Electronic Journal of Differential Equations , Vol . 2008 (2008) , No . 28 , pp . 1 – 1 1 .
ISSN : 1 72 - 6691 . URL : http : / / ejde . math . txstate . edu or http : / / ej de . math . unt . edu
ftp ejde . math . txstate . edu (login : ftp)

EXISTENCE RESULTS FOR IMPULSIVE EVOLUTION DIFFERENTIAL EQUATIONS WITH STATE - DEPENDENT DELAY

EDUARDO HERN ÁNDEZ, RATHINASAMY SAKTHIVEL , SUELI TANAKA AKI

ABSTRACT . We study the existence of mild solution for impulsive evolution abstract differential equations with state - dependent delay . A concrete appli - cation to partial delayed differential equations is considered .

1 . INTRODUCTION

In this work we discuss the existence of mild solutions for impulsive functional differential equations , with state - dependent delay , of the form

$$x'(t) = A(t)x(t) + f(t, x_{\rho(t,x_t)}), \quad t \in I = [0, a], \tag{1.1}$$

$$x_0 = \varphi \in \mathcal{B}, \tag{1.2}$$

$$\Delta x(t_i) = I_i(x_{t_i}), \quad i = 1, 2, ..., n, \tag{1.3}$$

where $A(t) : \mathcal{D} \subset X \to X, t \in I$, is a family of closed linear operators defined on a common domain $\mathcal{D}$ which is dense in a Banach space $(X, \| \cdot \|)$; the function $x_s : (-\infty, 0] \to X, x_s(\theta) = x(s + \theta),$ belongs to some abstract phase space $\mathcal{B}$

describedaxiomatically; $f : I \times \mathcal{B} \to X, \rho : I \times \mathcal{B} \to (-\infty, a], I_i : \mathcal{B} \to X,$

$i = 1, 2, ..., n,$ are appropriate functions ; $0 < t_1 < ....t_n < a$ are prefixed points and the symbol $\Delta\xi(t)$ represents the j ump of the function ξ at t, which is defined

$$\text{by}\Delta\xi(t) = \xi(t^+) - \xi(t^-).$$

Various evolutionary processes from fields as diverse as physics , population dy - namics , aeronautics , economics and engineering are characterized by the fact that they undergo abrupt changes of state at certain moments of time between intervals of continuous evolution . Because the duration of these changes are often negligible compared to the total duration of the process , such changes can be reasonably well - approximated as being instantaneous changes of state , or in the form of impulses . These process tend to more suitably modeled by impulsive differential equations , which allow for discontinuities in the evolution of the st ate . For more details on this theory and on its applications we refer to the monographs of Lakshmikantham et

2000 *Mathematics Subject Classification* . 35 R 10 , 34 K 5 .
Key words and phrases . State - dependent delay ; abstract Cauchy problem ; partial functional - differential equations ; evolution operators .

Submitted November 26 , 2007 . Published February 28 , 2008 .

al . [1 7] , and Samoilenko and Perestyuk [25] for the case of ordinary impulsive sys - tem and [1 8 , 23 , 24 , 1 5 , 1 6] for partial differential and partial functional differential equations with impulses .

On the other hand , functional differential equations with state - dependent delay appear frequently in applications as model of equations and for this reason the study of this type of equations has received great attention in the last years . There exists a extensive literature for ordinary state - dependent delay equations , see among another works , [2 , 1 , 3 , 4 , 6 , 7 , 8] . The study of partial differential equations with state dependent delay have been initiated recently , and concerning this matter we cite the pioneer works Rezounenko et al . [2 1] , Hern á ndez el al . [1 1] and the papers
[1 0 , 1 2 , 1 3 , 1 4 , 22] .

To the best of our knowledge , the study of the existence of solutions for systems described in the abstract form (1 . 1) – (1 . 2) is a untreated problem , and this fact , is the main motivation of this paper .

Throughout this paper , $(X, \| \cdot \|)$ is a Banach space , $\{A(t) : t \in \mathbb{R}\}$ is a family of closed linear operators defined on a common domain $\mathcal{D}$ which is dense in X, and we assume that the linear non - autonomous system

$$u'(t) = A^{(t)u(t),}_{u(s\ )=x} \in sX^{\leq}_{,}t \leq a, \tag{1.4}$$

has an associated evolution family of operators $\{U(t, s) : a \geq t \geq s \geq 0\}$. In the next definition , $\mathcal{L}(X)$ is the space of bounded linear operator from X into X
endowed with the uniform convergence topology .

Definition 1 . 1 . A family of linear operators $\{U(t, s) : a \geq t \geq s \geq 0\} \subset \mathcal{L}(X)$ is called an evolution family of operators for (1 . 4) if the following conditions hold :

(a) $U(t, s)U(s, r) = U(t, r)$ and $U(r, r)x = x$ for every $r \leq s \leq t$ and all $x \in X$;

(b) For each $x \in X$ the function $(t, s) \to U(t, s)x$ is continuous and $U(t, s) \in \mathcal{L}(X)$ for every $t \geq s$; and

(c) For $s \leq t \leq a$, the function $(s, t] \to \mathcal{L}(X), t \to U(t, s)$ is differentiable with

$$\frac{\partial}{\partial t}U(t, s) = A(t)U(t, s).$$

In the sequel , $\widetilde{M}$ is a positive constant such that $\| U(t, s) \| \leq \widetilde{M}$ for every $t \geq s$, and we always assume that $U(t, s)$ is a compact operator for every $t > s$. We refer the reader to [20] for additional details on evolution operator families .

