)\rangle = 0 \quad \text{forall} v \in B(0;\epsilon),$$
$$\text{thatis} \xi(v)(u-v) \in \mathbf{M}_\lambda. \quad \square$$

Lemma 3 . 2 . *For ea ch* $u \in \mathbf{M}^-_\lambda$, *the re exist* $\epsilon > 0$ *and a differentiable function* $\xi^- : B(0;\epsilon) \subset H^1(\Omega) \to \mathbb{R}^+$ *such that* $\xi^-(0) = 1$, *the function* $\xi^-(v)(u-v) \in \mathbf{M}^-_\lambda$ *and*

$$\langle (\xi^-)'(0), v\rangle = \frac{2\int_\Omega \nabla u \nabla v dx - \lambda q \int_\Omega f \mid u \mid^{q-2} uvdx - p\int_{\partial\Omega} g \mid u \mid^{p-2} uvds}{(2-q) \parallel u \parallel 2H_1 - (p-q)\int_{\partial\Omega} g \mid u \mid^p ds} \tag{3.2}$$

for al l $v \in H^1(\Omega)$.

Proof . Similar to the argument in Lemma 3 . 1 , there exist $\epsilon > 0$ and a differentiable function $\xi^- : B(0;\epsilon) \subset H^1(\Omega) \to \mathbb{R}$ such that $\xi^-(0) = 1$ and $\xi^-(v)(u-v) \in \mathbf{M}_\lambda$ for all $v \in B(0;\epsilon)$. Since

$$\langle \psi'_\lambda(u), u\rangle = (2-q) \parallel u \parallel^2_{H\ 1} - (p-q)\int_{\partial\Omega} g \mid u \mid^p ds < 0.$$

 Thus, by the continuity of the function ξ^-, we have

$$\langle \psi_\lambda'(\xi^-(v)(u-v)), \xi^-(v)(u-v)\rangle$$
$$= (2-q) \parallel \xi^-(v)(u-v) \parallel 2H_1 - (p-q) \int_{\partial\Omega} g \mid \xi^-(v)(u-v) \mid^p ds < 0$$

if ϵ sufficiently small, this implies that $\xi^-(v)(u-v) \in \mathbf{M}_\lambda^-$. □ **Proposition 3.3.** *Let* $\lambda_0 = \min\{\lambda_1, \lambda_2, line-p-minus^{1\}}_{p-q},$ *Then for* $\lambda \in (0, \lambda_0)$:

(i) *There exists a minimizing s equence* $\{u_n\} \subset \mathbf{M}_\lambda$ *such that*

$$J_\lambda(u_n) = \alpha_\lambda + o(1),$$
$$J_\lambda'(u_n) = o(1) \quad in H^*(\Omega);$$

(ii) *there exists a minimizing s equence* $\{u_n\} \subset \mathbf{M}_\lambda^-$ *such that*

$$J_\lambda(u_n) = \alpha_\lambda^- + o(1),$$
$$J_\lambda'(u_n) = o(1) \quad in H^*(\Omega).$$

Proof. (i) By Lemma 2.6 (ii) and the Ekeland variational principle [7], there exists a minimizing sequence $\{u_n\} \subset \mathbf{M}_\lambda$ such that

$$J_\lambda(u_n) < \alpha_\lambda + \frac{1}{n}, \tag{3.3}$$

$$J_\lambda(u_n) < J_\lambda(w) + \frac{1}{n} \parallel w - u_n \parallel H^1 \quad \text{foreach} w \in \mathbf{M}_\lambda. \tag{3.4}$$

By taking n large, from Lemma 2.6 (i), we have

$$J_\lambda(u_n) = <(\alpha \frac{1}{2}_\lambda - + H^{1\cdot} \underline{\qquad}_{p\frac{1}{1})<}^{n} {}^{\frac{\|u_n\ \|_{\beta\lambda}}{2}} {}^2 - (\frac{1}{q} - \underline{\qquad}_1^p)\lambda \int_\Omega f \mid u_n \mid^q dx \tag{3.5}$$

This implies

$$\parallel f \parallel Lp^* S_p^q \parallel u_n \parallel_{H^1}^q \geq \int_\Omega f \mid u_n \mid^q dx > \frac{-pq}{2\lambda(p-q)}\beta\lambda > 0. \tag{3.6}$$

Consequently, $u_n \neq 0$ and putting together (3.5), (3.6) and the H ö lder inequality, we obtain

$$\parallel u_n \parallel H^1 > [\frac{-pq}{2\lambda(p-q)}\beta\lambda^{S}{}_p^{-q} \parallel f \parallel -L_{p*}^1]1/q \tag{3.7}$$

$$\parallel u_n \parallel H^1 < [\frac{2(p-q)}{(p-2)q} \parallel f \parallel Lp^* S_p^q]1/(2-q) \tag{3.8}$$

