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# QUASILINEAR NONLOCAL INTEGRODIFFERENTIAL EQUATIONS IN BANACH SPACES

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Abstract . In this paper , we study the existence of mild solutions for quasi - linear integrodifferential equations with nonlocal conditions in Banach spaces . The results are established by using Hausdorff ' s measure of noncompactness .

## 1 . Introduction

In this paper , we discuss the existence of mild solution of the following nonlinear integrodifferential equation with nonlocal condition

$$\frac{du(t)}{dt} = A(t,u)u + \int_0^t f(t,s,u(s))ds, \quad t \in [0,b], \tag{1.1}$$

$$u(0) = g(u) + u_0, \tag{1.2}$$

where $f : [0,b] \times [0,b] \times \mathbb{X} \to \mathbb{X}$ and $A : [0,b] \times \mathbb{X} \to \mathbb{X}$ are continuous functions , $g : \mathcal{C}([0,b];\mathbb{X}) \to \mathbb{X}, u_0 \in \mathbb{X}$ and $\mathbb{X}$ is a real Banach space with norm $\| \cdot \|$ .

The notion of “ nonlocal condition ” has been introduced to extend the study of the classical initial value problems ; see , e . g . [ 4 , 8 , 1 0 , 1 1 , 1 9 ] . It is more precise for
describing nature phenomena than the classical condition since more information is taken into account , thereby decreasing the negative effects incurred by a possi - bly erroneous single measurement taken at the initial time . The study of abstract nonlocal initial value problems was initiated by Byszewski , we refer to some of the papers below . Byszewski [ 6 , 7 ] , Byszewski and Lasmikauthem [ 9 ] give the exis - tence and uniqueness of mild solutions and classical solutions when $f$ and $g$ satisfy Lipschitz - type conditions . Subsequently , many authors are devoted to studying of nonlocal problems . See [ 1 , 2 , 1 2 , 1 3 , 1 5 , 20 ] for the references and remarks about the advantage of the nonlocal problems over the classical initial value problems .

This article is motivated by the recent paper of Chandrasekaran [ 1 0 ] . We use some hypotheses in [ 1 0 ] , and using the method of Hausdorff ' s measure of noncom - pactness , we give the existence of mild solutions of quasilinear integrodifferential

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equations with nonlocal conditions ( 1 . 1 ) – ( 1 . 2 ) . Our results improve and extend some corresponding results in [ 2 , 7 , 8 , 1 0 , 1 5 ] .

## 2 . Preliminaries

Throughout this paper $\mathbb{X}$ will represent a Banach space with norm $\|\cdot\|$ . Denoted $\mathcal{C}([0,b];\mathbb{X})$ by the space of $\mathbb{X}-$ valued continuous functions on $[0,b]$ with the norm $\| u \| = \sup \{\| u(t) \|, t \in [0,b]\}$ for u $\in \mathcal{C}([0,b];\mathbb{X})$, and denoted $\mathcal{L}(0,b;\mathbb{X})$ by the space of $\mathbb{X}-$ valued Bochner integrable functions on $[0,b]$ with the norm $\| u \| \mathcal{L} = \int_0^b \| u(t) \| dt$.

The Hausdroff ' s measure of noncompactness $\beta\mathbb{Y}$ is defined by $\beta\mathbb{Y}(B) = \inf \{r > 0, B$ can be covered by finite number of balls with radii $r\}$ for bounded set $B$ in a Banach space $\mathbb{Y}$.

**Lemma 2 . 1** ( [ 3 ] ) . *Let $\mathbb{Y}$ be a real Banach space and $B, C \subseteq \mathbb{Y}$ be bounded , with the fo l lowing properties :*

(1) *$B$ is pre - compact if and only if $\beta\mathbb{X}(B) = 0$;*

(2) $\beta\mathbb{Y}(B) = \beta\mathbb{Y}(\overline{B}) = \beta\mathbb{Y}(convB)$, *where $\overline{B}$ and* conv $B$ *mean the closure and convex hull of $B$ respectively ;*

(3) $\beta\mathbb{Y}(B) \leq \beta\mathbb{Y}(C), where B \subseteq C$;