To consider the impulsive condition (1 . 3) , it is convenient to introduce some additional concepts and notations . We say that a function $u : [\sigma, \tau] \to X$ is a normalized piecewise continuous function on $[\sigma, \tau]$ if u is piecewise continuous and left continuous on $(\sigma, \tau]$. We denote by $\mathcal{PC}([\sigma, \tau]; X)$ the space formed by the normalized piecewise continuous functions from $[\sigma, \tau]$ into X. In particular , we introduce the space $\mathcal{PC}$ formed by all functions $u : [0, a] \to X$ such that u is continuous at $t \neq t_i, u(t_i^-) = u(t_i)$ and $u(t_i^+)$ exists , for all $i = 1, ..., n$. In this paper we always assume that $\mathcal{PC}$ is endowed with the norm $\| u \| \mathcal{PC} = \sup_{s \in I} \| u(s) \|$. It is clear that $(\mathcal{PC}, \| \cdot \| \mathcal{PC})$ is a Banach space .

To simplify the notations , we put $t_0 = 0, t_{n+1} = a$ and for $u \in \mathcal{PC}$ we denote by $\tilde{u}_i \in C([t_i, t_{i+1}]; X), i = 0, 1, ..., n,$ the function given by

$$\widetilde{u}_i(t) = \{^{u(t),}_{u(t_i^+),} \quad \text{for}^{\text{for}} t^t \in_= (^{t_i,}_{t_i} .t_{i+1}], \tag{1.5}$$

Moreover , for $B \subseteq \mathcal{PC}$ we denote by $\widetilde{B}_i, i = 0, 1, ..., n$, the set $\widetilde{B}_i = \{\tilde{u}_i : u \in B\}$.

Lemma 1 . 2 . *A s e t $B \subseteq \mathcal{PC}$ is relatively compact in $\mathcal{PC}$ if , and only if , the s e t $\widetilde{B}_i$ is relatively compact in $C([t_i, t_{i+1}]; X)$, for every $i = 0, 1, ..., n$.*

In this work we will employ an axiomatic definition for the phase space $\mathcal{B}$ which is similar to those introduced in [9] . Specifically , $\mathcal{B}$ will be a linear space of functions mapping $(-\infty, 0]$ into X endowed with a seminorm $\| \cdot \| \mathcal{B}$, and satisfies the following

conditions :

(A) If $x : (-\infty, \sigma + b] \to X, b > 0$, is such that $x |_{[\sigma,\sigma+ b]} \in \mathcal{PC}([\sigma, \sigma + b] : X)$ and $x_\sigma \in \mathcal{B}$, then for every $t \in [\sigma, \sigma + b]$ the following conditions hold :

(i) x_t is in $\mathcal{B}$,

(ii) $\| x(t) \| \leq H \| x_t \| \mathcal{B}$,

(i ii) $\| x_t \| \mathcal{B} \leq K(t - \sigma) \sup \{\| x(s) \| : \sigma \leq s \leq t\} + M(t - \sigma) \| x_\sigma \| \mathcal{B}$,

where $H > 0$ is a constant ; $K, M : [0, \infty) \to [1, \infty), K$ is continuous , M is lo cally bounded , and H, K, M are independent of $x(\cdot)$.

(B) The space $\mathcal{B}$ is complete .

Example 1 . 3 . Phase spaces $\mathcal{PC}_h(X), \mathcal{PC}_g^0(X)$. As usual , we say that $\psi : (-\infty, 0] \to X$ is normalized piecewise continuous , if ψ is left continuous and the restriction of ψ to any interval $[-r, 0]$ is piecewise continuous .

Let $g : (-\infty, 0] \to [1, \infty)$ be a continuous , nondecreasing function with $g(0) = 1$, which satisfies the conditions (g - 1) , (g - 2) of [9] . This means that $\lim_{\theta\to-\infty} g(\theta) = \infty$ and that the function $\Lambda(t) := \sup_{-\infty<\theta\leq-t} \frac{g(t+\theta)}{g(\theta)}$ is lo cally bounded for $t \geq 0$. Next , we modify slightly the definition of the spaces C_g, C_g^0 in [9] . We denote by $\mathcal{PC}_g(X)$ the space formed by the normalized piecewise continuous functions ψ such that $\frac{\psi}{g}$ is bounded on $(-\infty, 0]$ and by $\mathcal{PC}_g^0(X)$ the subspace of $\mathcal{PC}_g(X)$ formed by the functions ψ such that $\frac{\psi(\theta)}{g(\theta)} \to 0$ as $\theta \to -\infty$. It is easy to see that $\mathcal{PC}_g(X)$ and $\mathcal{PC}_g^0(X)$ endowed with the norm $\| \psi \| \mathcal{B} := \sup_{\theta\leq0} \frac{\|\psi(\theta)\|}{g(\theta)}$, are phase spaces in the sense considered in this work . Moreover , in these cases $K \equiv 1$.

Example 1 . 4 . Phase space $\mathcal{PC}_r \times L^2(g, X)$. Let $1 \leq p < \infty$, $0 \leq r < \infty$ and $g(\cdot)$ be a Borel nonnegative measurable function on $(-\infty, r)$ which satisfies the conditions (g - 5) - (g - 6) in the terminology of [9] . Briefly , this means that $g(\cdot)$ is lo cally integrable on $(-\infty, -r)$ and that there exists a nonnegative and lo cally bounded function Λ on $(-\infty, 0]$ such that $g(\xi + \theta) \leq \Lambda(\xi)g(\theta)$ for all $\xi \leq 0$ and $\theta \in (-\infty, -r) \setminus N_\xi$, where $N_\xi \subseteq (-\infty, -r)$ is a set with Lebesgue measure 0 .

