Electronic Journal of Differential Equations , Vol . 2007 (2007) , No . 69 , pp . 1 – 9 .
ISSN : 1 72 - 6691 . URL : http : / / ejde . math . txstate . edu or http : / / ej de . math . unt . edu
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A FIBE R – I NG MAP APPROACH TO A SEMILINEAR ELLIPTIC BOUNDARY VALUE PROBLEM

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Abstract . We prove the existence of at least two positive solutions for the semilinear elliptic boundary - value problem

$$-\Delta u(x) = \lambda a(x)u^q + b(x)u^p \quad \text{for} x \in \Omega; \quad u(x) = 0 \quad \text{for} x \in \partial\Omega$$

on a bounded region Ω by using the Nehari manifold and the fibering maps associated with the Euler functional for the problem . We show how knowledge of the fibering maps for the problem leads to very easy existence proofs .

1 . Introduction

We shall discuss the existence of positive solutions of the semilinear elliptic boundary - value problem

$$-\Delta u(x) = \lambda a(x)u^q + b(x)u^p \quad \text{for} x \in \Omega; \tag{1.1}$$

$$u(x) = 0 \quad \text{for} x \in \partial\Omega, \tag{1.2}$$

where Ω is a bounded region with smooth boundary in $\mathbb{R}^N, 0 < q < 1 < p < \frac{N+2}{N-2}$, $\lambda > 0$ and $a, b : \Omega \to \mathbb{R}$ are smooth functions which are somewhere positive but which may change sign on Ω. Equation (1 . 1) , (1 . 2) has been recently studied in [3] by using the Mountain Pass Lemma and in [5] and [7] using the Nehari manifold .

In [4] and [2] it was shown that the Nehari manifold for an equation such as (1 . 1) is closely related to the fibering maps for the problem . In this paper we show how a fairly complete knowledge of all possible forms of the fibering maps provides a very simple and comparatively elementary means of establishing results similar to those proved in [5] and [7] on the existence of multiple solutions of (1 . 1) , (1 . 2) . In section 2 we recall the properties which we shall require of fibering maps and of the Nehari manifold . In section 3 we give a fairly complete description of the fibering maps associated with (1 . 1) and in section 4 we use this information to give a very simple variational proof of the existence of at least two positive solutions of (1 . 1) , (1 . 2) for sufficiently small λ.

We shall throughout use the function space $W_0^{1,2}(\Omega)$ with norm

$$\| u \| = (\int_\Omega | \nabla u |^2 \, dx)1/2$$

2000 *Mathematics Subject Classification* . 35 J 20 , 36 J 65 .

Key words and phrases . Semilinear elliptic boundary value problem ; variational methods ; Nehari manifold ; fibering map .

Submitted Febraury 27 , 2007 . Published May 10 , 2007 .

 and the st andard $L^p(\Omega)$ spaces whose norms we denote by $\| u \|_p$.

2. Fibering Maps and the Nehari manifold

The Euler functional associated with (1.1), (1.2) is

$$J_\lambda(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\, dx - \frac{\lambda}{q+1}\int_\Omega a(x) |u|^{q+1}\, dx - \frac{1}{p+1}\int_\Omega b(x) |u|^{p+1}\, dx \quad \text{for all } u \in W_0^{1,2}(\Omega).$$

As J_λ is not bounded below on $W_0^{1,2}(\Omega)$, it is useful to consider the functional on the Nehari manifold

$$M_\lambda(\Omega) = \{u \in W_0^{1,2}(\Omega) : \langle J'_\lambda(u), u\rangle = 0\}$$

where $\langle , \rangle$ denotes the usual duality. Thus $u \in M_\lambda(\Omega)$ if and only if

$$\int_\Omega |\nabla u|^2\, dx - \lambda\int_\Omega a(x) |u|^{q+1}\, dx - \int_\Omega b(x) |u|^{p+1}\, dx = 0 \tag{2.1}$$

Clearly $M_\lambda(\Omega)$ is a much smaller set than $W_0^{1,2}(\Omega)$ and, as we shall show, J_λ is much better behaved on $M_\lambda(\Omega)$. In particular, on $M_\lambda(\Omega)$ we have that

$$J_\lambda(u) = \Big(\frac{1}{2} - \frac{1}{p+1}\Big)\int_\Omega |\nabla u|^2 - \lambda\Big(\frac{1}{q+1} - \frac{1}{p+1}\Big)\int_\Omega a(x)|u|^{q+1} \tag{2.2}$$

