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Connectedness of number theoretic tilings

Shigeki *Akiyama*[1†] and Nertila *Gjini*[2‡]

1 *Department ofMathematics , Faculty ofScience , Niigata University , Ikarashi 2 - 8050 , Niigata 950 - 2181 , Japan ,*

e - ma i l a ddress : akiyama @ ma t h . sc . n i i ga t a - u . a c . jp

2 *Department ofMathematics , University ofNew York Ti rana , Rr . Komuna e Parisit , Tirana , Albania ,*

e - ma i l a ddress : ngj in i @ unyt . edu . a l

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Let $T = T(A, D)$ be a self - affine ti le in $\mathbb{R}^n$ defined by an inte $g-r$ al expanding ma $r-t$ ix A and a digit set D. In connection wi $t-h$ canonical number systems , we s $u-t$ dy connectedness of T when D co $r-r$ esponds to $h-t_e$ set of consecutive integers
$\{0, 1, ..., |\det\ (A)| -1\}$. It i s shown that in $\mathbb{R}^3$ and $\mathbb{R}^4$, for any integral expanding matrix $A, T(A, D)$ i s connected .
We also s $t-u$ dy the connectedness of Pi sot dual tilings which play an important role in $h-t_e$ s $t-u$ dy of $\beta-$ expansion ,
substi $t-u$ tion and symbolic dynamical system . It i s shown $h-t_@$ each ti le generated by a Pi sot unit of de $g-r$ ee 3 i s
$a-r$ cwise connected . This i s na $u-t$ rally expected s ince the digit set consists of consecutive integers as above . However su $r-p$ ri singly , we found families of di sconnected Pisot dual tiles of degree 4 . Also we give a simple necessary and su $f-fi$ cient condition for $h-t_e$ connectedness of the Pi sot dual tiles of degree 4 . As a b $y-p$ roduct , a complete clas sification of $t-h_e\beta-$ expansion of 1 for qu^{r-a} tic Pisot units i s given .

Keywords : Tiling , Connectedness , Pi sot Number , Fractal

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1 I nt roductio n

A non empty set in $\mathbb{R}^n$ is called a *ti le* (i) if it coincides with the closure of its interior . If a finite set of tiles and the $i-r$ translations covers the space $\mathbb{R}^n$ without overlapping , then we say it fo $m-r$ s a *ti ling* . By
‘ without overlapping ’ we mean that the translated tiles are mutually disjoint up to an $n-$ dimensional set of Lebesgue measure zero .

In this paper , we will discuss the connectedness of tiles which $a-r$ ise from two different kinds of number
systems . Although the systems $a-r$ e pretty different in nature and could be separately discussed , we decided to put them together in a s ingle paper since the underlying ideas $a-r$ e close and the reader can find the sh $p-r-a$ contrast between them .

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(i) In some cases , it i s c $a-l$ led a *protile* when $h-t_e$ following tiling $prope^{r-t}$ ies $r-a$ e not yet proved .

1 . 1 Tiles associated to expanding in tegral matrices .

Let $M_n(\mathbb{Z})$ denotes the set of $n \times n$ matrices with entries in $\mathbb{Z}$. Let A be an expanding integral matrix in $M_n(\mathbb{Z})$. The word ' expanding ' means that all its eigenvalues have modulus $g-r$ eater than 1 . We also say that a monic polynomial in $\mathbb{Z}[x]$ is expanding if all roots have modulus greater than one . By definition , the ch $a-r$ acteristic polynomial of the expanding matrix is expanding and vice versa . Let $\mid detA \mid = q$ and let
$D = \{d_1, ...d_q\} \subset \mathbb{R}^n$ be a set of q distinct vectors , called a $q-$ *digit set* . If we let $S_j(x) = A - 1_{(x+d_j)}$, $1 \le j \le q$, then they are contractive maps under a suitable norm in $\mathbb{R}^n[28]$ and it is well $k - n_{own}$ that there is a unique compact set T satisfying $T = \bigcup_{j=1}^q S_j(T)[15, 22]$, which is explicitly given by

$$T := T(A, D) = \left\{ \begin{array}{c} \infty \\ \sum A^{-i} d_{ji} \quad : d_{ji} \quad \in D \\ i = 1 \end{array} \right\}.$$

$T_{\mu(T)}^{iscalled}$ is $anattractorpositive.Indeed^{of} this^{the} system\{S_j positivenes_{sisequivalent}^{\}_j^q=1,} {}^{andit} is_{to} the^{called} afactself_{thatT}^{-af-fi ne}{}_{and} tile$ a tiling . Basic questions and detailed studies on the tiling generated by T are found for example in J . C .Lag^{a-r} ias - Y . Wang [28] , R . Kenyon [25] , C . Bandt [1 0] , Y . Wang [43] , A . Vince [42] and their references .

One of the important aspects of self - affine tiles is connectedness . Hata [2 1] has shown that if $\{f_j\}_{1j \le lessequal-m}$
is a finite set of contractive maps (ii) of X, then the attractor $K = K(f1, \cdots, f_m)$ is a locally connected continuum if and only if , for any $1 \le i < j \le m$, there exists a sequence $\{r_0, r_1, \cdots, r_n, r_{n+1}\} \quad \subset \{1, 2, \cdots, m\}$ with $r_0 = i$ and $r_{n+1} = j$ such that $f_{r_k}(K) \cap f_{r_{k+1}}(K) \neq \varnothing$ for $k \in \{0, 1, \cdots, n\}$. Note that if a tile is connected then it must be arcwise connected . This is seen in the same proof by Hata [2 1] . Thus after all

Arcwise connectedness and connectedness are equivalent

in our framework . We will confirm this point also in the Pisot case in the proof of Theorem 4 . 1 on page 287 in a slightly different context , the graph d $i-r$ ected sets case (c . f . , Luo - Akiyama - Thuswaldner [29]) . Hacon - $Salda^{n-h}$ a - Veerman [20] have shown that , if $\mid$ det $A \mid = 2$ and $D = \{0, v\} \subset \mathbb{Z}^n$ is a complete set of coset representatives of the quotient $g-r$ oup $\mathbb{Z}^n / A\mathbb{Z}^n$, then $T(A, D)$ is a connected tile . Gr ö chenig - Haas [1 9] have proved the existence of connected self - similar lattice tilings for parabolic and elliptic dilations in dimension two . Kirat - Lau [26] , using a graph $a-r$ gument on D, have rediscovered Hata ' s above criterion of connectedness . Also they have shown the following su $f-fi$ cient criterion , which we will use in the proof of Theorem 3 . 1 on page 279 and Theorem 3 . 2 on page 28 1 . Afterwards we will call it a Kirat - Lau $Crite^{r-i}$ on .

Let $A \in M_n(\mathbb{Z})$ *be an expanding matrix with* $\mid \det A \mid = q$ *and* $p(x)$ *be i ts characteristic polynomial .*
Let $D = \{0, v, \cdots, (q-1)v\}$ *with* $v \in \mathbb{R}^n \setminus \{0\}$. *Suppose that there exists a polynomial* $g(x) \in \mathbb{Z}[x]$ *(which will be called multiplyingfactor) such that*

$$h(x) = g(x)p(x) = x^k + a_{k-1}x^{k-1} + a_{k-2}x^{k-2} + \cdots + a_1 x \pm q$$

with $\mid a_i \mid \le q-1$, *for* $1 \le i \le k-1$. *Then* $T(A, D)$ *is connected .*

The idea of this c $r-i$ terion is to find a common point on consecutive two tiles $T + kv$ and $T + (k+1)v$
and to apply Hata ' s type criterion mentioned above . As it is easy to desc $r-i$ be in this way all expanding

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(ii) Hata [2 1] studied ' we $a-k$ ' con $r-t$ actions , a slightly general concept .

polynomials of de $g-r$ ee 2 , K $r-i_{@}$ and Lau succeeded in proving the connectedness of a tile for a suitable digit set in dimension 2 .

In the first p^{a-r} t of this paper , we are interested in generalizing their results to higher dimensional cases
using the digit sets co $r-r$ esponding to consecutive integers $\{0,1,...,|\det(A)|-1\}$. We will show the following theorem , using the Schur - Cohn criterion reviewed in Section 2 on page 275 .