Let $\mathcal{B} := \mathcal{PC}_r \times L^p(g; X), r \geq 0, p > 1$, be the space formed of all classes of functions $\psi : (-\infty, 0] \to X$ such that $\psi |_{[-r, 0]} \in \mathcal{PC}([-r, 0], X)$, $\psi(\cdot)$ is Lebesgue - measurable on $(-\infty, -r]$ and $g \mid \psi \mid^p$ is Lebesgue integrable on $(-\infty, -r]$. The semi - norm in $\| \cdot \| \mathcal{B}$ is defined by

$$\| \psi \| \mathcal{B} := \sup_{\in_\theta[-r, 0]} \| \psi(\theta) \| +(-^{-r}_{\infty} g(\theta) \| \psi(\theta) \|^p \, d\theta)1/p.$$

Proceeding as in the proof of [9 , Theorem 1 . 3 . 8] it follows that $\mathcal{B}$ is a phase space which satisfies the axioms **A** and **B** . Moreover , for $r = 0$ and $p = 2$ this space coincides with $C_0 \times L^2(g, X), H = 1;$ $M(t) = \Lambda(-t)^{1/2}$ and $K(t) = 1+$

$$(\int_{-t}^{0} g(\tau) d\tau)^{1/2} \mathrm{for} t \geq 0.$$

Remark 1 . 5 . In retarded functional differential equations without impulses , the axioms of the abstract phase space $\mathcal{B}$ include the continuity of the function $t \to x_t$, see for instance [9] . Due to the impulsive effect , this property is not satisfied in impulsive delay systems and , for this reason , has been eliminated in our abstract description of $\mathcal{B}$.

The terminology and notations are those generally used in functional analysis . In particular , for Banach a space $(Z, \| \cdot \quad \| Z)$, the notation $B_r(x, Z)$ stands for the closed ball with center at x and radius $r > 0$ in Z.

To prove some of our results , we use a fixed point Theorem which is referred in the Literature as Leray Schauder Alternative Theorem , see [5 , Theorem 6 . 5 . 4] .

Theorem 1 . 6 . *Let D be a convex subset of a Banach space X and assume that $0 \in D$. Let $G : D \to D$ be a completely continuous map . Then the map G has a fixed point in D or the s e t $\{x \in D : x = \lambda G(x), 0 < \lambda < 1\}$ is unbounded .*

In the next section we study the existence of mild solutions for the abstract system (1 . 1) – (1 . 2) . In the last section an application is discussed .

2 . Existence Results

To prove our results on the existence of mild solutions for the abstract Cauchy problem (1 . 1) – (1 . 2) , we always assume that $\rho : I \times \mathcal{B} \to (-\infty, a]$ is continuous . In addition , we introduce the following conditions .

(H 0) Let $\mathcal{BPC}(\varphi) = \{u : (-\infty, a] \to X; u_0 = \varphi, u \mid I \in \mathcal{PC}\}$. The function $t \to \varphi t$ is continuous from $\mathcal{R}(\rho^-) = \{\rho(s, x_s) : \rho(s, x_s) \leq 0, x \in \mathcal{BPC}(\varphi), s \in [0, a]\}$ into $\mathcal{B}$ and there exists a continuous and bounded function $J^\varphi : \mathcal{R}(\rho^-) \to$ $(0, \infty)$ such that $\| \varphi t \| \mathcal{B} \leq J^\varphi(t) \| \varphi \| \mathcal{B}$ for every $t \in \mathcal{R}(\rho^-)$.

(H 1) The function $f : I \times \mathcal{B} \to X$ satisfies the following properties .

(a) The function $f(\cdot, \psi) : I \to X$ is strongly measurable for every $\psi \in \mathcal{B}$.

(b) The function $f(t, \cdot) : \mathcal{B} \to X$ is continuous for each $t \in I$.

(c) There exist an integrable function $m : I \to [0, \infty)$ and a continuous nondecreasing function $W \ : \ [0, \infty) \ \to \ (0, \infty)$ such that $\| f(t, \psi) \| \ \leq$

$$m(t)W(\| \psi \| \mathcal{B}), \text{forevery} (t, \psi) \in I \times \mathcal{B}.$$

(H 2) The maps I_i are completely continuous and there are positive constants $c - j_i$,
$j = 1, 2$, such that $\| I_i(\psi) \| \ \leq i_c^1 \| \psi \| \mathcal{B} + i_c^2, i = 1, 2, ..., n$, for every $\psi \in \mathcal{B}$.
(H 3) The function $I_i \quad : \mathcal{B} \to X$ is continuous and there are positive constants $L_i, i = 1, 2, ..., n$, such that $\| I_i(\psi 1) - I_i(\psi 2) \| \ \leq L_i \| \psi 1 - \psi 2 \| \mathcal{B}$, for every

$$\psi_j \in \mathcal{B}, j = 1, 2, i = 1, 2, ..., n.$$

Remark 2 . 1 . The condition (H 0) , is frequently verified by functions continuous and bounded . If , for instance , the space $\mathcal{B}$ verifies axiom C_2 in the nomenclature of [9] , then there exists a constant L > 0 such that $\| \varphi \| \mathcal{B} \leq$ L $\sup_{\theta \leq 0} \| \varphi(\theta) \|$ for every $\varphi \in \mathcal{B}$ continuous and bounded , see [9 , Proposition 7 . 1 . 1] for details . Consequently ,
$\| \varphi t \| \mathcal{B} \leq L \frac{\sup_{\theta \leq 0} \| \varphi(\theta) \|}{\| \varphi \| \mathcal{B}} \| \varphi \| \mathcal{B}$ for every continuous and bounded function $\varphi \in \mathcal{B} \backslash \{0\}$ and every $t \leq 0$. We note that the spaces $C_r \times L^p(g; X), C_g^0(X)$ verify axiom C_2, see [9 , p . 1 0] and [9 , p . 1 6] for details .

Remark 2 . 2 . Let $\varphi \in \mathcal{B}$ and $t \ \leq \ 0$. The notation φt represents the function defined by $\varphi t(\theta) = \varphi(t + \theta)$. Consequently , if the function $x(\cdot)$ in axiom **A** is such that $x_0 = \varphi$, then $x_t = \varphi t$. We also note that , in general , $\varphi telement{-}slash \mathcal{B}$. Consider for example the characteristic function $\mathcal{X}_{[-r,0]}, r > 0$, in the space $\mathbf{C_r} \times \mathbf{L^p}($ **g** ; **X**) .

 In this paper , we adopt the following concept of mild solution .