Theorem 2.1. *J_λ is coercive and bounded below on $M_\lambda(\Omega)$.*

Proof. It follows from (2.2) and the Sobolev embedding theorems that there exist positive constants c_1, c_2 and c_3 such that

$$J_\lambda(u) \geq c_1 \| u \|^2 - c_2\int_\Omega |u|^{q+1}\, dx \geq c_1 \| u \|^2 - c_3 \| u \|^{q+1}$$

and so J_λ is coercive and bounded below on $M_\lambda(\Omega)$. □

The Nehari manifold is closely linked to the behaviour of the functions of the form $\phi_u : t \to J_\lambda(tu)$ $(t > 0)$. Such maps are known as fibering maps and were introduced by Drabek and Pohozaev in [4] and are also discussed in Brown and Zhang [2]. If $u \in W_0^{1,2}(\Omega)$, we have

$$\phi_u(t) = \frac{1}{2}t^2\int_\Omega |\nabla u|^2 - \lambda\frac{t^{q+1}}{q+1}\int_\Omega a |u|^{q+1} - \frac{t^{p+1}}{p+1}\int_\Omega b |u|^{p+1} \tag{2.3}$$

$$\phi'_u(t) = t\int_\Omega |\nabla u|^2 - \lambda t^q\int_\Omega a |u|^{q+1} - t^p\int_\Omega b |u|^{p+1} \tag{2.4}$$

$$\phi''_u(t) = \int_\Omega |\nabla u|^2 - \lambda q t^{q-1}\int_\Omega a |u|^{q+1} - p t^{p-1}\int_\Omega b |u|^{p+1} \tag{2.5}$$

It is easy to see that $u \in M_\lambda(\Omega)$ if and only if $\phi'_u(1) = 0$ and, more generally, that $\phi'_u(t) = 0$ if and only if $tu \in M_\lambda(\Omega)$, i.e., elements in $M_\lambda(\Omega)$ correspond to stationary points of fibering maps. Thus it is natural to subdivide $M_\lambda(\Omega)$ into sets corresponding to lo cal minima , local maxima and points of inflection and so we define

$$M_\lambda^+(\Omega) = \{u \in M_\lambda(\Omega) : \phi_u''(1) > 0\},$$
$$M_\lambda^-(\Omega) = \{u \in M_\lambda(\Omega) : \phi_u''(1) < 0\},$$
$$M_\lambda^0(\Omega) = \{u \in M_\lambda(\Omega) : \phi_u''(1) = 0\},$$

and note that if $u \in M_\lambda(\Omega)$, i . e ., $\phi_u'(1) = 0$, then

$$\phi_u''(1) = (1-q)\int_\Omega |\nabla u|^2\,dx - (p-q)\int_\Omega b(x)|u|^{p+1}\,dx = (1-p)\int_\Omega |\nabla u|^2\,dx - \lambda(q-p)\int_\Omega a(x)|u|^{q+1}\,dx. \tag{2.6}$$

Also , as proved in Binding , Drabek and Huang [1] or in Brown and Zhang [2] , we have the following lemma .

Lemma 2 . 2 . *Suppose that* u_0 *is a local maximum o r minimum for* J_λ *on* $M_\lambda(\Omega)$.
Then , if $u_0 \notin M_\lambda^0(\Omega), u_0$ *is a critical point of* J_λ.