(4) $\beta\mathbb{Y}(B + C) \leq \beta\mathbb{Y}(B) + \beta\mathbb{Y}(C), where B + C = \{x + y : x \in B, y \in C\}$;

(5) $\beta\mathbb{Y}(B \cup C) \leq \max\{\beta\mathbb{Y}(B), \beta\mathbb{Y}(C)\}$;

(6) $\beta\mathbb{Y}(\lambda B) \leq | \lambda | \beta\mathbb{Y}(B) for any \lambda \in \mathbb{R}$;

( 7 ) *If the map $Q : D(Q) \subseteq \mathbb{Y} \to \mathbb{Z}$ is Lipschitz continuous with constant $k$, then $\beta\mathbb{Z}(QB) \leq k\beta\mathbb{Y}(B)$ for any bounded subset $B \subseteq D(Q)$, where $\mathbb{Z}$ be a Banach space ;*

(8) $\beta\mathbb{Y}(B) = \inf \{d_{\mathbb{Y}}(B, C); C \subseteq \mathbb{Y}$ *is precompact* $\} = \inf \{d_{\mathbb{Y}}(B, C); C \subseteq \mathbb{Y}$ *is finite valued* $\}$ *, where $d_{\mathbb{Y}}(B, C)$ means the nonsymmetric ( or symmetric ) Hausdorff distance between $B$ and $C$ in $\mathbb{Y}$;*

( 9 ) *If $\{W_n\}_{n=1}^{+\infty}$ is decreasing s equence of bounded clos ed nonempty subsets of $\mathbb{Y}$ and $\lim_{n\to\infty} \beta\mathbb{Y}(W_n) = 0$, then $\bigcap_{n=1}^{+\infty} W_n$ is nonempty and compact in $\mathbb{Y}$.*

The map $Q : W \subseteq \mathbb{Y} \to \mathbb{Y}$ is said to be a $\beta\mathbb{Y}-$ contraction if there exists a positive constant $k < 1$ such that $\beta\mathbb{Y}(Q(B)) \leq k\beta\mathbb{Y}(B)$ for any bounded closed subset $B \subseteq W$, where $\mathbb{Y}$ is a Bananch space .

**Lemma 2 . 2** ( Darbo - Sadovskii [ 3 ] ) . *If $W \subseteq \mathbb{Y}$ is bounded clos ed and convex , the continuous map $Q : W \to W$ is a $\beta\mathbb{Y}-$ contraction , th en the map $Q$ has at least one fixed point in $W$.*

In this paper we denote by $\beta$ the Hausdorff ' s measure of noncompactness of $\mathbb{X}$ and denote $\beta\mathcal{C}$ by the Hausdorff ' s measure of noncompactness of $\mathcal{C}([a,b];\mathbb{X})$. To discuss the existence , we need the following Lemmas in this paper .

**Lemma 2 . 3** ( [ 3 ] ) . *If $W \subseteq \mathcal{C}([0,b];\mathbb{X})$ is bounded , then $\beta(W(t)) \leq \beta\mathcal{C}(W)$ for al l $t \in [0,b]$, where $W(t) = \{u(t); u \in W\} \subseteq \mathbb{X}$. Furthermore if $W$ is equicontinuous on $[a,b]$, th en $\beta(W(t))$ is continuous on $[a,b]$ and $\beta\mathcal{C}(W) = \sup \{\beta(W(t)), \ t \in [a,b]\}$.*

**Lemma 2 . 4** ( [ 1 4 ] ) . *If $\{u_n\}_{n=1}^{\infty} \subset \mathcal{L}^1(a,b;\mathbb{X})$ is uniformly integrable , then the func - tio n $\beta(\{u_n(t)\}_{n=1}^{\infty})$ is measurable and*

$$\beta(\{\int_0^t u_n(s)ds\}_{n=1}^{\infty}) \leq 2\int_0^t \beta(\{u_n(s)\}_{n=1}^{\infty})ds. \tag{2.1}$$