Now, we show that

$$\parallel J_\lambda'(u_n) \parallel H - 1 \to 0 \quad \text{as} n \to \infty.$$

Applying Lemma 3.1 with u_n to obtain the functions $\xi_n : B(0; \epsilon_n) \to \mathbb{R}^+$ for some $\epsilon_n > 0$, such that $\xi_n(w)(u_n - w) \in \mathbf{M}_\lambda$. Choose $0 < \rho < \epsilon_n$. Let $u \in H^1(\Omega)$ with $u equivalence{-}negationslash 0$ and let $w_\rho = line{-}rho\|^u_{u\|H^1}.$ We set $\eta_\rho = \xi_n(w_\rho)(u_n - w_\rho)$. Since $\eta_\rho \in \mathbf{M}_\lambda$, we deduce from (3.4) that

$$J_\lambda(\eta_\rho) - J_\lambda(u_n) \geq -\frac{1}{n} \parallel \eta_\rho - u_n \parallel H^1$$

 and by the mean value theorem , we have

$$\langle J'_\lambda(u_n), \eta_\rho - u_n\rangle + o(\| \eta_\rho - u_n \| H^1) \geq -\frac{1}{n} \| \eta_\rho - u_n \| H^1.$$

Thus ,

$$\begin{aligned}&\langle J'_\lambda(u_n), -w_\rho\rangle + (\xi_n(w_\rho) - 1)\langle J'_\lambda(u_n), (u_n - w_\rho)\rangle \\ &\quad \geq -\frac{1}{n} \| \eta_\rho - u_n \| H^1 + o(\| \eta_\rho - u_n \| H^1).\end{aligned} \tag{3.9}$$

Since $\xi_n(w_\rho)(u_n - w_\rho) \in \mathbf{M}_\lambda$ and (3 . 9) it follows that

$$\begin{aligned}&-\rho\langle J'_\lambda(u_n), \frac{u}{\| u \| H^1}\rangle + (\xi_n(w_\rho) - 1)\langle J'_\lambda(u_n) - J'_\lambda(\eta_\rho), (u_n - w_\rho)\rangle \\ &\quad \geq -\frac{1}{n} \| \eta_\rho - u_n \| H^1 + o(\| \eta_\rho - u_n \| H^1).\end{aligned}$$

Thus ,

$$\langle J'_\lambda(u_n), \frac{u}{\| u \| H^1}\rangle \leq \rho^{1)}_{-\langle J_\lambda{}'(u_n) - J'_\lambda(\eta_\rho)} {}_{+\frac{\|\eta_\rho - u_n\| H^1}{(\xi_n(w_\rho)^{\rho}_{n\rho} -} + o(\|\eta \rho - u_n\|H^1)}, (u_n - w_\rho)\rangle. \tag{3.10}$$

Since $\| \eta_\rho - u_n \| H^1 \leq \rho \| \xi_n(w_\rho) \| \quad + \quad \| \xi_n(w_\rho) - 1 \| \| u_n \| H^1$ and

$$\lim_{\rho\to 0} \frac{\| \xi_n(w_\rho) - 1 \|}{\rho} \leq \| \xi'_n(0) \|,$$

if we let $\rho \to 0$ in (3 . 1 0) for a fixed n, then by (3 . 8) we can find a constant $C > 0$, independent of ρ, such that

$$\langle J'_\lambda(u_n), \frac{u}{\| u \| H^1}\rangle \leq \frac{C}{n}(1 + \quad \| \xi'_n(0) \|).$$

The proof will be complete once we show that $\| \xi'_n(0) \|$ is uniformly bounded in n. By (3 . 1) , (3 . 8) and the H ö lder inequality , we have

$$\langle \xi'_n(0), v\rangle \leq \frac{b \| v \| H^1}{| (2-q) \| u_n \| H^1 - (p-q) \int_{\partial\Omega} g \mid u_n \mid^p ds \mid} \quad \text{forsome} b > 0.$$

We only need to show that

$$| (2-q) \| u_n \| H^1 - (p-q) \int_{\partial\Omega} g \mid u_n \mid^p ds \mid > c \tag{3.11}$$

for some $c > 0$ and n large enough . We argue by contradiction . Assume that there exists a subsequence $\{u_n\}$, we have

$$(2-q) \| u_n \| H^1 - (p-q) \int_{\partial\Omega} g \mid u_n \mid^p ds = o(1). \tag{3.12}$$