3 . Analysis of the Fibering Maps

In this section we give a fairly complete description of the fibering maps as - so ciated with the problem . As we shall see the essential nature of the maps is determined by the signs of $\int a(x) \mid u \mid^{q+1} dx$ and $\int_\Omega b(x) \mid u \mid^{p+1} dx$. We will find it useful to consider the function

$$m_u(t) = t^{1-q}\int_\Omega \mid \nabla u \mid^2 dx - t^{p-q}\int_\Omega b(x) \mid u \mid^{p+1} dx.$$

Clearly , for $t > 0, tu \in M_\lambda(\Omega)$ if and only if t is a solution of

$$m_u(t) = \lambda\int_\Omega a(x) \mid u \mid^{q+1} dx. \tag{3.1}$$

Morever ,

$$m_u'(t) = (1-q)t^{-q}\int_\Omega \mid \nabla u \mid^2 dx - (p-q)t^{p-q-1}\int_\Omega b(x) \mid u \mid^{p+1} dx. \tag{3.2}$$

It is easy to see that m_u is a strictly increasing function for $t \geq 0$ whenever $\int_\Omega b(x) \mid u \mid^{p+1} dx \leq 0$ and m_u is initially increasing and eventually decreasing with a single turning point as in Figure 1 (b) when $\int_\Omega b(x) \mid u \mid^{p+1} dx > 0$.

Figure 1 . Possible forms of $m(u)$

Suppose $tu \in M_\lambda(\Omega)$. It follows from (2 . 6) and (3 . 2) that $\phi''_{tu}(1) = t^{q+2}m'_u(t)$ and so $tu \in M^+_\lambda(\Omega)(M^-_\lambda(\Omega))$ provided $m'_u(t) > 0(< 0)$.

We shall now describe the nature of the fibering maps for all possible signs of $\int_\Omega b(x) \mid u \mid^{p+1} dx$ and $\int_\Omega a(x) \mid u \mid^{q+1} dx$. If $\int_\Omega b(x) \mid u \mid^{p+1} dx \leq 0$ and $\int_\Omega a(x) \mid u \mid^{q+1} dx \leq 0$, clearly ϕ_u is an increasing function of t and so has graph as shown in Figure 2 (a) ; thus in this case no multiple of u lies in $M_\lambda(\Omega)$. If $\int_\Omega b(x) \mid u \mid^{p+1} dx \leq 0$ and $\int_\Omega a(x) \mid u \mid^{q+1} dx > 0$, then m_u has graph as in Figure 1 (a) , and it is clear that there
is exactly one solution of (3 . 1) . Thus there is a unique value $t(u) > 0$ such that $t(u)u \in M_\lambda(\Omega)$. Clearly $m'_u(t(u)) > 0$ and so $t(u)u \in M^+_\lambda(\Omega)$. Thus the fibering map ϕ_u has a unique critical point at $t = t(u)$ which is a lo cal minimum . Since $\lim_{t\to\infty} \phi_u(t) = \infty$, it follows that ϕ_u has graph as shown in Figure 2 (c) .

Suppose now $\int_\Omega b(x) \mid u \mid^{p+1} dx > 0$ and $\int_\Omega a(x) \mid u \mid^{q+1} dx \leq 0$. Then m_u has graph
as shown in Figure 1 (b) and it is clear that there is exactly one positive solution of (3 . 1) . Thus there is again a unique value $t(u) > 0$ such that $t(u)u \in M_\lambda(\Omega)$ and since $m'_u(t(u)) < 0$ in this case $t(u)u \in M^-_\lambda(\Omega)$. Hence the fibering map ϕ_u has a unique critical point which is a lo cal maximum . Since $\lim_{t\to\infty} \phi_u(t) = -\infty$, it follows that ϕ_u has graph as shown in Figure 2 (b) .

Finally we consider the case $\int_\Omega b(x) \mid u \mid^{p+1} dx > 0$ and $\int_\Omega a(x) \mid u \mid^{q+1} dx > 0$ where
the situation is more complicated . As in the previous case m_u has a graph as shown in Figure 1 (b) . If $\lambda > 0$ is sufficiently large , (3 . 1) has no solution and so ϕ_u has no critical points - in this case ϕ_u is a decreasing function . Hence no multiple of u lies in $M_\lambda(\Omega)$. If , on the other hand , $\lambda > 0$ is sufficiently small , there are exactly two solutions $t_1(u) < t_2(u)$ of (3 . 1) with $m'_u(t_1(u)) > 0$ and $m'_u(t_2(u)) < 0$.
Thus there are exactly two multiples of $u \in M_\lambda(\Omega)$, namely $t_1(u)u \in M^+_\lambda(\Omega)$ and $t_2(u)u \in M^-_\lambda(\Omega)$. It follows that ϕ_u has exactly two critical points - a lo cal minimum at $t = t_1(u)$ and a lo cal maximum at $t = t_2(u)$; moreover ϕ_u is decreasing in $(0, t_1)$, increasing in (t_1, t_2) and decreasing in (t_2, ∞) as in Figure 2 (d) .
The following result ensures that when λ is sufficiently small the graph of ϕ_u must be as shown in Figure 2 (d) for all non - zero u.