Definition 2 . 3 . A function $x : (-\infty, a] \to X$ is called a mild solution of the
abstract Cauchy problem (1 . 1) – (1 . 2) if $x_0 = \varphi, x_{\rho(s,x_s)} \in \mathcal{B}$ for every $s \in I$ and

$$x(t) = U(t,0)\varphi(0) + \int_0^t U(t,s)f(s, x_{\rho(s,x_s)})ds + \sum_{<_0 t_i<t} U(t,t_i)I_i(x_{t_i}), \quad t \in I.$$

The next result is a consequence of the phase space axioms . **Lemma 2 . 4 .** *If* $x : (-\infty, a] \to X$ *is a function such that* $x_0 = \varphi$ *and* $x \mid I \in$

$$\mathcal{PC}(I : X), then$$
$$\| x_s \| \mathcal{B} \leq (M_a + J^{\varphi}) \| \varphi \| \mathcal{B} + K_a \sup\{\| x(\theta) \|; \theta \in [0, \max\{0, s\}]\}, \quad s \in \mathcal{R}(\rho^-) \cup I,$$
$$K_a = \sup_{t\in I} K(t).$$
$$where J^{\varphi} = \sup_{t\in\mathcal{R}(\rho^-)} J^{\varphi}(t), M_a = \sup_{t\in I} M(t) and$$

Remark 2 . 5 . In the rest of this work , $y : (-\infty, a] \to X$ is the function defined by

$$y0 = \varphi \text{and} y(t) = U(t,0)\varphi(0) \text{for} t \in I.$$

Now , we can prove our first existence result . **Theorem 2 . 6 .** *Let conditions* (H 0) – (H 3) *be satisfied and assume that*

$$1 \quad > K_a^{\widetilde{M}}(\lim_{\xi\to\infty^+} \inf \frac{W(\xi)}{\xi} \int_0^a m(s)ds + \sum_n^{i=1} L_i). \tag{2.1}$$

Then there exists a mild s o lution of (1 . 1) – (1 . 2) . *Proof .* On the space $Y = \{u \in \mathcal{PC} : u(0) = \varphi(0)\}$ endowed with the uniform convergence norm ($\| \cdot \| \infty$), we define the operator $\Gamma : Y \to Y$ defined by

$$\Gamma x(t) = U(t,0)\varphi(0) + \int_0^t U(t,s)f(s, \bar{x}_{\rho(s,\bar{x}s)})ds + \sum_{<_0 t_i<t} U(t,t_i)I_i(\bar{x}_{t_i}), \quad t \in I,$$

where $\bar{x} : (-\infty, a] \to X$ is such that $\bar{x}_0 = \varphi$ and $\bar{x} = x$ on I. From our assumptions , it is easy to see that $\Gamma x(\cdot) \in Y$.

Let $\bar{\varphi} : (-\infty, a] \to X$ be the extension of φ to $(-\infty, a]$ such that $\bar{\varphi}(\theta) = \varphi(0)$ on I and $\widetilde{J}^{\varphi} = \sup \{J^{\varphi}(s) : s \in \mathcal{R}(\rho^-)\}$. By using Lemma 2 . 4 , for $r > 0$ and

$x^r \in B_r(\bar{\varphi} \mid I, Y)$weobtain

$$\| \Gamma x^r - \varphi(0) \|$$

$$\leq (\widetilde{M}+1)H \| \varphi \| \mathcal{B} + \widetilde{M} \int_0^a m(s)W(\| \bar{x}^r_{\rho(s,\bar{x}^r_s)} \| \mathcal{B})ds + \widetilde{M} \sum_{i=1}^{n} (L_i \| \bar{x}_{t_i} \| \mathcal{B} + \| I_i(0) \|)$$

$$\leq (\widetilde{M}+1)H \| \varphi \| \mathcal{B} + \widetilde{M} \int_0^a m(s)W((M_a + \widetilde{J}^{\varphi}) \| \varphi \| \mathcal{B} + K_a \sup_{\theta \in [0,a]} \| \bar{x}^r(\theta) \|)ds + \widetilde{M} \sum_{i=1}^{n} L_i(\| \bar{x}_{t_i} - \varphi \| \mathcal{B} + \| \varphi \| \mathcal{B} + \| I_i(0) \|)$$

$$\leq (\widetilde{M}+1)H \| \varphi \| \mathcal{B} + \widetilde{M}W((M_a + \widetilde{J}^{\varphi}) \| \varphi \| \mathcal{B} + K_a(r + \| \varphi(0) \|)) \int_0^a m(s)ds, + \widetilde{M} \sum_{i=1}^{n} L_i(K_a r + \| \varphi \| \mathcal{B} + \| I_i(0) \|)$$

which from (2 . 1) implies that $\| \Gamma x^r - \varphi(0) \|_\infty \leq r$ for r large enough .

Let $r > 0$ be such that $\Gamma(B_r(\bar{\varphi} \mid I, Y)) \subset B_r(\bar{\varphi} \mid I, Y)$. Next , we will prove that $\Gamma(\cdot)$ is completely continuous from $B_r(\bar{\varphi} \mid I, Y)$ into $B_r(\bar{\varphi} \mid I, Y)$. To this end , we introduce the decomposition $\Gamma = \Gamma_1 + \Gamma_2$ where $(\Gamma_1 x)0 = \varphi, (\Gamma_2 x)0 = 0$, and

$$\Gamma_1 x(t) = U(t,0)\varphi(0) + \int_0^t U(t,s)f(s, \bar{x}_{\rho(s,\bar{x}_s)})ds, \quad t \in I$$

$$\Gamma_2 x(t) = \sum_{0 < t_i < t} U(t,t_i)I_i(\bar{x}_{t_i}), \quad t \in I.$$

To begin , we prove that the set $\Gamma_1(B_r(\bar{\varphi} \mid I, Y))(t) = \{\Gamma_1 x(t) : x \in B_r(\bar{\varphi} \mid I, Y)\}$ is relatively compact in X for every $t \in I$.