Theorem 1 . 1 *Let* $d = 3,4$ *and* $A \in M_d(\mathbb{Z})$ *be an expanding matrix with* $|\det A| = q$ *and* $D =$
$\{0,v,\cdots,(q-1)v\}$ *with* $v\in\mathbb{R}^d\setminus\{0\}$. *Then* $T(A,D)$ *is connected .*

The proofs are settled separately in Theorem 3 . 1 on page 279 and Theorem 3 . 2 on page 28 1 . They are almost done by b $r-u$ te force and are quite complicated having lots of subcases . However this result gives an evidence of a widely believed speculation that all such ' consecutive integer digit tiles ' may be connected . This is in good contrast to the later part of this paper .

We do not intend to consider general digit sets but only use digits which co $r-r$ espond to consecutive integers . One reason of this restriction is that this case is essential and widely s $t-u$ died in relation to canonical number systems . For canonical number systems and attached tilings , see K á tai - K $hungarumlaut-o$ rnyei [24] , Kov á cs - Peth $hungarumlaut-o[27]$, Gilbe $r-t[18]$. Recent progress on topological s $u-t$ dies on this tiling are seen in Akiyama - Thuswaldner [6 , 7] .

Another reason is as follows . As it is easy to find a disconnected tile when we choose ' scattered ' digit sets , an interesting d $i-r$ ection is to find a connected tile for a given expanding matrix A. Thus it may be just awkward to consider general digit sets in higher dimensional cases , since we are already able to show the connectedness only by using consecutive integers .

1 . 2 Tiles associated to Pisot units .

Now we will explain the later p^{a-r} t of this paper . Let $\beta>1$ be a real number which is not an integer . A
greedy expansion of a positive real x in base β is an expansion of the form :

$$x = P^{\infty}_{i=-N_0} a_{-i}\beta^{-i} = a_{N_0}, a_{N_0-1}, \cdots a_0.a_{-1}a_{-2}...$$

with $a_{-i}\in\mathcal{A}_\beta=[0,\beta)\cap\mathbb{Z}$ and a *greedy condition*

$$0 \le x - P^{N}_{-N_0} a_{-i}\beta^{-i} < \beta^{-N} \quad \forall N \ge -N_0.$$

$l^{The}_{a-r}ly^{integerpart}_{.\quad Thisexpansion}{}^{ofx}for^{is}x^{a_N}{}^{0}_{\in,[}\,a_{N_0}0^{-\ 1,}_{,1)isproduced}\cdots a_0. = \sum^{i^0=-N}_{by}{}_{iterating}0a-_i\beta^{-i}the_\beta and the fractional trans for$
is defined simi -

$$U_\beta : x \to \beta x - \lfloor \beta x \rfloor$$

keeping track its c $a-r-r$ ies $\lfloor\beta x\rfloor \in\mathcal{A}_\beta$. Basic properties of this expansion are summ $r-a$ ized in [30] . To fix our notations we briefly review them . Denote by $\mathcal{A}^*_\beta$(resp .$\mathcal{A}^\omega_\beta$) the set of finite words on $\mathcal{A}_\beta$(resp . the
set of right infinite words on $\mathcal{A}_\beta$). Let $1=d_{-1}\beta^{-1}+d_{-2}\beta^{-2}+\cdots$ be an expansion of 1 defined by the algorithm

$$c_{-i} = \beta c_{-i+1} - \lfloor \beta c_{-i+1} \rfloor, \quad d_{-i} = \lfloor \beta c_{-i+1} \rfloor$$

with $c_0=1$, where $\lfloor x\rfloor$ denotes the maximal integer not exceeding x. In other words , this expansion is

$achieved^{as}_{Letu^\omega\in}\mathcal{A}^{the_\omega}_\beta trajecto^{r-y}_{denotethe} of U^n_\beta right^{(1)}_{infinite}{}^{(n} = 1, word2,...).d_\beta generated^{(1)}_{by} = .d_{-1}, repetition d'_{-of^2_u},\cdots isthat$

shown that the $\beta-$ expansion of 1 can be ch $a-r$ acterized by the conditions of lexicographic order , as follows :

 Let $d = (d_{-i})i \geq 1$ be a sequence of nonnegative integers different from $1, 0^\omega$, such that $\sum_{i \geq 1} d_{-i} \beta^{-i} = 1$,
with $d_{-1} \geq 1$ and for $i \geq 2, d_{-i} \leq d_{-1}$, then d is the $\beta-$ expansion of 1 if and only if :

$$\forall p \geq 1, \sigma^p(d) <_{lex} d, \tag{1.1}$$

where σ is the shift defined by $\sigma((x_i)i \leq M) \quad = \quad (x_{i-1})i \leq M$. He also has shown that a sequence $x \quad = x_1, x_2, \cdots$ of nonnegative integers is realized as a $\beta-$ expansion of some positive real number if and only if it satisfies the following lexicographical condition :

$$\forall p \geq 0, \quad \sigma^p(x) <_{lex} d^*(1) \quad (1.2)$$

$$with d^*(1) = \begin{cases} d_\beta(1), & if d_\beta(1) is infinite; \\ (d_{-1}, d_{-2}, \cdots, d_{-n+1}, (d_{-n} - 1))^\omega, & if d_\beta(1) = d_{-1}, \cdots, d_{-n}. \end{cases}$$

In this case this sequence $x = x_1, x_2, \cdots$ is called *admissible* .

Hereafter let β be a *Pisot number* which is an algebraic integer greater than 1 whose Galois conjugates other than itself have modulus smaller than 1 . Let $\mathbb{Q}(\beta) \geq 0$ be nonnegative elements of the minimum field containing the rational numbers $\mathbb{Q}$ and β. Bertrand [1 2] and Schmidt [36] showed that any greedy
expansion of $x \in \mathbb{Q}(\beta) \geq 0$ is $even^{t-u}$ ally periodic , which means that there exists a positive integer L such that $a_{-N} = a_{-N-L}$ for su $f - fi$ ciently large N. We call a *Pisot unit* a Pisot number which is also a unit of the integer ring of $\mathbb{Q}(\beta)$. The symbolic dynamical system X_β attached to $\beta-$ expansion is the subshift
of the full shift $\mathcal{A}_\beta^{\mathbb{N}}$ whose language consists of all admissible words in $\mathcal{A}_{\beta.}^*$ X_β is sofic if and only if the expansion of 1 is eventually periodic (see [1 3]) . Especially when β is a Pisot number it gives a sofic system . Thurston [4 1] introduced an idea to const $r - u$ ct a self - affine tiling generated by a Pisot unit β which is a geometric realization of this sofic system X_β. Akiyama [2] and $P - r$ aggastis [34] studied in detail such self - affine tilings . G . Rauzy [35] already const $r - u$ cted this $i - k$ nd of tiling in a different approach closely related to substitutions . This tiling has a strong connection to the explicit const $r - u$ ction of Markov partitions of dynamical systems , hopefully toral automo $r - p$ hisms . See also P . Arnoux - Sh . Ito [9] .
Let us recall this tiling by Pisot units , which is called *dual tiling* , following the notation of [2] . Let

$$\beta = \beta^{(1)}, \beta^{(2)}, \cdots, \beta^{(r_1)} and \beta^{(r_1+1)}, \overline{\qquad} \beta^{(r}1^{+1)}, \cdots, \beta^{(r_1+r_2)}, \overline{\qquad} \beta^{(r}1^{+r}2)$$

$be_{the} the_{cor-responding}$ real and the $complex_{conjugates} conjugates^{of \beta,}_{of x \in \mathbb{Q}(\beta).} respectively^{We}_{Define a map} \Phi also^{denote by}_{:\mathbb{Q}(\beta) \quad \to \mathbb{R}^{r_1}} x +^{(j)}_{2r_2} - (j$
$by^{1,}_{:} 2, \cdots, r_1 + 2r_2)$

$$\Phi(x) = “x^{(2)}, \cdots, x^{(r_1)}, \Re(x^{(r_1+1)}), \Im(x^{(r_1+1)}), \cdots, \Re(x^{(r_1+r_2)}), \Im(x^{(r_1+r_2)})”.$$

Let $A = a_{-1}, a_{-2}, \cdots$ be a greedy expansion in base β of an element $\mathbb{Z}[\beta] \cap [0, 1)$. Define S_A to be the set of elements of $\mathbb{Z}[\beta] \geq 0$ whose $g - r$ eedy expansion has the su $fi - f_x A$. In other words we classify all

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elements of $\mathbb{Z}[\beta] \geq 0$ by their fractional p^{a-r} t and map them via Φ to have a tile $T_A = \Phi(S_A)$. An empty word is designated by λ and the tile T_λ is called the central tile . As already noticed in Thurston [4 1] , the Pisot condition gu^{a-r} antees that T_A is compact and the rest $i - r$ ction to units is necess $a - r$

y to have a tiling by this const $r-u$ ction . Therefore we restrict ourselves to Pisot units . Under this restriction , it is not so easy to
show that these T_A give a tiling of the space $\mathbb{R}^r 1^{+2r} 2-1$ though we expect it is always valid . Let Fin (β)
be the set of all finite beta expansions . This is obviously a subset of $\mathbb{Z}[1/\beta] \geq 0$. If β s atisfies

$$Fin(\beta) = \mathbb{Z}[1/\beta] \geq 0,$$

then we say that β has finitely expansible property $parenleft-F)$. This property (F) implies that β is a Pisot number (see [1 6]) . It is comparatively easy to const $r-u$ ct a tiling defined by Pisot units with (F) , in the above sense ([2]) . In [5] , we introduced a wider class of Pisot units with this tiling property called *weaklyfiniteness* . It is conjectured that this property holds even for all Pisot numbers (c . f . [8] , [3 8] , [39]) . In this paper , we do not discus s further this tiling property .