Lemma 3 . 1 . *There exists* $\lambda_1 > 0$ *such that , when* $\lambda < \lambda_1, \phi_u$ *takes on positive*
values for al l non - zero $u \in W^{1,2}_0(\Omega)$.

Proof . If $\int_\Omega b(x) \mid u \mid^{p+1} dx \leq 0$, then $\phi_u(t) > 0$ for t sufficiently large . Suppose

$$u \in W^{1,2}_0(\Omega) \text{and} \int_\Omega b(x) \mid u \mid^{p+1} dx > 0.\text{Let}$$

$$h_u(t) = \frac{t^2}{2}\int_\Omega \mid \nabla u \mid^2 dx - p\frac{t^{p+1}}{+1}\int_\Omega b(x) \mid u \mid^{p+1} dx.$$

Then elementary calculus shows that h_u takes on a maximum value of

$$\frac{p-1}{2(p+1)}\{\frac{(\int_\Omega \mid \nabla u \mid^2 dx)^{p+1}}{(\int_\Omega b(x) \mid u \mid^{p+1} dx)^2}\}\frac{1}{p-1} \quad \text{when} t = t_{\max} = (\frac{\int_\Omega \mid \nabla u \mid^2 dx}{\int_\Omega b(x) \mid u \mid^{p+1} dx}{}^{)}\frac{1}{p-1}.$$

However

$$\frac{(\int_\Omega \mid \nabla u \mid^2 dx)^{p+1}}{(\int_\Omega \mid u \mid^{p+1} dx)^2} \geq \frac{1}{S^{2(p+1)}_{p+1}}$$

where S_{p+1} denotes the Sobolev constant of the embedding of $W_0^{1,2}(\Omega)$ into $L^{p+1}(\Omega)$. Hence

$$h_u(t_{\max}) \geq \frac{p-1}{2(p+1)}\Big(\frac{1}{\| b^+ \|_\infty^2 \, S_{p+1}^{2(p+1)}}\Big)^{\frac{1}{p-1}} = \delta$$

FIGURE 2 . Possible forms of fibering maps where δ is independent of u.

We shall now show that there exists $\lambda_1 > 0$ such that $\phi_u(t_{\max}) > 0$, i . e . ,

$$h_u(t_{\max}) - \frac{\lambda(t_{\max})^{q+1}}{q+1}\int_\Omega a(x) \mid u \mid^{q+1} dx > 0$$

for all $u \in W_0^{1,2}(\Omega) - \{0\}$ provided $\lambda < \lambda_1$. We have

$$\frac{(t_{\max})^{q+1}}{q+1}\int_\Omega a(x) \mid u \mid^{q+1} dx$$

$$\leq \frac{1}{q+1} \parallel a \parallel_\infty S_{q+1}^{q+1}(\frac{\int_\Omega \mid \nabla u \mid^2 dx}{\int_\Omega b(x) \mid u \mid^{p+1} dx}{}^{)}\frac{q+1}{p-1}(\int_\Omega \mid \nabla u \mid^2 dx)\frac{q+1}{2}$$

$$= \frac{1}{q+1} \parallel a \parallel \infty S_{q+1}^{q+1}\{\frac{(\int_\Omega \mid \nabla u \mid^2 dx)^{p+1}}{(\int_\Omega b(x) \mid u \mid^{p+1} dx)^2}\}^{line-parenleft-minus2q_p+1_1}$$

$$= \frac{1}{q+1} \parallel a \parallel_\infty S_{q+1}^{q+1}[\frac{2(p+1)}{p-1}{}^{]}\frac{q+1}{2}h_u(t_{\max})\frac{q+1}{2} = ch_u(t_{\max})\frac{q+1}{2}$$

where c is independent of u. Hence

$$\phi_u(t_{\max}) \geq h_u(t_{\max}) - \lambda ch_u(t_{\max})\frac{q+1}{2} = h_u(t_{\max})\frac{q+1}{2}(h_u(t_{\max})\frac{1-q}{2} - \lambda c)$$

and so , since $h_u(t_{\max}) \geq \delta$ for all $u \in W_0^{1,2}(\Omega) - \{0\}$, it follows that $\phi_u(t_{\max}) > 0$ for all non - zero u provided $\lambda < \delta\frac{1-q}{2}{}_{/2c} = \lambda_1$. This completes the proof . □