 **Lemma 2 . 5** ( [ 3 ] ) **.** *If* $W \subseteq \mathcal{C}([0,b];\mathbb{X})$ *is bounded and equicontinuous , then* $\beta(W(s))$ *is continuous and*

$$\beta(\int_0^t W(s)ds) \leq \int_0^t \beta(W(s))ds. \tag{2.2}$$

From [ 1 0 ] , we know that for any fixed $u \in \mathcal{C}([0,b];\mathbb{X})$ there exist a unique con - tinuous function $U_u :\quad [0,b]\times[0,b] \to B(\mathbb{X})$ defined on $[0,b]\times[0,b]$ such that

$$U_u(t,s) = I + \int_s^t A_u(\omega)U_u(\omega,s)d\omega, \tag{2.3}$$

where $B(\mathbb{X})$ denote the Banach space of bounded linear operators from $\mathbb{X}$ to $\mathbb{X}$ with the norm $\| Q \| = \sup\{\| Qu \| \quad : \quad \| u \| \quad = 1\}$, and $I$ stands for the identity operator on $\mathbb{X}$, $A_u(t) = A(t,u(t))$. From ( 2 . 3 ) , we have

$$U_u(t,t) = I, \quad U_u(t,s)U_u(s,r) = U_u(t,r), \quad (t,s,r) \in [0,b]\times[0,b]\times[0,b],$$

$\frac{\partial U_u(t,s)}{\partial t} = A_u(t)U_u(t,s)$ for almost all $t \in [0,b], \forall s \in [0,b]$. **Definition 2 . 6 .** A continuous function $u(t) \in \mathcal{C}([0,b];\mathbb{X})$ such that

$$u(t) = U_u(t,0)u_0 + U_u(t,0)g(u) + \int_0^t U_u(t,s)\int_0^s f(s,\tau,u(\tau))dsd\tau \tag{2.4}$$

and $u(0) = g(u) + u_0$ is called a mild solution of ( 1 . 1 ) – ( 1 . 2 ) .

The evolution family $\{U_u(t,s)\}0 \leq s \leq t \leq b$ is said to be equicontinuous if $(t,s) \to \{U_u(t,s)x : \quad x \in B\}$ is equicontinuous for $t > 0$ and for all bounded subset $B$ in $\mathbb{X}$. The following Lemma is obvious .

**Lemma 2 . 7 .** *If th e evolution family* $\{U_u(t,s)\}0 \leq s \leq t \leq b$ *is equicontinuous and* $\eta \in \mathcal{L}(0,b;\mathbb{R}^+)$, *then the s e t* $\{\int_0^t U_u(t-s,s)u(s)ds, \| u(s) \| \leq \eta(s)$ *for a . e .* $s \in [0,b]\}$ *is*

$$equicontinuous for t \in [0,b].$$

In section 3 , we give some existence results when $g$ is compact and $f$ satisfies the conditions with respect to Hauadorff ' s measure of noncompactness . In section 4 , we use the different method to discuss the case when $g$ is Lipschitz continuous and $f$ satisfies the conditions with the Hauadorff ' s measure of noncompactness .

In this paper , we denote $M = \sup\{\| U_u(t,s) \| : (t,s) \in [0,b]\times[0,b]\}$ for all $u \in \mathbb{X}$. Without loss of generality , we let $u_0 = 0$.

## 3 . The existence results for compact $g$

In this section by using the usual techniques of the Hausdorff ' s measure of non - compactness and its applications in differential equations in Banach spaces ( see , e . g . [ 3 , 5 , 1 4 ] ) , we give some existence results of the nonlocal problem ( 1 . 1 ) – ( 1 . 2 ) . Here we list the following hypotheses :

( HA ) : The evolution family $\{U_u(t,s)\}0 \leq s \leq t \leq b$ generated by $A(t,u)$ is equicon - tinuous , and $\| U_u(t,s) \| \leq M$ for almost all $t,s \in [0,b]$.

( Hg ) (1) $g : \mathcal{C}([0,b];\mathbb{X}) \to \mathbb{X}$ is continuous and compact ;

( 2 ) There exist $N > 0$ such that $\| g(u) \| \leq N$ for all $u \in \mathcal{C}([0,b];\mathbb{X})$.