The second aim of this paper is to explore the problem of connectedness of Pisot dual tiles of low degree using again the Schur - Cohn crite $r-i_{on}$ discussed in Section 2 on page 275 . A general arcwise connectedness criterion for Pisot dual tiles is established in Theorem 4 . 1 on page 287 .

Furthermore we can prove the following theorem .

Theorem 1 . 2 *Each tile corresponding to a Pisot unit* β *is arcwise connected if* $d_\beta(1)$ *terminates with* 1 *.*

The proof is found after the one of Theorem 4 . 1 on page 287 . Our conjecture is that for all Pisot units
with finite $\beta-$ expansion of 1 , the last non zero digit of $d_\beta(1)$ must be one . The conjecture is t $r-u$ e especially
for cubic Pisot units β with finite $\beta-$ expansion of 1 , (see [4] , [1 1]) and as we prove in Theorem 4 . 9 on page 307 it is also t $r-u$ e for qu^{r-a} tic Pisot units β with finite $\beta-$ expansion of 1 .

To treat all Pisot units , Theorem 1 . 2 is not enough since the $\beta-$ expansion of 1 is not finite in general .
Let p be the characte $r-i$ stic polynomial of β. If $p(0)=1$ then the $\beta-$ expansion of 1 cannot be finite (see
$P-r$ oposition 1 of [1]) . Even when $p(0)=-1$ there $r-a$ e many such cases . Including these cases , we can generalize the above conjecture :

Conjecture 1 *Let* β *be a Pisot unit and consider i ts eventually periodic* $\beta-$ *expansion of* $1:\quad d_\beta(1)\quad=$

$$.d_{-1},\cdots,d_{-n},(d_{-n-1},\cdots,d_{-n-k})^\omega.Then$$
$$d_{-n-k}-d_{-n}=\pm 1.$$

This conjecture is shown to be valid for de $g-r$ ee less than 5 in this paper . More challenging would be the following $conjec^{t-u}$ re :

Conjecture 2 *Let* $\beta>1$ *be a real number and assume that i ts* $\beta-$ *expansion of* 1 *is eventually periodic with* $d_\beta(1)=.d_{-1},\cdots,d_{-n},(d_{-n-1},\cdots,d_{-n-k})^\omega.$ *Then* $\mid d_{-n-k}-d_{-n}\mid$ *coincides with the absolute value of the norm of* β*.*

This $conjec^{t-u}$ re was first fo $m-r$ ulated in [3] . Strong numerical evidence exists for this conjecture . How - ever , unfortunately the Pisot dual tile can be disconnected even if this conjecture is t $r-u$ e . We summa $i-r$ ze our main results in the following theorem .

Theorem 1 . 3 *Let* β *be a Pisot unit of degree 3 or 4 defined by the monic polynomial* $p(x)\in\mathbb{Z}[x]$*. If* $\deg\beta=3$ *or* $p(0)\quad=\quad 1$ *then each tile is connected . If* $\deg\beta=4$ *and* $p(0)\quad=\quad -1$ *then each tile is*
connected if and only if

$$a+c-2\lfloor\beta\rfloor=1$$
$$forp(x)=x^4-ax^3-bx^2-cx-1.$$

These s tatements $a-r$ e a combination of Theorem 4 . 4 on page 29 1 , Theorem 4 . 5 on page 292 , Theorem 4 . 7 on page 301 and Theorem 4 . 8 on page 307 . In spite of the quite simple na $t-u_{re}$ of the s tatement , the proof is pretty involved having lots of subcases . However we may say that this result gives us a bre $a-k_t r-h$ ough .

In fact , if deg $\beta=4, p(0)=-1$ and $a+c-2[\beta]=1,$ there exists a disconnected tile . As f $a-r$ as we know , no example of disconnected Pisot dual tiles was known before . As these tiles $a-r$ e generated by consecutive integers , it was even expected that Pisot dual tiles $a-r$ e always connected . Thus this result gives an unfortunate su $r-p$ rise that there exists a concrete family of Pisot units one of whose dual tiles is disconnected . (See a rem $a-r$ k after Theorem 4 . 8 on page 307 .)

Fig . 1 : The proj ection of the central tile (disconnected) generated by $t-h_e$ Pisot unit β wi $h-t$ minimal equation x^4-

$$3x^3-7x^2-6x-1=0$$

$$W-h_{en}\beta- \text{ expansion of 1 is eventually periodic }, w^{i-r} \text{ te it as}$$

$$d_\beta(1)=c_{-1},...c_{-M}(c_{-M-1}...c_{-M-L})^\omega$$

with $c_{-M}\neq c_{-M-L}$. We say that the period (resp . preperiod) of $\beta-$ expansion of 1 is $L($ resp $.M)$.

As a byproduct , we will give a complete classification of the $\beta-$ expansion of 1 for cubic and qua^{r-t} ic Pisot units in Theorem 4 . 3 on page 290 , Theorem 4 . 9 on page 307 and Theorem 4 . 6 on page 298 which are na $t-u$ rally proven during our proofs . Theorem 4 . 3 on page 290 was proved by Bassino [1 1] . She computed the $\beta-$ expansion of 1 for any cubic Pisot number , including non units . In view of the prominent role of the expansion of 1 in symbolic dynamics of beta expansion , it is worthy to state independently Theorem 4 . 9 on page 307 and Theorem 4 . 6 on page 298 . It is also an unfo $r-t$ unate su $p-r$ rise that there is no uniform bound on the length of the expansion of 1 for qua^{r-t} ic Pisot units with finite $\beta-$ expansion of 1 . Also , there is no uniform bound on period and preperiod of the expansion of 1 for qu^{a-r} tic Pisot units with infinite $\beta-$ expansion of 1 . The next table m $a-k_{es}$ the situation cle $r-a$ er .

Further study of connectedness may be explored in a different setting . Pisot dual tilings under a certain condition $a-r$ e fo $m-r$ ulated as a geometric realization of substi $t-u$ tive dynamical system . Cante $i-r$ ni [14] s $t-u$ died connectedness of such substitutive tilings and gave general criteria which works for these tiles . It

Table ignored!

Tab . 1 : Length bounds related to the expansion of 1 .

may be f $r-u$ itful to extend the above conjectures to his si $u-t$ ation and to study the connectedness of a family of substitutive tiles .

This paper is organized as follows : In Section 2 , we $prep^{a-r}$ e some results related to the Schur - Cohn criteria to count the number of roots inside / outside the unit c $i-r$ cle . Section 3 on page 279 is devoted to the connectedness of tiles associated to expanding integral matrices of low degree by the K $i-r$ at - Lau crite $i-r$ on . Tiles associated to Pisot numbers $a-r$ e treated in Section 4 on page 287 . The beginning of Section 4 on page 287 is of importance . We give a proof of Theorem 1 . 2 on page 273 and describe a method to prove connectedness of Pisot dual tiles . This is more complicated than the one in Section 3 on page 279 but the underlying sp^{i-r} it is simil $r-a$. Then we show in the subsections 4 . 1 and 4 . 2 the connectedness for quadratic and cubic Pisot units . Later subsections are for the qu^{a-r} tic Pisot units . The idea of the proof of disconnectedness is found in Lemma 3 on page 300 in this las t section . In few words , we show the disconnectedness of a proj ection of the tile along the d $i-r$ ection of the negative real root and use the forbidden words for beta expansions in $\mathcal{A}_\beta^*$ to ' cut ' the tile . Convenient lists $a-r$ e found in Figure 2 on page 309 and Figure 3 on page 3 1 0 . In the shaded box , the expansion of one is not written in a fixed length . Readers find the explicit form in Theorem 4 . 9 on page 307 and Theorem 4 . 6 on page 298 . The four disconnected cases are also indicated in Figure 3 on page 3 1 0 .