It follows from the above lemma that when $\lambda < \lambda_1$, $\int_\Omega a(x) \mid u \mid^{q+1} dx > 0$ and $\int_\Omega b(x) \mid u \mid^{p+1} dx > 0$ then ϕ_u must have exactly two critical points as discussed in the remarks preceding the lemma .

Thus when $\lambda < \lambda_1$ we have obtained a complete knowledge of the number of critical points of ϕ_u, of the intervals on which ϕ_u is increasing and decreasing and of the multiples of u which lie in $M_\lambda(\Omega)$ for every possible choice of signs of $\int_\Omega b(x) \mid u \mid^{p+1} dx$ and $\int_\Omega a(x) \mid u \mid^{q+1} dx$. In particular we have the following result .

$$M^0_\lambda(\Omega) = \varnothing when 0 < \lambda < \lambda_1. \qquad \textbf{Corollary} 3.2.$$

Corollary 3 . 3 . *If* $\lambda < \lambda_1$, *th en the re exists* $\delta_1 > 0$ *such that* $J_\lambda(u) \geq \delta_1$ *for al l*

$$u \in M^-_\lambda(\Omega).$$

Proof . Consider $u \in M^-_\lambda(\Omega)$. Then ϕ_u has a positive global maximum at $t = 1$ and

$$\int b(x) \mid u \mid^{p+1} dx > 0. \text{Thus}$$

$$\begin{aligned} J_\lambda(u) = \phi_u(1) \geq \phi_u(t_{\max}) \\ \geq h_u(t_{\max})\frac{q+1}{2}(h_u(t_{\max})\frac{1-q}{2} - \lambda c) \\ \geq \delta\frac{q+1}{2}(\delta\frac{1-q}{2} - \lambda c) \end{aligned}$$

and the left hand side is uniformly bounded away from 0 provided that $\lambda < \lambda_1$. □

4 . Existence of Positive Solutions

In this section using the properties of fibering maps we shall give simple proofs of the existence of two positive solutions , one in $M^+_\lambda(\Omega)$ and one in $M^-_\lambda(\Omega)$. **Theorem 4 . 1 .** *If* $\lambda < \lambda_1$, *there exists a minimizer of* J_λ *on* $M^+_\lambda(\Omega)$. *Proof .* Since J_λ is bounded below on $M_\lambda(\Omega)$ and so on $M^+_\lambda(\Omega)$, there exists a minimizing sequence $\{u_n\} \subseteq M^+_\lambda(\Omega)$ such that

$$\lim_{n\to\infty} J_\lambda(u_n) = u \in \inf_{M^+_\lambda(\Omega)} J_\lambda(u).$$

Since J_λ is coercive , $\{u_n\}$ is bounded in $W^{1,2}_0(\Omega)$. Thus we may assume , without loss of generality , that $u_n \rightharpoonup u_0$ in $W^{1,2}_0(\Omega)$ and $u_n \to u_0$ in $L^r(\Omega)$ for $1 < r < \frac{2N}{N-2}$. If we choose $u \in W^{1,2}_0(\Omega)$ such that $\int_\Omega a(x) \mid u \mid^{q+1} dx > 0$, then the graph of the fibering map ϕ_u must be of one of the forms shown in Figure 2 (c) or (d) and so there exists $t_1(u)$ such that $t_1(u)u \in M^+_\lambda(\Omega)$ and $J_\lambda(t_1(u)u) < 0$. Hence ,

$$\inf_{u\in M_\lambda} {+}_{(\Omega)} J_\lambda(u) < 0. \text{By}(2.2),$$

$$J_\lambda(u_n) = (\frac{1}{2} - p\frac{1}{+1})\int_\Omega \mid \nabla u_n \mid^2 dx - \lambda(\frac{1}{q+1} - p\frac{1}{+1})\int_\Omega a(x) \mid u_n \mid^{q+1} dx$$

and so

$$\lambda(\frac{1}{q+1} - p\frac{1}{+1})\int_\Omega a(x) \mid u_n \mid^{q+1} dx = (\frac{1}{2} - p\frac{1}{+1})\int_\Omega \mid \nabla u_n \mid^2 dx - J_\lambda(u_n).$$

Letting $n \to \infty$, we see that $\int_\Omega a(x) \mid u_0 \mid^{q+1} dx > 0$.