The case $t = 0$ is obvious . Let $0 < \varepsilon < t \leq a$. If $x \in B_r(\bar{\varphi} \mid I, Y)$, from Lemma 2 . 4 follows that $\| \bar{x}_{\rho(t,\bar{x}_t)} \| \mathcal{B} \leq r^* := (M_a + \widetilde{J}^{\varphi}) \| \varphi \| \mathcal{B} + K_a(r + \| \varphi(0) \|)$ which implies

$$\| \int_0^\tau U(\tau,s)f(s, \bar{x}_{\rho(s,\bar{x}_s)})ds \| \leq r^{**} := \widetilde{M}W(r^*) \int_0^a m(s)ds, \quad \tau \in I. \tag{2.2}$$

From the above inequality , we find that

$$\begin{aligned}\Gamma_1 x(t) &= U(t,0)\varphi(0) + U(t,t-\varepsilon) \int_0^{t-\varepsilon} U(t-\varepsilon,s)f(s, \bar{x}_{\rho(s,\bar{x}_s)})ds \\ &\quad + \int_{t-\varepsilon}^t U(t,s)f(s, \bar{x}_{\rho(s,\bar{x}_s)})ds \\ &\in \{U(t,0)\varphi(0)\} + U(t,t-\varepsilon)B_{r^{**}}(0,X) + C_\varepsilon,\end{aligned}$$

where diam $(C_\varepsilon) \leq 2\widetilde{M}W(r^*)\int_{t-\varepsilon}^{t} m(s)ds \to 0$ as $\varepsilon \to 0$, which allows us to conclude that $\Gamma_1(B_r(\bar{\varphi} \mid I, Y))(t)$ is relatively compact in X.

Now , we prove that $\Gamma_1(B_r(\bar{\varphi} \mid I, Y))$ is equicontinuous on I. Let $0 < t < a$ and $\varepsilon > 0$. Since the set $\Gamma_1(B_r(\bar{\varphi} \mid I, Y))(t)$ is relatively compact compact in X, from the properties of the evolution family $U(t, s)$, there exists $0 < \delta \leq a - t$ such that

 $\| U(t+h,t)x - x \| < \varepsilon$, for every $x \in \Gamma_1(B_r(\bar\varphi \mid I,Y))(t)$ and all $0 < h < \delta$. Under these conditions , for $x \in B_r(\bar\varphi \mid I,Y)$ and $0<h<\delta$ we obtain

$$\begin{aligned} \| \Gamma_1 x(t+h) - \Gamma_1 x(t) \| &\leq \| U(t+h,0)\varphi(0) - U(t,0)\varphi(0) \| \\ &+ \| (U(t+h,t) - I) \int_0^t U(t,s) f(s, \bar x_{\rho(s,\bar x_s)}) ds \| \\ &+ \widetilde{M} \int_t^{t+h} m(s) W(r^*) ds \\ &\leq 2\varepsilon + \widetilde{M} W(r^*) \int_t^{t+h} m(s) ds, \end{aligned}$$

which proves that $\Gamma_1(B_r(\bar\varphi \mid I,Y))$ is right equicontinuous at $t \in (0,a)$. A simi - lar procedure shows that $\Gamma_1(B_r(\bar\varphi \mid I,Y))$ is right equicontinuous at zero and left equicontinuous at $t \in (0,a]$. Thus , the set $\Gamma_1(B_r(\bar\varphi \mid I,Y))$ is equicontinuos on I.

Using the same arguments as in [1 1 , Theorem 2 . 2] , it follows that Γ_1 is a con - tinuous map , which complete the proof that Γ_1 is completely continuous . On the other hand , from the assumptions and the phase space axioms it follows that

$$\| \Gamma_2 x - \Gamma_2 y \| \infty \leq K_a^{\widetilde{M}} \sum_{i=1}^{n} L_i \| x - y \|_\infty$$

which proves that Γ_2 is a contraction on $B_r(\bar\varphi \mid I,Y)$ and that Γ is a condensing map

$$\text{on} B_r(\bar\varphi \mid I,Y).$$

Finally , the existence of a mild solutions is a consequence of [1 9 , Theorem 4 . 3 . 2] .

The proof is complete . □

In the next result , $\mathcal{BPC}(\varphi)$ is the set introduced in assumption (H 0) . **Theorem 2 . 7 .** *Let (H 0) – (H 2) be satisfied . If* $\rho(t,x_t) \leq t$ *for every* $(t,x) \in I \times$

$$\mathcal{BPC}(\varphi), \mu = 1 - K_a^{\widetilde{M}} \sum_{i=1}^{n} c_i > 0 and$$
$$K_a^{\widetilde{M}} \int_0^a m(s) ds < \int_C^\infty \frac{ds}{W(s)},$$

where

$$C = (M_a + J^\varphi + \widetilde{M} H K_a) \| \varphi \| \mathcal{B} + \frac{\widetilde{M} K_a}{\mu} \sum_{n}^{i=1} [i_c^1 (M_a + \widetilde{M} H K_a) \| \varphi \| \mathcal{B} + i_c^2]$$

then there exists a mild s o lution of (1 . 1) – (1 . 2) . Proof . On the space $\mathcal{BPC} = \{u : (-\infty, a] \to X; u_0 = 0, u \mid I \in \mathcal{PC}\}$ provided with the sup - norm $\| \cdot \|_\infty$, we define the operator $\Gamma : \mathcal{BPC} \to \mathcal{BPC}$ by $(\Gamma u)0 = 0$ and

$$\Gamma x(t) = \int_0^t U(t,s) f(s, \bar x_{\rho(s,\bar x_s)}) ds + \sum_{<_0 t_i < t} U(t,t_i) I_i(\bar x_{t_i}), \quad t \in I,$$