( Hf ) (1) $f:\ [0,b]\times[0,b]\times\mathbb{X}\to\mathbb{X}$ satisfies the *Carath é odory* - *type* condition ; i . e ., $f(\cdot,\cdot,u)$ is measurable for all $u\in\mathbb{X}$ and $f(t,s,\cdot)$ is continuous for

$$\text{a.e.}t,s\in[a,b];$$

( 2 ) There exist two functions $h : [0, b] \times \mathbb{R}^+ \to \mathbb{R}^+$ and $q : [0, b] \times \mathbb{R}^+ \to \mathbb{R}^+$ such that $h(\cdot, r) \in \mathcal{L}(0, b; \mathbb{R}^+)$ for every $r \geq 0, h(t, \cdot)$ is continuous and increasing , $q(s) \in \mathcal{L}(0, b; \mathbb{R}^+)$, and $\| f(t, s, u) \| \leq q(t)h(s, \| u \|)$ for a . e . $t \in [0, b]$, and all $u \in \mathcal{C}([0, b]; \mathbb{X})$, and for all positive constants $K_1, K_2$, the scalar equation

$$m(t) = K_1 + K_2 \int_0^t h(s, m(s))ds, \quad t \in [0, b] \tag{3.1}$$

has at least one solution ;

( 3 ) There exist $\eta \in \mathcal{L}(0, b; \mathbb{R}^+), \zeta \in \mathcal{L}(0, b; \mathbb{R}^+)$ such that $\beta(f(t, s, D)) \leq \eta(t)\zeta(s)\beta(D)$ for a . e . $t, s \in [0, b]$, and for any bounded subset $D \subset \mathcal{C}([0, b], \mathbb{X})$. Here we let $\int_0^t \eta(s)ds \leq K$

Now , we give an existence result under the above hypotheses .

**Theorem 3 . 1 .** *Assume the hypotheses* ( HA ) , ( Hf ) , ( Hg ) *are satisfied , then the nonlocal initial value problem* ( 1 . 1 ) – ( 1 . 2 ) *has at least one mild s o lutio n .*

*Proof .* Let $m(t)$ be a solution of the scalar equation

$$m(t) = MN + RM \int_0^t h(s, m(s))ds, \tag{3.2}$$

where $R = \int_0^t q(s)ds$. Defined a map $Q : \mathcal{C}([0, b]; \mathbb{X}) \to \mathcal{C}([0, b]; \mathbb{X})$ by

$$(Qu)(t) = U_u(t, 0)g(u) + \int_0^t U_u(t, s) \int_0^s f(s, \tau, u(\tau))d\tau ds, \quad t \in [0, b] \tag{3.3}$$

for all $u \in \mathcal{C}([0, b]; \mathbb{X})$. We can show that $Q$ is continuous by the usual techniques ( see , e . g . [ 1 6 , 1 7 ] ) .

We denote by $W_0 = \{u \in \mathcal{C}([0, b]; \mathbb{X}), \| u(t) \| \leq m(t)$ for all $t \in [0, b]\}$. Then $W_0 \subseteq \mathcal{C}([0, b]; \mathbb{X})$ is bounded and convex .

Define $W_1 = \overline{\phantom{xx}}_{\text{conv}} K(W_0)$, where $\overline{\phantom{x}}\text{conv}$ means the closure of the convex hull in $\mathcal{C}([0, b]; \mathbb{X})$. As $U_u(t, s)$ is equicontinuous , $g$ is compact and $W_0 \subseteq \mathcal{C}([0, b]; \mathbb{X})$ is bounded , due to Lemma 2 . 7 and hypothesis ( Hf )(2), $W_1 \subseteq \mathcal{C}([0, b]; \mathbb{X})$ is bounded closed convex nonempty and equicontinuous on $[0, b]$.

For any $u \in Q(W_0)$, we know

$$\begin{aligned} \| u(t) \| &\leq MN + M \int_0^t \int_0^s q(s)h(\tau, m(\tau))d\tau ds \\ &\leq MN + M \int_0^t h(\tau, m(\tau))d\tau \int_0^t q(s)ds \\ &\leq MN + MR \int_0^t h(s, m(s))ds \\ &= m(t) \end{aligned}$$

for $t \in [0, b]$. It follows that $W_1 \subset W_0$.