2 Expanding po l ynomials and P i sot po l ynomials

Let $f(x)=\sum_{i=0}^n a_i x^{n-i}$ be a polynomial with *complex* coefficients a_i within this section . Admitting an abuse of terminology , we say that $f(x)$ is an *expanding polynomial* if each root has modulus $g-r$ eater than one . A monic *real* polynomial f is a *Pisot polynomial* if it has a real root greater than one and other roots $a-r$ e inside the unit c $r-i$ cle and additionally $|a_n| \geq 1.$ These definitions agree with the original si $t-u$ ation when $f(x)$ is the i $r-r$ educible polynomial over $\mathbb{Z}$ of an algebraic integer .

We briefly review the Schur - Cohn crite $r-i_{on}$ to count the number of zeros inside / outside the unit c $r-i$ cle . In the literature , the Schur - Cohn c $r-i$ terion is often explained in the simplest case that all the dete $m-r$ inants $a-r$ e non zero (iii). In general , this restriction leads us to a difficulty to characte $i-r$ ze polynomials with prescribed location of zeros , in terms of a single family of polynomial inequalities . However for expanding polynomials , such a characterization is well known . Further a ch $a-r$ acterization of Pisot polynomials will be given (Theorem 2 . 2 and Coroll $a-r$ y 2 . 2 on page 278) , which will be used later on .

The reciprocal polynomial of f is defined by $f^*(x)=x^{\deg} f_{f(1/x)}.$ Let $D_n=D_n(f)$ be the dete $r-m$ i - nant of following $2n\times 2n$ matrix with coefficients :

$$b_{i,j} = eight-less-colon \quad \begin{array}{ll} \frac{a_{j-i},}{a_{i-j},} & for^{for} 1_n \leq +i_1 \leq \leq niand^{i}_{\leq 2n} \leq jand \leq ii_- +^n_{n\leq} j \leq i \\ 0, & oh-terwise \end{array}$$

(iii) A cle $a-r$ and original description including such degenerate cases i s found in [40] . An e $r-a$ lier version of $t-h_{is}$ section was based on $t-h_{is}$ Japanese book , without $noticing^{h-t}$ e stand $a-r$ d name after Schur - Cohn .

 We will write it in the following form where the empty ent $r-i_{es}$ represent 0 .

Table ignored!

$$\downarrow \quad \downarrow$$
$$n \quad 2n$$

which is the resultant of f and f^*. Hence $D_n = 0$ if and only of there exists an *inversible root* β, that is , $f(\beta) = f(1/\overline{\beta}) = 0$. Especially if a real polynomial f has a root on the unit circle then $D_n = 0$. By definition , $D_n \neq 0$ for expanding polynomials and Pisot polynomials with $n \geq 3$, s ince $| a_n | \geq 1$ does not allow an inversible root . Delete the $n-$ th , $2n-$ th rows and columns from D_n to get a $2(n-1) \times 2(n-1)$
matrix with dete $r-m$ inant D_{n-1}. From D_{n-1} we create D_{n-2} in the same way . Continue l $k-i_e$ this till we
get

Table ignored!

Then the famous Schur - Cohn ' s crite $r-i_{on}$(c . f . [3 1]) is

Theorem 2 . 1 *Assume that* $D_i \neq 0 (i = 1, ..., n)$ *and le t* p *be the number ofsign changes ofthe sequence* $1, -D_1, D_2, ..., (-1)^n D_n$. *Then* $f(x) \in \mathbb{C}[x]$ *has* p *zeros inside the unit circle and no zeros on the unit circle .*

A technical problem arises from the non vanishing assumption on D_i.

Example 1 *We have* $(D_0, D_1, ..., D_5) = (1, 0, 0, 0, 1, 5)$ *for* $x^5 - 2x^4 - 2x^3 - x^2 - 2x + 1$ *and* $(1, 0, 0, 0, 1, -5)$ *for* $x^5 - 2x^4 - x^3 - 2x^2 - 2x + 1$. *However the situation ofzeros is the same : there are exactly wo roots in the unit circle and three outside for both polynomials . When consecutive zeros appear in* $D_1, D_2, ..., D_n$, *the number of sign changes of* $1, -D_1, D_2, ..., (-1)^n D_n$ *does not te ll how many roots lie in the unit circle .*

The clas sical theo $r-y$ of Schur - Cohn assures that there is a way to escape from such a situation by t $k-a$ ing different principal minors of the co $r-r$ esponding quadratic form (c . f . [40]) , or by replacing f with other polynomials which have as many zeros as f(c . f . Theorem 45 . 1 and Theorem 45 . 2 of [3 1]) .

However this is not convenient in practice . As we wish to derive results on families of polynomials , exceptional treatments should be reduced to a minimum . For this pu $r-p$ ose , we *prep*$^{r-a}$ e some necess $a-r$ y and sufficient conditions of expanding polynomials and Pisot polynomials .

Corollary 2 . 1 *Thepolynomial* $f(x) \in \mathbb{C}[x]$ *is expanding ifand only if* sgn $(D_i) = (-1)^i$ *for* $i = 1, ..., n,$

 which is also called the Schur - Cohn criterion . Here we define

$$\mathrm{sgn}(x) = braceex - braceleftmid - braceex - braceleftbt \quad \begin{matrix} 1 & x > 0 \\ -1 & x < 0 \\ 0 & x = 0. \end{matrix}$$

The origin of this Coroll $a-r$ y dates back to Hermite and Hurwitz who connected the root distribution problem with the inv $a-r$ iants of He $m-r$ itian fo $r-m$ s . The determinants D_i do not vanish because they are principal minors of a positive definite Hermitian forms . We derive this Coroll $a-r$ y 2 . 1 by slightly extending
Marden ' s argument in page 1 94 – 200 of [3 1] (c . f . [1 7]) . Define $f0(x) = f(x)$ and $f_{j+}1(x) = \overline{\quad}_{a_n}^{(j)}{}_{-j} f_j(x) -$
$0_a^{(j)} f_j^*(x)$ for $j = 0, 1, ..., n-1$ with $f_j(x) = \sum_{k=0}^{n-j} k_a^{(j)} x^{n-j-k}$. D $i-r$ ect determinant computation yields

$$fk+1(0)D_k = -f1(0)...fk(0)D_{k+1}$$

and hence

$$\mathrm{sgn}(D_k D_{k+1}) = -\mathrm{sgn}(f1(0)...fk+1(0)) \tag{2.1}$$

provided $f1(0)...fk+1(0) \neq 0$, which is (43 . 4) in [3 1] (iv). A c $r-u$ cial fact is *If* f_j *has* p_j *zeros inside the unit circle and* $f_{j+}1(0) \neq 0$, *then* $f_{j+}1$ *has*

$$p_{j+}1 = \{_{n}^{p_j} -j - p_j \quad if^{if \, f_{j+}1(0)}_{\quad f_{j+}1(0)} <^{>} 0^0 \tag{2.2}$$

zeros inside the unit circle . The set ofzeros on the unit circle of f_j *coincides with that of* $f_{j+}1$.
which is a consequence of Rouch é ' s theorem for circles of radius $1 + \varepsilon$ with small ε ' s , using the equality
$| f(z) |=| f^*(\overline{\quad\quad} z) |$ valid on the unit c $i-r$ cle .