Combining (3 . 1 2) with (3 . 7) , we can find a suitable constant $d > 0$ such that $\int_{\partial\Omega} g \mid u_n \mid^p ds \geq d$ for n sufficiently large . (3 . 1 3) In addition (3 . 1 2) , and the fact that $u_n \in \mathbf{M}_\lambda$ also give

$$\lambda \int_\Omega f \mid u_n \mid^q dx = \| u_n \|^2_{H^1} - \int_{\partial\Omega} g \mid u_n \mid^p ds = p\frac{-2}{2-q} \int_{\partial\Omega} g \mid u_n \mid^p ds + o(1)$$

 and

 Thus ,

$$J_\lambda(u_n) = \frac{1}{2} \parallel u_n \parallel_{H\ 1}^2 - \frac{\lambda}{q}\int_\Omega f \mid u_n \mid^q dx - \frac{}{\ \ }_1^p \int_{\partial\Omega} g \mid u_n \mid^p ds$$

$$= (\frac{1}{2} - \frac{}{\ \ }_1^p) \int_{\partial\Omega} g \mid u_n \mid^p ds + o(1)$$

$$= (\frac{1}{2} - \frac{}{\ \ }_1^p) \int_{\partial\Omega} g \mid 0_u^+ \mid^p ds \quad \mathrm{as} n \to \infty,$$

this contradicts $J_\lambda(u_n) \to \alpha_\lambda < 0$ as $n \to \infty$. Moreover ,

$$o(1) = \langle J_\lambda'(u_n), \phi\rangle = \langle J_\lambda'(u_0), \phi\rangle + o(1) \quad \mathrm{forall}\phi \in H^1(\Omega).$$

Thus , $0_u^+ \in \mathbf{M}_\lambda$ is a nonzero solution of equation (1 . 1) and $J_\lambda(0_u^+) \geq \alpha_\lambda$. Now we prove that $u_n \to 0_u^+$ strongly in $H^1(\Omega)$. Suppose otherwise , then $\parallel 0_u^+ \parallel H^1 < \liminf_{n\to\infty} \parallel u_n \parallel H^1$ and so

$$\parallel 0_u^+ \parallel_{H\ 1}^2 - \lambda \int_\Omega f \mid 0_u^+ \mid^q dx - \int_{\partial\Omega} g \mid 0_u^+ \mid^p ds$$

$$< \lim_{n\to\infty} \inf(\parallel u_n \parallel 2H_1 - \lambda \int_\Omega f \mid u_n \mid^q dx - \int_{\partial\Omega} g \mid u_n \mid^p ds) = 0,$$

this contradicts $0_u^+ \in \mathbf{M}_\lambda$. Hence $u_n \to 0_u^+$ strongly in $H^1(\Omega)$ and

$$J_\lambda(u_n) \to J_\lambda(0_u^+) = \alpha_\lambda \quad \mathrm{as} n \to \infty.$$

Moreover , we have $0_u^+ \in \mathbf{M}_\lambda^+$. If not , then $0_u^+ \in \mathbf{M}_\lambda^-$ and by Lemma 2 . 4 , there are unique t_0^+ and t_0^- such that $t_0^+ 0_u^+ \in \mathbf{M}_\lambda^+$ and $t_0^- 0_u^+ \in \mathbf{M}_\lambda^-$. In particular , we have

$$t_0^+ < t_0^- = 1.\mathrm{Since}$$

$$\frac{d}{dt} J_\lambda(t_0^+ 0_u^+) = 0 \quad \text{and} \quad \frac{d^2}{dt^2} J_\lambda(t_0^+ 0_u^+) > 0,$$

there exists $t_0^+ < \bar{t} \leq t_0^-$ such that $J_\lambda(t_0^+ 0_u^+) < J_\lambda(^{\bar{t}} 0_u^+)$. By Lemma 2 . 4 ,

$$J_\lambda(t_0^+ 0_u^+) < J_\lambda(^{\bar{t}} 0_u^+) \leq J_\lambda(t_0^- 0_u^+) = J_\lambda(0_u^+),$$

which is a contradiction . Since $J_\lambda(0_u^+) = J_\lambda(\mid 0_u^+ \mid)$ and $\mid 0_u^+ \mid \in \mathbf{M}_\lambda^+$, by Lemma 2 . 2 we may assume that 0_u^+ is a nontrivial nonnegative solution of equation (1 . 1) . From Lemma 2 . 6 it follows that