Suppose $u_n negationslash - arrowright u_0$ in $W_0^{1,2}(\Omega)$. We shall obtain a contradiction by discussing the fibering map ϕ_{u_0}. Since $\int_\Omega a(x) \mid u_0 \mid^{q+1} dx > 0$, the graph of ϕ_{u_0} must be either of the form shown in Figure 2 (c) or (d) . Hence there exists $t_0 > 0$ such that $t_0 u_0 \in M_\lambda^+(\Omega)$ and ϕ_{u_0} is decreasing on $(0, t_0)$ with $\phi'_{u_0}(t_0) = 0$.

Since $u_n negationslash - arrowright u_0$ in $W_0^{1,2}(\Omega)$, $\int_\Omega \mid \nabla u_0 \mid^2 dx < \liminf_{n\to\infty} \int_\Omega \mid \nabla u_n \mid^2 dx$. Thus , as

$$\phi'_{u_n}(t) = t \int_\Omega \mid \nabla u_n \mid^2 dx - \lambda t^q \int_\Omega a(x) \mid u_n \mid^{q+1} dx - t^p \int_\Omega b(x) \mid u_n \mid^{p+1} dx$$

 and

$$\phi'_{u_0}(t) = t\int_\Omega |\nabla u_0|^2\,dx - \lambda t^q \int_\Omega a(x) | u_0 |^{q+1}\,dx - t^p \int_\Omega b(x) | u_0 |^{p+1}\,dx,$$

it follows that $\phi'_{u_n}(t_0) > 0$ for n sufficiently large . Since $\{u_n\} \subseteq M^+_\lambda(\Omega)$, by considering the possible fibering maps it is easy to see that $\phi'_{u_n}(t) < 0$ for $0 < t < 1$ and $\phi'_{u_n}(1) = 0$ for all n. Hence we must have $t_0 > 1$. But $t_0 u_0 \in M^+_\lambda(\Omega)$ and so

$$J_\lambda(t_0u_0) < J_\lambda(u_0) < \lim_{n\to\infty} J_\lambda(u_n) = \inf_{M^+_{\in_u}\lambda(\Omega)} J_\lambda(u)$$

and this is a contradiction . Hence $u_n \to u_0$ in $W^{1,2}_0(\Omega)$ and so

$$J_\lambda(u_0) = \lim_{n\to\infty} J_\lambda(u_n) = \inf_{M^+_{\in_u}\lambda(\Omega)} J_\lambda(u).$$

Thus u_0 is a minimizer for J_λ on $M^+_\lambda(\Omega)$. □

Theorem 4 . 2 . *If $\lambda < \lambda_1$, there exists a minimizer of J_λ on $M^-_\lambda(\Omega)$. Proof .* By Corollary 3 . 3 we have $J_\lambda(u) \geq \delta_1 > 0$ for all $u \in M^-_\lambda(\Omega)$ and so $\inf_{u\in M^-_\lambda(\Omega)} J_\lambda(u) \geq \delta_1$. Hence there exists a minimizing sequence $\{u_n\} \subseteq M^-_\lambda(\Omega)$ such that

$$\lim_{n\to\infty} J_\lambda(u_n) = \inf_{M^-_{\in_u}\lambda(\Omega)} J_\lambda(u) > 0.$$