where $\bar x = x + y$ on $(-\infty, a]$ and $y(\cdot)$ is the function defined in Remark 2 . 5 . To use Theorem 1 . 6 , we establish *a priori* estimates for the solutions of the integral equation $z = \lambda \Gamma z, \lambda \in (0,1)$. Let x^{λ} be a solution of $z = \lambda \Gamma z, \lambda \in (0,1)$. By using Lemma 2 . 4 , the notation $\alpha^{\lambda}(s) = \sup_{\theta \in [0,s]} \| x^{\lambda}(\theta) \|$, and the fact that $\rho(s, \_(x^{\lambda})_s) \leq$

 s, for each $s \in I$, we find that

$$\| x^{\lambda}(t) \| \leq \widetilde{M} \int_0^t m(s) W((M_a + J^{\varphi} + \widetilde{M} H K_a) \| \varphi \| \mathcal{B} + K_a \alpha^{\lambda}(s)) ds$$

$$+ \widetilde{M} \sum_{<0 t_i \leq t} i_c^1 [(M_a + \widetilde{M} H K_a) \| \varphi \| \mathcal{B} + K_a \alpha^{\lambda}(t)] + \widetilde{M} \sum_{i=1}^{n} i_c^2,$$

and so ,

$$\alpha^{\lambda}(t) \leq \widetilde{M} \sum [i_c^1 (M_a + \widetilde{M} H K_a) \| \varphi \| \mathcal{B} + i_c^2] + K_a^{\widetilde{M}} \sum_{i=1}^{n} \sum_{0 < t_i \leq t} i_c^1 \alpha^{\lambda}(t)$$

$$+ \widetilde{M} \int_0^t m(s) W((M_a + J^{\varphi} + \widetilde{M} H K_a) \| \varphi \| \mathcal{B} + K_a \alpha^{\lambda}(s)) ds,$$

which implies

$$\alpha^{\lambda}(t) \leq \frac{\widetilde{M}}{\mu} \sum_{n}^{i=1} [i_c^1 (M_a + \widetilde{M} H K_a) \| \varphi \| \mathcal{B} + i_c^2]$$

$$+ \frac{\widetilde{M}}{\mu} \int_0^t m(s) W((M_a + J^{\varphi} + \widetilde{M} H K_a) \| \varphi \| \mathcal{B} + K_a \alpha^{\lambda}(s)) ds,$$

for every $t \in [0, a]$. By defining $\xi^{\lambda}(t) = (M_a + J^{\varphi} + \widetilde{M} H K_a) \| \varphi \| \mathcal{B} + K_a \alpha^{\lambda}(t)$, we find that

$$\xi^{\lambda}(t) \leq (M_a + J^{\varphi} + \widetilde{M} H K_a) \| \varphi \| \mathcal{B} + \frac{\widetilde{M} K_a}{\mu} \sum_{n}^{i=1} [i_c^1 (M_a + \widetilde{M} H K_a) \| \varphi \| \mathcal{B} + i_c^2]$$

$$+ \frac{\widetilde{M} K_a}{\mu} \int_0^t m(s) W(\xi^{\lambda}(s)) ds.$$

Denoting by $\beta\lambda(t)$ the right hand side of the last inequality , if follows that

$$\beta_{\lambda}'(t) \leq \frac{\widetilde{M} K_a}{\mu} m(t) W(\beta\lambda(t))$$

and hence

$$\int_{\beta\lambda(0)=C}^{\beta\lambda(t)} \frac{ds}{W(s)} \leq \frac{\widetilde{M} K_a}{\mu} \int_0^a m(s) ds < \int_C^{\infty} \frac{ds}{W(s)},$$

which implies that the set of functions $\{\beta\lambda(\cdot) : \lambda \in (0, 1)\}$ is bounded in $C(I, \mathbb{R})$. This show that the set $\{x^{\lambda}(\cdot) : \lambda \in (0, 1)\}$ is bounded in $\mathcal{BPC}$.

To prove that the map Γ is completely continuous , we consider the decomposition

$$\Gamma = \Gamma_1 + \Gamma_2 \text{where} (\Gamma_i x) 0 = 0, i = 1, 2, \text{and}$$

$$\Gamma_1 x(t) = \int_0^t U(t, s) f(s, \bar{x}_{\rho(s, \bar{x}_s)}) ds, \quad t \in I,$$

$$\Gamma_2 x(t) = \sum_{0 < t_i < t} U(t, t_i) I_i(\bar{x}_{t_i}), \quad t \in I.$$

Proceeding as in the proof of Theorem 2 . 6 we can prove that Γ_1 is completely continuous . The continuity of Γ_2 can be proven using the phase space axioms .

 To prove that Γ_2 is also completely continuous , we use Lemma 1 . 2 . For $r > 0$, $t \in [t_i, t_{i+1}] \cap (0, a], i \geq 1$, and $u \in B_r = B_r(0, \mathcal{BPC})$ we find that

$$\widetilde{\Gamma}_{2^u}(t) \in braceleftbt-braceex-braceex-braceleftmid-braceex-braceex \sum \sum_{i-1_{j=0}^{j=1}}^{j=1_i^i} U(t_{i+1}, t_j) I_j(B_{r^*}(0, X))_{U(t,t_j)}^{U(t_i,t_j)}$$

where $r^* := (M_a +^{\widetilde{M}} HK_a) \parallel \varphi \parallel \mathcal{B} + K_a r$, which proves that $[\Gamma_{\widetilde{2}}(B_r)]i(t)$ is relatively compact in X for every $t \in [t_i, t_{i+1}]$, since the maps I_j are completely continuous . Moreover , using the compactness of the operators I_i and properties of the evolution family $U(\cdot)$, we can prove that $[\Gamma_{\widetilde{2}}(B_r)]i(t)$ is equicontinuous at t, for every $t \in [t_i, t_{i+1}]$ and each $i = 1, 2, ..., n$, which complete the proof that Γ_2 is completely continuous .