Proof of Corolla $r-y$2.1. The su $fi-f$ ciency of the condition sgn $(D_i) = (-1)^i$ is a direct consequence of
Theorem 2 . 1 . Let us prove the necessity . We claim that that if $f_{j+}1$ has a root in the closed unit disk then
f_j also does . To show this , we divide the situation into t $h-r_{ee}$ cases . If $| n_a^{(j)}{}_{-j} |>\ | 0_a^{(j)} |$ then (2 . 2) gives
$p_j = p_{j+}1 > 0.$ If $| n_a^{(j)}{}_{-j} | < | 0_a^{(j)} |$ then $p_j = n - j - p_{j+}1 > 0$ s ince $p_{j+}1 \leq n - j - 1.$ Finally
if $| n_a^{(j)}{}_{-j} |=| 0_a^{(j)} |$ then the leading coefficient and the constant te $r-m$ of f_j have the same absolute value ,
proving that at least one root of f_j is in the closed unit disk . This shows the claim . As f is expanding , this claim shows that f_j is also expanding for $j = 1, ..., n.$ Therefore $f_j(0)$ can not vanish for $j = 1, ...n.$ Observing (2 . 2) again , since $p_j = 0$ for $j = 0, ..., n,$ we have $f_j(0) > 0$ for $j = 1, ..., n.$ The relation (2 . 1) implies that sgn $(D_k D_{k+1}) = -1,$ which shows the as sertion . $a50$

We give a characterization of Pisot polynomials , which does not seem to have been w^{r-i} tten down elsewhere although it follows from the above reviewed results .

$$(iv) D_k = (-1)^k \Delta_k in[31].$$

Theorem 2 . 2 *Each Pisot polynomial satisfies* $f(1) < 0$ *and* $D_i \leq 0(i = 2, ..., n)$. *Conversely a polynomial* $f(x) = x^n + a_1x^{n-1} + \cdots + a_n \in \mathbb{R}[x]$ *is a Pisot polynomial if* $f(1) < 0$ *and* $D_i < 0(i = 2, ..., n)$. *If* $a_n \neq \pm 1$ *then every Pisot polynomial satisfies* $f(1) < 0$ *and* $D_i < 0(i = 2, ..., n)$.

In other words , provided $a_n \neq \pm 1$, a Pisot polynomial is characterized by a system of inequalities $f(1) < 0$ and $D_i < 0(i = 2, ..., n)$. It is likely that this ch $a - r$ acterization is also valid for $a_n = \pm 1$. We prove some cases of low degree in Coroll $a - r$ y 2 . 2 .

Proof : Assume that a monic $f \in \mathbb{R}[x]$ is a Pisot polynomial with $| a_n | > 1$. As there is only one real root greater than 1 , we have $f(1) < 0$. Using $f1(0) =| a_n |^2 -1 > 0$ and (2.2), $f1$ and f have the same number of roots inside the unit c $i - r$ cle . As $f1$ is of de $r - g$ ee $n - 1, f_1^*$ must be an expanding polynomial .

Thus Corollary 2 . 1 reads sgn $(D_j(f_1^*)) = (-1)^j$ and thus sgn $(D_j(f1)) = (-1)^j$ sgn $(D_j(f_1^*)) = 1$ for $j = 1, ..., n-1$. Employing the formula (43 . 3) in [3 1] :

$$f1(0)^{j+2}D_j(f) = -D_{j-1}(f1)$$

with $f1(0) =| a_n |^2 -1 > 0$, we get $D_j = D_j(f) < 0$ for $j = 2, ..., n$, proving the las t statement . Now we consider the case $a_n = \pm 1$. We replace a_i by $a_i + \varepsilon_i$ with small ε_i ' s , and we write the co $r - r$ esponding Schur - Cohn determinants as $i_D^{(\varepsilon_1,...,\varepsilon_n)}$. If $| a_n + \varepsilon_n |> 1$ then $i_D^{(\varepsilon_1,...,\varepsilon_n)} < 0$ by the above discussion . As $i_D^{(\varepsilon_1,...,\varepsilon_n)} \to D_i$ when $(\varepsilon_1, ..., \varepsilon_n)$ tends to 0 , we have $D_i \leq 0$ for $i = 2, ..., n$. This proves the first s tatement of the Theorem .

It remains to show that $f(1) < 0$ and $D_i < 0(i = 2, ..., n)$ is a sufficient condition to have a Pisot polynomial . Let us st $a - r$ t with the case $| a_n | > 1$. S ince $f(x)$ is a monic polynomial in $\mathbb{R}[x]$ and $D_i < 0(i = 1, ..., n)$, Theorem 2 . 1 implies that there $a - r$ e exactly $n - 1$ zeros inside the unit c $r - i$ cle . $f(1) < 0$ shows the existence of at least one positive root greater than 1 , proving that f is a Pisot polynomial . Finally let us as sume that $f(x) \in \mathbb{R}[x], | a_n |= 1, D_i < 0(i = 2, ..., n)$ and $f(1) < 0$.

Choose a small real ε such that $| a_n + \varepsilon |^2 -1 > 0$. Substitute a_n by $a_n + \varepsilon$ and denote by $i_D^{(\varepsilon)}$ the co $r - r$ esponding Schur - Cohn determinants . Then following the same discussion , $i_D^{(\varepsilon)} < 0$ for $i = 1, 2, ...n$ implies that $f(x) + \varepsilon$ is a Pisot polynomial . On the other hand , $D_n \neq 0$ implies there $a - r$ e no zeros of f on the unit c $i - r$ cle , because , by definition , D_n is the resultant of f and f^*. As the roots are continuous $f - u$ nctions with respect to coefficients , this shows that f is a Pisot polynomial . $a50$

Corollary 2 . 2 *If* $n = 3$ *or* $n = 4$ *then a monic polynomial* $f(x) = x^n + a_1x^{n-1} + \cdots + a_n \in \mathbb{R}[x]$ *is a Pisot polynomial if and only if* $f(1) < 0$ *and* $D_i < 0(i = 2, ..., n)$.

Proof : According to Theorem 2 . 2 , it remains to show that if f is a Pisot polynomial with $a_n = \pm 1$, then $D_i \neq 0(i = 2, ..., n)$. Recall that $D_n \neq 0$ for Pisot polynomials with $n \geq 3$. Note that

$$D_2 = \begin{vmatrix} 1 & a_1 & a_n \\ 1 & a_{n-1} & a_n \\ a_n & a_{n-1} & 1 \\ a_n & a_1 & 1 \end{vmatrix} = (-1 + n_a^2 + a_{n-1} - a_na_1)(-1 + n_a^2 - a_{n-1} + a_na_1).$$

$D_2 = 0$ implies $a_{n-1} = a_na_1$. From the two equalities $a_n = \pm 1$ and $a_{n-1} = a_na_1$ we deduce $D_3 = 0$, which shows the case for $n = 3$. For the quartic case , we have

$$\begin{aligned} D_4 = &\quad -(a_1 - a_2 + a_3)(a_1 + a_2 + a_3)(2_{a_1} - 4a_2 - 3^2_a)^2 \\ D_3 = &\quad -(a_1 + a_3)^2(2_{a_1} - 4a_2 - a_3^2) \quad for a_4 = -1 \\ D_2 = &\quad -(a_1 + a_3)^2 \end{aligned}$$

and

$$\begin{aligned} D_4 = &\quad -(a_1 - a_3)^4(-2 + a_1 - a_2 + a_3)(2 + a_1 + a_2 + a_3) \\ D_3 = &\quad -(a_1 - a_3)^3(a_1 + a_3) \quad for a_4 = 1. \\ D_2 = &\quad -(a_1 - a_3)^2 \end{aligned}$$

If $a_4 = -1$, then $D_2 = 0$ or $D_3 = 0$ happens only when $a_3 = -a_1$, since $D_4 \neq 0$. But this implies $D_4 = 164_{a_2} \geq 0$. Together with the fact that Theorem 2 . 2 gives $D_4 \leq 0$, we have $D_4 = 0$, a contradiction .

If $a_4 = 1$, then $D_2 = 0$ or $D_3 = 0$ happens only when $a_3 = -a_1$. This gives $D_4 = 164_{a_1}(2 + a_2)^2 \geq 0$ which leads us to the same contradiction . $a50$

3 Connectedness of self - af fi ne t i l i ngs generated by an expanding matrix

In this section we shall prove connectedness of tiles generated by an expanding matrix , up to de $g - r$ ee 4 .

3 . 1 Connectedness of $sel^{f-hyphen}$ *af fine tilings generated by an expanding cubic ma - trix*

The next lemma is an explicit fo $r - m$ of Coroll $r - a$ y 2 . 1 on page 276 .