We define $W_{n+1} = \overline{\phantom{xx}}_{\text{conv}} Q(W_n)$, for $n = 1, 2, \ldots$. Form above we know that $\{W_n\}_{n=1}^{\infty}$ is a decreasing sequence of bounded , closed , convex , equicontinuous on [ 0 , b ] and nonempty subsets in $\mathcal{C}([0, b], \mathbb{X})$.

Now for $n \geq 1$ and $t \in [0, b], W_n(t)$ and $Q(W_n(t))$ are bounded subsets of $\mathbb{X}$, hence , for any $\varepsilon > 0$, there is a sequence $\{u_k\}_{k=1}^{\infty} \subset W_n$ such that ( see , e . g . [ 5 ] , pp . 1 25 )

$$\beta(W_{n+1}(t) = \beta(QW_n(t))$$
$$\leq 2\beta(\int_0^t U_u(t,s)\int_0^s f(s,\tau,\{u_k(\tau)\}_{k=1}^{\infty})d\tau ds) + \varepsilon$$
$$\leq 2M\int_0^t \beta(\int_0^s f(s,\tau,\{u_k(\tau)\}_{k=1}^{\infty})d\tau)ds + \varepsilon$$
$$\leq 4M\int_0^t\int_0^s \beta(f(s,\tau,\{u_k(\tau)\}_{k=1}^{\infty}))d\tau ds + \varepsilon$$
$$\leq 4M\int_0^t\int_0^s \eta(s)\zeta(\tau)\beta(\{u_k(\tau)\}_{k=1}^{\infty})d\tau ds + \varepsilon$$
$$\leq 4M\int_0^t \zeta(\tau)\beta(W_n(\tau))d\tau\int_0^t \eta(s)ds + \varepsilon$$
$$\leq 4MK\int_0^t \zeta(s)\beta(W_n(s))ds + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary , it follows from the above inequality that

$$\beta(QW_{n+}1(t)) \leq 4MK\int_0^t \zeta(s)\beta(W_n(s))ds \tag{3.4}$$

for all $t \in [0,b]$. Because $W_n$ is decreasing for $n$, we define

$$\alpha(t) = \lim_{n\to\infty}\beta(W_n(t))$$

for all $t \in [0,b]$. From ( 3 . 4 ) , we have

$$\alpha(t) \leq 4MK\int_0^t \zeta(s)\alpha(s)ds$$

for $t \in [0,b]$, which implies that $\alpha(t) = 0$ for all $t \in [0,b]$. By Lemma 2 . 3 , we know that $\lim_{n\to\infty}\beta\mathcal{C}(W_n) = 0$. Using Lemma 2 . 1 , we know that $W = \bigcap_{n=1}^{\infty} W_n$ is convex compact and nonempty in $\mathcal{C}([0,b];\mathbb{X})$ and $Q(W) \subset W$. By the famous Schauder ' s fixed point theorem , there exists at least one mild solution $u$ of the initial value problem ( 1 . 1 ) – ( 1 . 2 ) , where $u \in W$ is a fixed point of the continuous

map$Q$. □

**Remark 3 . 2 .** If the function $f$ is compact or Lipschitz continuous ( see , e . g . [ 6 , 1 6 , 1 8 ] ) , then ( Hf ) ( 3 ) is automatically satisfied .

In some of the early related results in references and above result , it is supposed that the map $g$ is uniformly bounded . We indicate here that this condition can be released . In fact , if $g$ is compact , then it must be bounded on bounded set . Here we give an existence result under another growth condition of $f$( see , [ 1 1 , 20 ] ) , when $g$ is not uniformly bounded . Precisely , we replace the hypothesis ( Hf ) ( 2 ) by

( Hf ) ( 2 ' ) There exists a function $p \in \mathcal{L}(0,b;\mathbb{R}^+)$ and a increasing function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\| f(t,s,u) \| \leq p(t)\psi(\| u \|)$, for a . e . $t \in [0,b]$, and all $u \in \mathcal{C}([0,b];\mathbb{X})$.