The existence of a mild solution is now a consequence of Theorem 1 . 6 . The proof is complete . □

3 . Applications

In this section we consider an application of our abstract results . Consider the partial differential equation

$$\frac{\partial u(t,\xi)}{\partial t} = + \frac{\partial^2 u(t,\xi)}{integraldisplay - minus^{t}{}^{\infty}_{\partial\xi^2} a} {}^{+}_{1(} {}_s a^{0\ (t,\ \xi) u(t,\ \xi)}_{-t) u(s\ \ -\rho 1(t) \rho 2(\int_0^\pi} a_2(\theta) \mid u(t,\theta) \mid^2 d\theta), \xi) ds \tag{3.1}$$

for $t \in I = [0, a], \xi \in [0, \pi]$. The above equation is subject to the conditions

$$u(t, 0) = u(t, \pi) = 0, \quad t \geq 0, \tag{3.2}$$

$$u(\tau, \xi) = \varphi(\tau, \xi), \quad \tau \leq 0, 0 \leq \xi \leq \pi. \tag{3.3}$$

$$\Delta u(t_j, \xi) = -{}^{tj}_{\infty} \gamma_j(s - t_j) u(s, \xi) ds, \quad j = 1, 2, ..., n. \tag{3.4}$$

To study this system , we consider the space $X = L^2([0, \pi])$ and the opera - tor $A : D(A) \subset X \to X$ given by $Ax = x''$ with $D(A) := \{x \in X : x'' \in X,\ x(0) = x(\pi) = 0\}$. It is well known that A is the infinitesimal generator of an analytic semigroup $(T(t))t \geq 0$ on X. Furthermore , A has discrete spectrum with eigenvalues $-n^2$, $n \in \mathbb{N}$, and corresponding normalized eigenfunctions given by $z_n(\xi) = (\frac{2}{\pi})^{1/2} \sin\ (n\xi)$. In addition , $\{z_n : n \in \mathbb{N}\}$ is an orthonormal basis of X and $T(t)x = \sum_{n=1}^{\infty} e^{-n^2 t} \langle x, z_n \rangle z_n$ for $x \in X$ and $t \geq 0$. It follows from this rep - resentation that $T(t)$ is compact for every $t > 0$ and that $\parallel T(t) \parallel \leq e^{-t}$ for every

$t \geq 0$.

On the domain $D(A)$, we define the operators $A(t) : D(A) \subset X \to X$ by $A(t)x(\xi) = Ax(\xi) + a_0(t, \xi)x(\xi)$. By assuming that $a_0(\cdot)$ is continuous and that $a_0(t, \xi) \leq -\delta_0(\delta_0 > 0)$ for every $t \in \mathbb{R}, \xi \in [0, \pi]$, it follows that the system

$$u'(t) = A(t)u(t) \quad t \geq s,$$
$$u(s) = x \in X,$$

has an associated evolution family given by $U(t,s)x(\xi) = [T(t-s)e^{Rt}_s a_0(\tau,\xi)d\tau_x](\xi)$. From this expression , it follows that $U(t,s)$ is a compact linear operator and that $\| U(t,s) \| \leq e^{-(1+\delta_0)(t-s)}$ for every $t, s \in I$ with $t > s$.

Proposition 3 . 1 . *Let* $\mathcal{B} = \mathcal{PC}_0 \times L^2(g,X)$ *and* $\varphi \in \mathcal{B}$. *Assume that condition (H 0) holds ,* $\rho i : [0,\infty) \to [0,\infty), i = 1,2,$ *are continuous and that the following conditions are verified .*

(a) *The functions* $a_i : \mathbb{R} \to \mathbb{R}$ *are continuous and* $L_f = (\int_{-\infty}^0 \frac{(a_1(s))^2}{g(s)} ds)^{1/2}$ *is finite .*

(b) *The functions* $\gamma i : \mathbb{R} \to \mathbb{R}, i = 1,2,...,n,$ *are continuous , bounded and*

$$L_i := (\int_{-\infty}^0 \frac{(\gamma i(s))^2}{g(s)} ds)1/2 < \infty foreveryi = 1,2,...,n.$$

Then there exists a mild so lution of (3 . 1) – (3 . 3) . *Proof .* From the assumptions , we have that

$$f(t,\psi)(\xi) = integraldisplay - minus^0_\infty a_1(s)\psi(s,\xi)ds,$$

$$\rho(s,\psi) = s - \rho 1(s)\rho 2(\int_0^\pi a_2(\theta) \mid \psi(0,\xi) \mid^2 d\theta),$$

$$I_i(\psi)(\xi) = integraldisplay - minus^0_\infty \gamma i(s)\psi(s,\xi)ds, \quad i = 1,2,...,n,$$

are well defined functions , which permit to transform system (3 . 1) – (3 . 3) into the abstract system (1 . 1) – (1 . 2) . Moreover , the functions f, I_i are bounded linear oper - ator , $\| f \| \leq L_1$ and $\| I_i \| \leq L_i$ for every $i = 1,2,...n.$ Now , the existence of a mild solutions can be deduced from a direct application of Theorem 2 . 7 . The proof is complete . □

From Remark 2 . 1 we have the following result . .

Corollary 3 . 2 . *Let* $\varphi \in \mathcal{B}$ *be continuous and bounded . Then there exists a mild s o lutio n of (3 . 1) – (3 . 3) on* I.

Acknowledgements . The authors are grateful to the anonymous referees for their comments and suggestions .

References

[1] Aiello , Walter G . ; Freedman , H . I . ; Wu , J . ; Analysis of a model representing stage - structured population growth with state - dependent t ime delay . *SIAM J . Appl . Math .* 52 (3) (1 992) , 855 – 869 .

[2] Arino , Ovide ; Boushaba , Khalid ; Boussouar , Ahmed A mathematical model of the dynamics of the phytoplankton - nutrient system . Spatial heterogeneity in ecological models (Alcal de Henares , 1 998) . *Nonlinear Analysis RWA .* 1 (1) (2000) , 69 – 87 .