Lemma 1 *The polynomial* $p(x) = x^3 + ax^2 + bx + c$ *with integer coe* $f - fi$ *cients is expanding if and only if*

$$\jmath \,|^{|}\, b^b - +ac1_{|}\, |_{<} < |a^{c^2} - +1^{,}_{c|}. \tag{3.1}$$

Theorem 3 . 1 *Let* $A \in M_3(\mathbb{Z})$ *be an expanding matrix with* $\mid detA \mid= q$ *and* $D = \{0, v, \cdots, (q-1)v\}$ *with* $v \in \mathbb{R}^3 \setminus \{0\}$. *Then* $T(A, D)$ *is connected .*

Proof : Let $p(x) = x^3 + ax^2 + bx + c$ with $a, b, c \in \mathbb{Z}$ be the ch $a - r$ acteristic polynomial of A, which is expanding . We s $t - u$ dy the following two systems of inequalities , equivalent to (3 . 1) :

$$eight - greater - greater - less - greater - greater - colon \quad a^{b^{a-}_{b-}} - +^{ac_{ac}} b^b +^{+}_{-} c^{c^c_c 2_2}_{\geq\geq} \leq_{2,\overset{0}{\geq},2} -2,, \quad and \quad colon - gr \tag{3.2}$$

From the first one , we get the following bounds for the coefficients :

$$c \geq 2 \quad -2c + 2, \leq b \leq 2c - 1, \quad -c + 1 \leq a \leq c + 1,$$

while from the second we have :

$$c \leq -2, \quad 2c + 2 \leq b \leq -2c - 1, \quad c - 1 \leq a \leq -c - 1.$$

To show the connectedness of $T(A,D)$, we use the K $i-r$ at - Lau Criterion . Since the way of finding the multiplying factor is the same for both systems , here we solve only the first system . We can divide the classification into the following cases : $Case^1_-$. Suppose that $-2c+2 \le b \le -c$. From the system (3 . 2) in this case we get $-b-c \le a \le c-1$

$$and 0 \le 1+a < c.$$

♦ If $a > -b-c$ then $-c+1 \le a+b \le 0$. We also have that $-c < b+c \le 0$. So the required

$$polynomial is h(x) = (1+x)p(x) = x^4+(1+a)x^3+(a+b)x^2+(b+c)x+c.$$

♦ If $a = -b-c$ then the required polynomial $h(x)$ is :

$$x^5+(1+a)x^4+(1-c)x^3+(b+c)x+c = (x^2+x+1)p(x).$$

$Case^2_-$. Suppose that $-c+1 \le b \le -1$. From the system (3 . 2) in this case we get $-b-c \le a \le c-1$ which implies that $-c+1 \le a \le c-1$. So in this case the multiplying factor is $g(x) \equiv 1$. $Case^3_-$. Suppose that $0 \le b \le c-1$. From the system (3 . 2) in this case we get $2+b-c \le a \le c$ which

$$implies that -c+2 \le a \le c.$$

♦ If $a \le c-1$ the multiplying factor is $g(x) \equiv 1$. ♦ If $a = c$ then $b > 1$ and $1-c < b-c < 0$, so the polynomial $h(x)$ is

$$x^4+(c-1)x^3+(b-c)x^2+(c-b)x-c = (x-1)p(x).$$

$Case^4_-$. Suppose that $c \le b \le 2c-1$ which implies that $-c < c-b \le 0$. From the system (3 . 2) in this case we get $-1 \le b-a \le c-2$ and $1 \le 1-a \le c$.

♦ If $a < 1+c$ the required polynomial $h(x)$ is

$$x^4+(a-1)x^3+(b-a)x^2+(c-b)x-c = (x-1)p(x).$$

$$♦ If a = 1+c then b \ge c+2, -c+1 < b-2c < 0, -2c+2 \le 2c-2b+1 \le 0.$$

◊ If $-c+1 \le 2c-2b+1 \le 0$ then the $requ^{i-r}$ ed polynomial $h(x)$ is

$$x^5+(c-1)x^4+(b-2c-1)x^3+(2c-2b+1)x^2+(b-2c)x+c = (x-1)^2p(x).$$

◊ If $-2c+2 \le 2c-2b+1 \le -c$ then $-c+1 < 3c-2b+1 \le 0$ and $-c \le 2b-4c-1 < -1$. ⋄ If $2b-4c-1 > -c$ then the required polynomial $h(x)$ is

$$x^7+(c-1)x^6+(b-2c)x^5+(3c-2b)x^4+(2b-4c-1)x^3+(3c-2b+1)x^2+(b-2c)x+c = (x^2+1)(x-1)^{\mathrm{two-p}}(x).$$

⋄ If $2b-4c-1 = -c$ then the required polynomial $h(x)$ is

$$x^6+(c-1)x^5+(b-2c)x^4+(2c-b)x-c = (3_{x-}2x^2+2x-1)p(x).$$

$a50$

3 . 2 Connectedness of $sel^{f-hyphen}$ *affine tilings generated by an expanding quartic ma - trix*

From Coroll $a-r$ y 2 . 1 on page 276 , we deduce

 Lemma 2 *The polynomial* $p(x) = x^4 + ax^3 + bx^2 + cx + d$ *with integer coe f − fi cients is expanding if and only if*

$$eight-greater-greater-less-greater-greater-colon \quad \begin{array}{l} d \geq 2, \\ \mid c - ad \mid \leq d^2 - 2, \\ \mid a + c \mid < 1 + b + d, \\ -1 + b - ac - c^2 + d - a^2 d - 2bd - acd - d^2 + bd^2 - d^3 < 0 \end{array}$$

or (3 . 3)

$$eight-greater-greater-less-greater-greater-colon \quad \begin{array}{l} d \leq -2, \\ \mid c - ad \mid \leq d^2 - 2, \\ \mid a + c \mid < -1 - b - d, \\ -1 + b - ac - c^2 + d - a^2 d - 2bd - acd - d^2 + bd^2 - d^3 > 0. \end{array}$$

Proof : From Coroll $a - r$ y 2 . 1 on page 276 we observe that :

$$D_2 \gtrless 0 \iff \mid c - ad \mid \leq d^2 - 2,$$
$$(d^2 - 1)(1 - b + d) + (a + c)(c - ad) < 0 \quad acd + d^2 + bd^2 - d^3) > 0$$
$$D_3 < 0 \iff \quad or$$
$$(d^2 - 1)(1 - b + d) + (a + c)(c - ad) > 0 \quad acd + d^2 + bd^2 - d^3) < 0,$$
$$D_4 > 0 \iff \mid a + c \mid < 1 + b + d \quad or \quad \mid a + c \mid < -1 - b - d.$$

and that :

$$if \ \mid c - ad \mid \leq d^2 - 2 \ then \ (d^2 - 1)(1 + b + d) + (a + c)(c - ad) \lessgtr 0.$$

Second , since for the expanding polynomial $p(0), p(1)$ and $p(-1)$ have the same sign ,

$$d \geq 2 \quad \Longrightarrow \mid a + c \mid < 1 + b + d$$
$$d \leq -2 \quad \Longrightarrow \mid a + c \mid < -(1 + b + d).$$

We get the des $i - r$ ed result (3.3). $a50$

Theorem 3 . 2 *Let* $A \in M_4(\mathbb{Z})$ *be an expanding matrix with* $\mid det A \mid = d$ *and* $D = \{0, v, \cdots, (d-1)v\}$ *with* $v \in \mathbb{R}^4 \setminus \{0\}$. *Then* $T(A, D)$ *is connected .* Proof : Let $p(x) = x^4 + ax^3 + bx^2 + c - x + d$ with $a, b, c, d \in \mathbb{Z}$ be the characte $i - r$ stic polynomial of A, which is expanding . From the systems of inequalities (3 . 3) we get the following bounds for the coe $f - fi$ cients :

$$\mid d \mid \geq 2, \quad - \mid d \mid \leq a \leq \mid d \mid, \quad -3 \mid d \mid +8 \leq b \leq 3 \mid d \mid -8, \quad -3 \mid d \mid +6 \leq c \leq 3 \mid d \mid -6.$$

 We can divide the classification into the following cases :

$$Conditions_{-}^{1}eight-greater-greater-less-greater-greater-colon \quad d\le -2, \; 2,^{\text{plus}-a^2d-2bd-}_{ad|\le d^{2-acc-plus^2}}$$

$$Conditions_{-}^{2}eight-greater-greater-less-greater-greater-colon \quad d\ge 2, \; ad\,|\le d^2-2,^{a-minus c\text{plus}-c_+^2 da-plus^2 d-2bd-}_{|c_{|a}^{-1+}{}_{-+b}{}^{c|}<1+b+d,}$$

Since the matrix A is expanding if and only if $-A$ is expanding and the characteristic polynomials of both
matrices $a-r$ e monic polynomials , we may choose $p(x)$ or $p(-x)$ appropriately in the proof , which enables us to assume that $a\ge 0$. Now we use the K $i-r$ at - Lau Crite $i-r$ on again .