$$0 > J_\lambda(0_u^+) \geq -\lambda(\frac{(p-q)(2-q)}{2pq}^) (\parallel f \parallel Lp^* S_p^q) \frac{2}{2-q}$$

and so $J_\lambda(0_u^+) \to 0$ as $\lambda \to 0$. □

Next , we establish the existence of a lo cal minimum for J_λ on $\mathbf{M}_\lambda^-$. **Theorem 3 . 5 .** *Let* $\lambda_0 > 0$ *as in Proposition 3 . 3 . Then for* $\lambda \in (0, \lambda_0)$ *the functional* J_λ *has a minimizer* u_0^- *in* $\mathbf{M}_\lambda^-$ *and satisfies*

$$\text{(i)} \quad J_\lambda(u_0^-) = \alpha_\lambda^-;$$

(i i) u_0^- *is a nontrivial nonnegative s o lutio n of equation* (1 . 1) . *Proof* . By Proposition 3 . 3 (i i) , there exists a minimizing sequence $\{u_n\}$ for J_λ on $\mathbf{M}_\lambda^-$ such that

$$J_\lambda(u_n) = \alpha_\lambda^- + o(1) \quad \text{and} \quad J_\lambda'(u_n) = o(1) \quad \mathrm{in} H^*(\Omega).$$

 By Lemma 2 . 6 and the compact imbedding theorem , there exist a subsequence $\{u_n\}$
and $u_0^- \in H^1(\Omega)$ such that

$$u_n \rightharpoonup u_0^- \quad \text{weakly in} H^1(\Omega),$$
$$u_n \to u_0^- \quad \text{strongly in} L^p(\partial\Omega),$$
$$u_n \to u_0^- \quad \text{strongly in} L^q(\Omega).$$

Since $(2-q) \parallel u_n \parallel_{H^1}^2 -(p-q)\int_{\partial\Omega} g \mid u_n \mid^p ds < 0$, by the Sobolev trace inequality there exists $C > 0$ such that $\int_{\partial\Omega} g \mid u_n \mid^p ds > C$. Moreover ,

$$o(1) = \langle J_\lambda'(u_n), \phi\rangle = \langle J_\lambda'(u_0), \phi\rangle + o(1) \quad \text{for all} \phi \in H^1(\Omega)$$

and

$$(2-q) \parallel u_0 \parallel_{H\ 1}^2 - (p-q)\int_{\partial\Omega} g \mid u_0 \mid^p ds$$
$$\leq \lim_{n\to\infty} \inf((2-q) \parallel u_n \parallel_{H\ 1}^2 - (p-q)\int_{\partial\Omega} g \mid u_n \mid^p ds) \leq 0.$$

Thus , $u_0^- \in \mathbf{M}_\lambda^-$ is a nonzero solution of equation (1 . 1) . Now we prove that $u_n \to u_0^-$ strongly in $H^1(\Omega)$. Suppose otherwise , then $\parallel u_0^- \parallel H^1 < \lim\inf_{n\to\infty} \parallel u_n \parallel H^1$ and so

$$\parallel u_0^- \parallel_{H\ 1}^2 - \lambda\int_\Omega f \mid u_0^- \mid^q dx - \int_{\partial\Omega} g \mid u_0^- \mid^p ds$$
$$< \lim_{n\to\infty} \inf(\parallel u_n \parallel 2H_1 - \lambda\int_\Omega f \mid u_n \mid^q dx - \int_{\partial\Omega} g \mid u_n \mid^p ds) = 0,$$

this contradicts $u_0^- \in \mathbf{M}_\lambda^-$. Hence $u_n \to u_0^-$ strongly in $H^1(\Omega)$. This implies

$$J_\lambda(u_n) \to J_\lambda(u_0^-) = \alpha_\lambda^- \quad \text{as} n \to \infty.$$

Since $J_\lambda(u_0^-) = J_\lambda(\mid u_0^- \mid)$ and $\mid u_0^- \mid \in \mathbf{M}_\lambda^-$, by Lemma 2 . 2 we may assume that u_0^- is a nontrivial nonnegative solution of equation (1.1). $\square$

Now , we complete the proof of Theorem 1 . 1 . By Theorems 3 . 4 , 3 . 5 , we obtain
equation (1 . 1) has two nontrivial nonnegative solutions 0_u^+ and u_0^- such that $0_u^+ \in \mathbf{M}_\lambda^+$ and $u_0^- \in \mathbf{M}_\lambda^-$. Since $\mathbf{M}_\lambda^+ \cap \mathbf{M}_\lambda^- = \phi$, this implies that 0_u^+ and u_0^- are different .

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