 **Theorem 3 . 3 .** *Suppose that* ( HA ) , ( Hf ) ( 1 ) , ( Hf ) ( 2 ' ) , ( Hf ) ( 3 ) , ( Hg ) ( 1 ) *are satisfied .*
*Then the equation* ( 1 . 1 ) – ( 1 . 2 ) *has at least one mild s o lutio n if*

$$\lim_{k\to\infty} \sup \frac{M}{k}(\varphi(k) + b\psi(k)\int_0^b p(s)ds) < 1, \tag{3.5}$$
$$where \varphi(k) = \sup\{\| g(u) \|, \quad \| u \|\leq k\}.$$

*Proof .* The inequality ( 3 . 5 ) implies that there exists a constant $k > 0$ such that

$$M(\varphi(k) + b\psi(k)\int_0^b p(s)ds) < k.$$

Just as in the proof of Theorem 3 . 1 , let $W_0 = \{u \in \mathcal{C}([0, b]; \mathbb{X}) :\| u(t) \|\leq k\}$ and $W_1 = \overline{\phantom{xx}}_{\text{conv}} QW0$. Then for any $u \in W_1$, we have

$$\| u(t) \|\leq M\varphi(k) + M\int_0^t \int_0^s p(\tau)\psi(k)d\tau ds$$
$$\leq M\varphi(k) + bM\psi(k)\int_0^b p(s)ds < k$$

for $t \in [0, b]$. It means that $W_1 \subset W_0$. So we can complete the proof similarly to Theorem 3.1 □

## 4 . Existence results for Lipschitz $g$

In the previous section , we obtained the existence results when $g$ is compact but without the compactness of $\{U_u(t, s)\}0 \leq s \leq t \leq b$ or $f$. In this section , we discuss the equation ( 1 . 1 ) – ( 1 . 2 ) when $g$ is Lipschitz and $f$ is not Lipschitz . Precisely , we replace ( Hg ) ( 1 ) by
( Hg ) ( 1 ' ) There exist a constant $L \in (0, \frac{1}{M})$ such that $\| g(u) - g(v) \| \leq L \| u - v \|$ for

$$\text{every} u, v \in \mathcal{C}([0, b]; \mathbb{X}).$$

**Theorem 4 . 1 .** *Let* ( HA ) , ( Hg ) ( 1 ' ) ( 2 ) , ( Hf ) *be satisfied . Then the equation* ( 1 . 1 ) – ( 1 . 2 ) *has at least one mild s o lution provided that*

$$ML + 4MK\int_0^b \zeta(s)ds < 1. \tag{4.1}$$

*Proof .* We define $Q1, Q2 : \mathcal{C}([0, B]; \mathbb{X}) \to \mathcal{C}([0, B]; \mathbb{X})$ by

$$(Q1^u)(t) = U_u(t, 0)g(u),$$
$$(Q2^u)(t) = \int_0^t U_u(t, s)\int_0^s f(s, \tau, u(\tau))d\tau ds$$

for $u \in \mathcal{C}([0, B]; \mathbb{X})$. Note that $Q1 + Q2 = Q$, as defined in the proof of Theorem 3 . 1 . We define $W_0 = \{u \in \mathcal{C}([0, B]; \mathbb{X}) : \| u(t) \| \leq m(t) \ \forall t \in [0, b]\}$, and let $W = \overline{\phantom{xx}}_{\text{conv}} QW0$. Then from the proof of Theorem 3 . 1 we know that $W$ is a bounded closed convex and equicontinuous subset of $\mathcal{C}([0, B]; \mathbb{X})$ and $QW \subset W$. We shall prove that $Q$ is $\beta\mathcal{C}-$ contraction on $W$. Then Darbo - Sadovskii ' s fixed point theorem can be used to get a fixed point of $Q$ in $W$, which is a mild solution of ( 1 . 1 ) – ( 1 . 2 ) . First , for every bounded subset $B \subset W$, from the ( Hg ) ( 1 ' ) and Lemma 2 . 1 we have

$$\beta\mathcal{C}(Q1^B) = \beta\mathcal{C}(U_B(t,0)g(B)) \leq M\beta C(g(B)) \leq ML\beta\mathcal{C}(B). \tag{4.2}$$