As in the previous proof , since J_λ is coercive , $\{u_n\}$ is bounded in $W^{1,2}_0(\Omega)$ and we may assume , without loss of generality , that $u_n \rightharpoonup u_0$ in $W^{1,2}_0(\Omega)$ and $u_n \to u_0$ in

$$L^r(\Omega) \text{for} 1 < r < \frac{2N}{N-2}. \text{By}(2.2)$$

$$J_\lambda(u_n) = (\frac{1}{2} - \frac{1}{q+1})\int_\Omega |\nabla u_n|^2\,dx + (\frac{1}{q+1} - p\frac{1}{+1})\int_\Omega b(x) | u_n |^{p+1}\,dx$$

and , since $\lim_{n\to\infty} J_\lambda(u_n) > 0$ and

$$\lim_{n\to\infty}\int_\Omega b(x) | u_n |^{p+1}\,dx = \int_\Omega b(x) | u_0(x) |^{p+1}\,dx,$$

we must have that $\int_\Omega b(x) | u_0(x) |^{p+1}\,dx > 0$. Hence the fibering map ϕ_{u_0} must have graph as shown in Figure 2 (b) or (d) and so there exists $\hat{t} > 0$ such that

$$\hat{t}_{u_0} \in M^-_\lambda(\Omega).$$

Suppose $u_n arrowright - negationslash u_0$ in $W^{1,2}_0(\Omega)$. Using the facts that

$$\int_\Omega |\nabla u_0|^2\,dx < \liminf_{n\to\infty}\int_\Omega |\nabla u_n|^2\,dx$$

 and that , since $u_n \in M_\lambda^-(\Omega), J(u_n) \geq J(su_n)$ for all $s \geq 0$, we have

$$\begin{aligned} J(\hat{t}u_0) &= \frac{1}{2}\hat{t}^2\int_\Omega |\nabla u_0|^2\,dx - \frac{\lambda\hat{t}^{q+1}}{q+1}\int_\Omega a(x)\,|u_0|^{q+1}\,dx - p\frac{\hat{t}^{p+1}}{+1}\int_\Omega b(x)\,|u_0|^{p+1}\,dx \\ &< \lim_{n\to\infty}[\frac{1}{2}\hat{t}^2\int_\Omega |\nabla u_n|^2\,dx - \frac{\lambda\hat{t}^{q+1}}{q+1}\int_\Omega a(x)\,|u_n|^{q+1}\,dx \\ &\quad -p\frac{\hat{t}^{p+1}}{+1}\int_\Omega b(x)\,|u_n|^{p+1}\,dx] \\ &= \lim_{n\to\infty} J(\hat{t}u_n) \\ &\leq \lim_{n\to\infty} J(u_n) = \inf_{M^-_{\in u}\lambda(\Omega)} J_\lambda(u) \end{aligned}$$

which is a contradiction . Hence $u_n \to u_0$ in $W_0^{1,2}(\Omega)$ and the proof can be completed as in the previous theorem . □ **Corollary 4 . 3 .** *Equation* (1 . 1) , (1 . 2) *has at least two positive s o lutions whenever*

$$0 < \lambda < \lambda_1.$$

Proo f − period By Theorems 4 . 1 and 4 . 2 there exist $u^+ \in M_\lambda(\Omega)$ and $u^- \in M_\lambda^-(\Omega)$

such$_{J(u^\pm)}$that$^{J(u^+)}_{=J}$ $_{(|u^\pm|)}$ = $\inf_{u\in}$ and$^M_{|u^\pm|}$+$\lambda\in^{(\Omega)}_{M_\lambda}$ $J^{(u)}_{\pm(\Omega)}$ and$^{\text{and}}$ $J^{(u^-)}_{\text{so}}$ $\overset{=}{\text{wemay}}$ $\overset{u\in M}{\inf}_{\text{assume}}$ $-\lambda^{(\Omega)}_{u^\pm}$ $J^{(u)}_{\geq}$ 0. MoreoverByLemma

2.2$u^\pm$ are critical points of J on $W_0^{1,2}(\Omega)$ and hence are weak solutions (and so by standard regularity results classical solutions) of (1 . 1) , (1 . 2) . Finally , by the Harnack inequality due to Trudinger [6] , we obtain that $u^\pm$ are positive solutions of (1.1), (1.2). □

Acknowledgement . We would like to thank the referee for bringing [5] to our attention and for making the important observation that the fact that $M_\lambda^0(\Omega) = \varnothing$ follows from sufficient knowledge of the fibering maps .

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