*First_* suppose that the coefficients of the polynomial $p(x)$ satisfy Conditions 1 with $a\ge 0$. Here we have 2 possibilities : Case 1$greater-greater-greater-less-greater-greater-greater-colon$ $d\le -2,$ $b+d+1<a+c$ $1-d^2<c-ad$ $-1+ba-minu$

or Case 2$greater-greater-greater-less-greater-greater-greater-colon$ $d\le -2,$ $0<a+c\le -b-d-1,$ $1-d^2<c-ad<d^2-1,$ $b-acc-plus_+^2d\text{plus}-a^2d-2bd-$

Let us see the range of the coefficients in Case 1 . We get that

$$greater-greater-greater-less-greater-greater-greater-colon \quad d_0\le\le -a^{2,}_{\le}-d, \; (^a_1-^-d^{)(}_{a+b}{}^{1}{}_{}+^+d^{d)}<^< b^{\le -2}_{c\le -a.}-d,$$

• For $a=-d$ we get that $b\ge 0$. So the required polynomial $h(x)$ is

$$6_{x-}(1+d)x^5+(b+d+1)x^4+(c-b-d)x^3+(b+d-c)x^2+(c-d)x+d,$$

where the multiplying factor is x^2-x+1. • For $0\le a\le -d-1$ we have that $2d\le c\le -d-1$ and $d-2\le b\le a-2$.

– If $c=2d$ then $d\le -7, b=d-2, a=0$. The required polynomial $h(x)$ is

$$9_{x-}2x^8+(d+1)x^7+6_{x-}4x^5+(d+5)4_{x-}(d+4)x^3+2x^2-d,$$

where the multiplying factor is $(x^2+1)(x-1)(x^2-x+1)$. – If $2d+1\le c\le d$ we get $d-2\le b\le minus-d\_2$.

∗ If $b\ge d+1$ the $requ^{i-r}$ ed polynomial $h(x)$ is

$$x^6+(a-1)x^5+(1+b-a)x^4+(a-b+c)x^3+(b-c+d)x^2+(c-d)x+d,$$

where the multiplying factor is x^2-x+1.

$*$ If $d-2\leq b\leq d$ then $a=0$ or $a=1$.

$\diamond$ For $b-a=d$ we have that the polynomial $h(x)$ is

$$x^7+(a-1)x^6+(d+1)x^5+(c-1-d)x^4+(2d-c)x^3+(c-b-d)x^2+(d-c)x-d,$$

where the multiplying factor is $(x^2+1)(x-1)$.

$\diamond$ For $b-a\leq d-1, a=0$ and $b=d-2$ the multiplying factor is $(x^2-x+1)(x^2+1)(x-1)$.

For $b-a\leq d-1, a=0$ and $b=d-1$ the multiplying factor is $(x^2-x+1)(x-1)$.

For $b-a\leq d-1, a=1$, and $b=d$ the multiplying factor is $(x^2-x+1)(x^2+1)(x-1)$.

– If $c\geq d+1$ then $\mid a\mid,\mid b\mid,\mid c\mid a-r$ e less than $\mid d\mid$ so the multiplying factor is $g(x)\equiv 1$.

Now let us see the Case 2 of the Conditions 1 which leads to :

$$eight-greater-greater-less-greater-greater-colon \;\; \begin{matrix} d=-2, \\ ab =^{=} 0'_{-1}, \\ c=1, \end{matrix} \quad or \;\; colon-greater-greater-less-greater-greate$$

In this case we have two subcases :

⧫ For $a=minus-d$ we have that $b\geq 1$ and $d+1\leq c\leq -3$. So the polynomial $h(x)$ is

$$x^6-(1+d)x^5+(1+b+d)x^4+(c-b-d)x^3+(b-c+d)x^2+(c-d)x+d,$$

where the multiplying factor is x^2-x+1.

⧫ For $0\leq a\leq minus-d_{-}1$, we have that $3d+8\leq b\leq d-minus_{-}3$ and $2+d\leq c\leq -3d-6$. • If $-2d\leq c\leq -3d-6$ we have that $d\leq 2+2a+b\leq -d-1$

– If $2+2a+b\geq d+1$ the polynomial $h(x)$ is

$$x^7+(2+a)x^6+(2+2a+b)x^5+(1+2a+2b+c)x^4+(a+2b+2c+d)x^3+(b+2c+2d)x^2+(c+2d)x+d,$$

where the multiplying factor is $(x^2+x+1)(x+1)$. – If $2+2a+b=d$ then the polynomial $h(x)$ is

$$x^9+(2+a)x^8+(1+d)x^7+(a+b+c+d+1)x^6+(a+2b+2c+2d)x^5+(2b+3c+3d-1)x^4+(a+2b+3c+3d)x^3+(b+2c+3d)x^2+(c+2d)x+d,$$

where the multiplying factor is $(x^2+x+1)(x^2+1)(x+1)$.

• If $-d\leq c\leq -2d-1$ we have that $2d+3\leq b\leq -2$ and $a\leq -d-2$. – If $d+1\leq b\leq -2$ then the polynomial $h(x)$ is

$$x^5+(1+a)x^4+(a+b)x^3+(b+c)x^2+(c+d)x+d,$$

where the multiplying factor is $x+1$.

$$--If2d+3\leq b\leq dthen2d+3\leq a+b\leq -2.$$

$*$ If $a+b\geq d+1$ then the polynomial $h(x)$ is

$$x^5+(1+a)x^4+(a+b)x^3+(b+c)x^2+(c+d)x+d,$$

where the multiplying factor is $x+1$.

$*$ If $a+b \leq d$ then $a \leq -d-3, c \geq -d+1$ and $d \leq 2a+b+2 \leq -1$.

$\diamond$ If $d+1 \leq 2a+b+2$ then the polynomial $h(x)$ is

$$x^7 + (2\text{plus} - \text{a})x^6 + (2a+b+2)x^5 + (2a+2b+c+1)x^4 + (a+2\text{plus} - \text{b2}c\text{plus} - \text{d})x^3 + (b+2c+2d)x^2 + (c+2d)xd -$$

where the multiplying factor is $(x^2+x+1)(x+1)$. $\diamond$ If $2a+b+2=d$ then the polynomial $h(x)$ is

$$x^9 + (a+2)x^8 + (d+1)x^7 + (1+a+b+c+d)x^6 + (a+2b+2c+2d)x^5 + (2b+3c+3d-1)x^4 + (a+2b+3c+3d)x^3 + (b+2c+3d)x^2 + (c+2d)x + d,$$

where the multiplying factor is $(x^2+x+1)(x^2+1)(x+1)$.

$\bullet If d+2 \leq c \leq -d-1 then 2d+6 \leq b \leq -3-d.$

– If $b \leq d$ then $a \leq -d-2$, and the polynomial $h(x)$ is

$$x^5 + (1+a)x^4 + (a+b)x^3 + (b+c)x^2 + (d+c)x + d,$$

where the multiplying factor is $x+1$.

– If $b \geq d+1$ then the multiplying factor is $g(x) \equiv 1$.

*Second_* suppose that the coefficients of the polynomial $p(x)$ satisfy Conditions 2 with $a \geq 0$. Here we have 2 possibilities :

Case 1$greater-greater-greater-less-greater-greater-greater-colon$ $d \geq 2,$ $-2^{b}_{-}{}^{-}d^{d_2} \leq \leq c^a +^{c\leq 0}_{-ad\leq}{}^{,}_{d^2} - 2,$ $b - acc - plus^2_+ \text{dplus} - \text{a}^2 d - 2bd - a$

or

$reater-greater-greater-less-greater-greater-greater-colon$ $d_1 \geq \leq 2^{,}_{a} + c \leq b+d,$ $1-d^2 < c - ad < d^2 - 1,$ $ba - minus\text{cplus} - \text{c}^2_+ da - plus^2 d - 2bd - acd\text{plus} - \text{d}^2_+ bd^2 -$

In Case 1 we get 3 subcases :

- $d \geq 2, 0 \leq a \leq d-2, b = minus - d_, c = a - minus_.$ Here the polynomial $h(x)$ is

$$x^6 + ax^5 + (1-d)4_{x-}ax + d,$$

where the multiplying factor is x^2+1.

- $a+1 \leq d \leq -b2^{,}_{\leq} -3^{-b-d\leq c\leq}_{greater-greater-greater-less-greater-greater-greater-colon}{}^{d_0}_{-\frac{d}{-}a}{}^{\geq\leq 2,} -^{a}_{-a} \cdot -d - ad + d^2,$

In this case we get that the bounds for the coefficients $a - r$ e

$$-d+1 \leq b \leq 2d-3 \quad and \quad -2d+3 \leq c \leq 0.$$

 – If $-2d+3 \leq c \leq -d$ then $d \geq 3, b \geq -a$, and $-2d+2 \leq b+c \leq 0$.