 Next , for every bounded subset $B \subset W$, for $t \in [0, b]$ and every $\varepsilon > 0$, there is a sequence $\{u_k\}_{k=1}^{\infty} \subset B$, such that

$$\beta(Q2^B(t)) \le 2\beta(\{Q2^u k(t)\}_{n=1}^{\infty}) + \varepsilon.$$

Note that $B$ and $Q2^B$ are equicontinuous , we can get from Lemma 2 . 1 , Lemma 2 . 4 , Lemma 2 . 5 and ( Hf ) ( 3 ) that

$$\begin{aligned}
\beta(Q2^B(t)) &\le 2M \int_0^t \beta(\int_0^s f(s,\tau,\{u_k(\tau)\}_{k=1}^{\infty})d\tau)ds + \varepsilon \\
&\le 4M \int_0^t \int_0^s \beta(f(s,\tau,\{u_k(\tau)\}_{k=1}^{\infty}))d\tau ds + \varepsilon \\
&\le 4M \int_0^t \int_0^s \eta(s)\zeta(\tau)\beta(\{u_k(\tau)\}_{k=1}^{\infty})d\tau ds + \varepsilon \\
&\le 4M \int_0^t \zeta(\tau)\beta(B(\tau))d\tau \int_0^t \eta(s)ds + \varepsilon. \\
&\le 4MK \int_0^t \zeta(\tau)\beta(B(\tau))d\tau + \varepsilon \\
&\le 4MK\beta\mathcal{C}(B) \int_0^b \zeta(s)ds + \varepsilon
\end{aligned}$$

Taking supremum in $t \in [0, b]$, we have

$$\beta\mathcal{C}(Q2^B) \le 4MK\beta\mathcal{C}(B) \int_0^b \zeta(s)ds + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary , we have

$$\beta\mathcal{C}(Q2^B) \le 4MK\beta\mathcal{C}(B) \int_0^b \zeta(s)ds \tag{4.3}$$

for any bounded $B \subset W$.

Now , for any subset $B \subset W$, due to Lemma 2 . 1 , ( 4 . 2 ) and ( 4 . 3 ) we have

$$\begin{aligned}
&\beta\mathcal{C}(QB) \le \beta\mathcal{C}(Q1^B) + \beta\mathcal{C}(Q2^B) \\
&\le (ML + 4MK \int_0^b \zeta(s)ds)\beta\mathcal{C}(B).
\end{aligned}$$

By ( 4 . 1 ) we know that $Q$ is a $\beta\mathcal{C}-$ contraction on $W$. By Lemma 2 . 2 , there is a fixed point $u$ of $Q$ in $W$, which is a solution of ( 1 . 1 ) – ( 1 . 2 ) . This completes the proof . □

Now we give an existence result without the uniform boundedness of $g$.

**Theorem 4 . 2 .** *Suppose that* ( HA ) , ( Hf ) ( 1 ) , ( Hf ) ( 2 ' ) , ( Hf ) ( 3 ) , ( Hg ) ( 1 ' ) *are satis - fied . Then th e equation* ( 1 . 1 ) – ( 1 . 2 ) *has at least one mild s o lutio n if* ( 4 . 1 ) *and the following condition are satisfied*

$$ML + bM \int_0^b p(s)ds \limsup_{k\to\infty} \frac{\psi(k)}{k} < 1. \tag{4.4}$$

*Proof .* From ( 4 . 4 ) and the fact that $L < 1$, there exists a constant $k > 0$ such that

$$M(kL + bM\int_0^b p(s)ds\psi(k) + \parallel g(0) \parallel) < k.$$

 We define $W_0 = \{u \in \mathcal{C}([0,b]); \mathbb{X} :\| u(t) \| \leq k, \forall t \in [0,b]\}$. Then for every $u \in W_0$, we have

$$\| Qu(t) \| \leq M(\| g(u) \| + \psi(k) \int_0^t \int_0^s p(\tau) d\tau ds)$$
$$\leq M(\| g(u) - g(0) + g(0) \| \quad + b\psi(k) \int_0^t p(s) ds)$$
$$\leq M(kL + \| g(0) \| + b\psi(k) \int_0^t p(\tau) d\tau) < k$$

for $t \in [0,b]$. This means that $QW0 \subset W_0$. Define $W = \overline{\phantom{xxx}}_{\text{conv}} QW0$. The above proof also implies that $QW \subset W$. So we can prove the theorem similar with Theorem 4.1. □

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