[3] Cao , Yulin ; Fan , Jiangping ; Gard , Thomas C . ; The effects of state - dependent t ime delay on a stage - structured population growth model . *Nonlinear Analysis TMA .* , 1 9 (2) (1 992) , 95 – 105 . [4] Alexander Domoshnitsky , Michael Drakhlin and Elena Litsyn ; On equations with delay de - pending on solution . *Nonlinear Analysis TMA .* , 49 (5) (2002) , 689 - 70 1 .

[5] Granas , A . ; Dugundji , J . ; *Fixed Point Theory* . Springer - Verlag , New York , 2003 .

[6] Hartung , Ferenc , Linearized stability in periodic functional differential equations with state - dependent delays . *J . Comput . Appl . Math .* , 1 74 (2) (2005) , 20 1 - 2 1 1 .

[7] Hartung , Ferenc ; Herdman , Terry L . ; Turi , Janos ; Parameter identification in classes of neutral differential equations with state - dependent delays . *Nonlinear Analysis TMA . Ser . A : Theory Methods* , 39 (3) (2000) , 305 – 325 .

[8] Hartung , Ferenc ; Turi , Janos ; Identification of parameters in delay equations with state - dependent delays . *Nonlinear Analysis TMA .* , 29 (1 1) (1 997) , 1 303 - 1 3 18 .

[9] Hino , Yoshiyuki ; Murakami , Satoru ; Naito , Toshiki ; *Functional - differential equations with infinite delay . Lecture Notes in Mathematics ,* 1473 . Springer - Verlag , Berlin , 1 991 . [10] Hern á ndez , E ; Mark A . Mckibben . ; On state - dependent delay partial neutral functional dif - ferential equations . *Appl . math . Comput .* 186 (1) (2006) , 294 - 30 1 . [1 1] Hern á ndez , E ; Prokopczyk , A ; Ladeira , Luiz ; A note on partial functional differential equa - t ions with state - dependent delay . *Nonlinear Analysis : Real World Applications* , 7 (2006) , 5 10 - 5 1 9 . [1 2] Hern á ndez , E ; Mallika Arjunan , A . Anguraj ; Existence Results for an Impulsive Neutral Functional Differential Equation with State - Dependent Delay . *Appl . Anal .* , 86 (7) (2007) , 86 1 - 872

[1 3] Hern á ndez , E ; Existence of Solutions for a Second order Abstract Functional Differential Equation with State - Dependent Delay . *Electronic Journal of Differential Equations* , (2007) , No . 2 1 pp . 1 - 10 . [14] Hern á ndez , E ; M . Pierri and G . Goncalves . ; Existence results for an impulsive abstract partial differential equation with state - dependent delay . *Comput . Appl . Math .* , 52 (2006) , 41 1 - 420 . [1 5] Hern á ndez , Eduardo ; Henriquez , Hernan R ; Impulsive partial neutral differential equations , *Appl . Math . Lett .* , 1 9 (3) (2006) , 2 1 5 - 222 . [1 6] Hern á ndez , Eduardo ; Henriquez , Hernan R ; Marco Rabello ; Existence of solutions for a class of impulsive partial neutral functional differential equations , *J . Math . Anal . Appl .* , 33 1 (2) (2007) , 1 1 35 - 1 1 58 . [17] V . Lakshmi k − a ntham , D . D . Bainov , and P . S . Simeonov ; *Theory of Impulsive Differential Equations* , World Scientific , Singapore , 1 989 . [18] Liu , James H . ; Nonlinear impulsive evolution equations , *Dynam . Contin . Discrete Impuls . Systems* 6 (1) (1 999) , 77 - 85 . [1 9] Martin , R . H . , *Nonlinear Operators and Differential Equations in Banach Spaces ,* Robert E . Krieger Publ . Co . , Florida , 1 987 . [20] Pazy , A . ; *Semigroups of linear operators and applications to partial differential equations .* Applied Mathematical Sciences , 44 . Springer - Verlag , New York - Berlin , 1 983 . [2 1] Rezounenko , Alexander V . ; Wu , Jianhong ; A non - local PDE model for population dynamics with state - selective delay : Local theory and global attractors , *J . Comput . Appl . Math .* , 1 90 (1 - 2) (2006) , 99 - 1 1 3 . [22] Alexander V . Rezounenko ; Partial differential equations with discrete and distributed state - dependent delays , *J . Math . Anal . Appl .* , 326 (2) (2007) , 103 1 - 1045 . [23] Rogovchenko , Yuri V . ; Impulsive evolution systems : main results and new trends , *Dynam . Contin . Discrete Impuls . Systems* , 3 (1) (1 997) , 57 - 88 . [24] Rogovchenko , Yuri V . ; Nonlinear impulse evolution systems and applications to population models , *J . Math . Anal . Appl .* , 207 (2) (1 997) , 300 - 3 1 5 . [25] A . M . Samoilenko and N . A . Perestyuk ; *Impulsive Differential Equations* , World Scientific , Singapore , 1 995 .

Eduardo Hern $\acute{a}_{ndez}$, Sueli Tanaka Aki Departamento de Matem á tica , I . C . M . C . Universidade de S $\tilde{a}_o$ Paulo , Caixa Postal 6 68 , 1 3 560 - 970 , S $\tilde{a}_o$ Carlos SP , Brazil

E - mail address : lalo h − m@ i cmc . sc . usp . br

Rathinasamy Sakthivel

Department of Mechanical Engineering , Pohang University of Science and Technology , Pohang - 79 0 - 784 , South Korea

E - mail address : krsakthivel @y^{a-h} oo . com

Sueli Tanaka Aki

Departamento de Matem á tica , I . C . M . C . Universidade de S $\tilde{a}_o$ Paulo , Caixa Postal 668 , 1 3 560 - 970 , S $\tilde{a}_o$ Carlos SP , Brazil

E - mail address : smtanaka @ i cmc . sc . usp . br