$*$ If $b+c \leq -d$ then the polynomial $h(x)$ is

$$x^7+(1+a)x^6+(1+a+b)x^5+(1+a+b+c)x^4+(d+a+b+c)x^3+(d+b+c)x^2+(c+d)x+d,$$

where the multiplying factor is $(x^2+1)(x+1)$. $*$ If $b+c \geq -d+1$ then the polynomial $h(x)$ is

$$x^5+(1+a)x^4+(a+b)x^3+(b+c)x^2+(c+d)x+d,$$

where the multiplying factor is $x+1$. – If $-d+1 \leq c \leq 0$ we have that $-d+1 \leq b \leq d$. $*$ If $b<d$ the multiplying factor is $g(x) \equiv 1$.

$*$ If $b=d$ then the polynomial $h(x)$ is

$$x^6+ax^5+(d-1)x^4+(c-a)x^3-cx-d,$$

where the multiplying factor is x^2-1.

- *greater – greater – greater – less – greater – greater – greater – colon* $0_d^b \leq\geq\geq a \leq d-2,_{2,}^{-2-a-d} -ad+d^2,$ $2+ad-d^2 \leq c \leq -a.$

This case is possible only if $2 \leq d \leq 4$ and the multiplying factor is $g(x) \equiv 1$ except when $d=2$, $a=0, b=2, c=0$. In this case the multiplying factor is x^2+1.

Now let us consider the Case 2 of the Conditions 2 . Here we get that $0 \leq a \leq d, -d+1 \leq b \leq 3d-3$, and $-d+1 \leq c \leq 3d-3$.

- If $c \geq 2d$ then $d \geq 3, b \geq a+d$ and $c \leq -a+b+d$ and for $d=3,4$ we have that $b=a+d$ and $c=2d$. In this case the polynomial $h(x)$ is

$$x^7+(a-2)x^6+(b+2-2a)x^5+(2a+c-2b-1)x^4+(2b+d-a-2c)x^3+(2c-b-2d)x^2+(2d-c)x-d,$$

where the multiplying factor is $(x^2-x+1)(x-1)$.

- If $d \leq c \leq 2d-1$ then $b \geq a$ and there $a-r$ e three cases to be s $u-t$ died : – If $b=a$ then $c=d$ and the polynomial $h(x)$ is

$$x^5+(a-1)x^4+(d-a)2_{x-}d,$$

where the multiplying factor is $x-1$. – If $a+1 \leq b \leq a+d-1$ then $d \leq c \leq minus-a+b+d$. Here we see that $b-c \geq d-minus_.$ $*$ If $b-c = minus-d$ then $a=0, c=b+d$ and $b \leq d-3$. The polynomial $h(x)$ is

$$x^7-x^6+(b+1)x^5+(d-1)x^4-bx-d,$$

where the multiplying factor is $(x^2+1)(x-1)$. $*$ If $b-c \geq minus-d_+1$ then the polynomial $h(x)$ is

$$x^5+(a-1)x^4+(b-a)x^3+(c-b)x^2+(d-c)x-d,$$

 where the multiplying factor is $x-1$.

$$--Ifb \geq a+dthenc \geq d+1, b \leq 2d, -2d+2 \leq b+d-2c \leq d-1.$$

$*$ If $b+d-2c \geq minus-d_{+}1$ then $a \geq 2$ and $-2d+1 \leq a+c-2b+1 \leq 0$. $\diamond$ If $a+c-2b \geq minus-d$ the polynomial $h(x)$ is

$$x^9+(a-2)x^8+(1-2a+b)x^7+(a+c+1-2b)x^6+(b+d-2c+a-2)x^5+(1-2a+c+b-2d)x^4+(a-2b+c+d)x^3+(b+d-2c)x^2+(c-2d)x+d,$$

where the multiplying factor is $(x^3+1)(x-1)^2$. $\diamond$ If $a+c-2b+1 \leq minus-d$ then the polynomial $h(x)$ is

$$x^{11}+(a-2)x^{10}+(b+2-2a)x^9+(2a+c-2b-1)x^8+(2b-a-2c+d-1)x^7+(2-a+2c-b-2d)x^6+(2a-b-c+2d-2)x^5+(1-2a+2b-c-d)x^4+(a+2c-2b-d)x^3+(b+2d-2c)x^2+(c-2d)x+d,$$

where the multiplying factor is $(x^3+1)(x^2+1)(x-1)^2$. $*$ If $b+d-2c \leq minus-d$ then $d \geq 5, a \leq d-1$ and the polynomial $h(x)$ is

$$x^7+(a-2)x^6+(b-2a+2)x^5+(2a-2b+c-1)x^4+(2b-a-2c+d)x^3+(2c-b-2d)x^2+(2d-c)x-,d$$

where the multiplying factor is $(x^2-x+1)(x-1)$.

$$\bullet If -d+1 \leq c \leq d-1 then -d+1 \leq b \leq 2d-1.$$

– If $b \geq d$ then $a \leq d-1$ and $c \geq 2-d$.
$*$ If $c \leq 1$ then $a \geq 1-c$ and $b=d$. The polynomial $h(x)$ is

$$x^6+ax^5+(d-1)x^4+(c-a)x^3-cx-d,$$

where the multiplying factor is x^2-1.
$*$ If $c \geq 2$ then $d \geq 3$ and $-d \leq a-b \leq 0$.
$\diamond$ If $a-b \geq -d+1$ the polynomial $h(x)$ is

$$x^5+(a-1)x^4+(b-a)x^3+(c-b)x^2+(d-c)x-d,$$

where the multiplying factor is $x-1$. $\diamond$ If $a-b=-d$ then the polynomial $h(x)$ is

$$x^7+(a-1)x^6+(d-1)x^5+(1-2a-d+c)4_{x-}cx^3+(a-c)x^2+(c-d)x+d,$$

where the multiplying factor is $(x-1)^2(x+1)$. – If $b \leq d-1$ and $a \leq d$ then the multiplying factor is $g(x) \equiv 1$. – If $b \leq d-1$ and $a=d$ then the polynomial $h(x)$ is

$$x^5+(d-1)x^4+(b-d)x^3+(c-b)x^2+(d-c)x-d,$$

where the multiplying factor is $x-1$.

$a50$

Remark 1 *Here we do not restrict ourselves only in the case when the characteristic polynomial of the matrix A is irreducible . Thisfact is in contrast with thefollowing section .*

4 Connectedness of self - af fi ne t i l i ngs generated by a P i sot u n i t .

We give a sufficient condition for the tiling generated by a Pisot unit to be $a-r$ cwise connected . Let β be a
Pisot unit whose minimal polynomial is $p(x) = x^n - a_1x^{n-1} - \cdots - a_{n-1}x - a_n \in \mathbb{Z}[x]$ with $a_n = \pm 1$. It follows immediately from Thurston ' s const $r-u$ ction that there $r-a$ e only finitely many tiles up to translation , that the number of tiles coincides with that of different suffix of the $\beta-$ expansion of 1 , i . e . , the c $a-r$ dinality

$of_{g-reedy}\{U_\beta^k(_{expansion}1)\}k = 1, 2, ...\cup of^{\{0\}.Recall the}_{elements\ of\mathbb{Z}[\beta]}\cap[0,1), whichisidentifiedwithai-rghtinfinite(orfinite)admissi$

word in $\mathcal{A}_\beta^* \cup \mathcal{A}_\beta^\omega$. The tile T_A was defined as $\Phi(S_A)$. Symbolically the set T_A is the collection of left infinite admissible sequences

$$...a_3a_2a_1a_0 \oplus A = ...a_3a_2a_1a_0.A$$

$realized_{b\in\mathcal{A}_\beta^* and}$ by $the_{say} map_{that} \Phi^{into}_{aleft} infinite^{\mathbb{R}^{n-1}}\dot{}_{word}{}^{Here} is^{we} denoteadmissible^{by} a\oplus whenall^b finite^{theconcatenation}$ suffixes $a-r$ e $admissible^{ofwordsa}_{.} \in The^{\mathcal{A}}*\beta andinterval\ [^{0,1)is}_{depends} subdivided_{ontheinter-val}$ by $\{^{U_\beta^k(1)\}k=1,2,...}_{[t\ \ i,\ \ \ \ t_{i+1})}\ whereA\cup$ $\{0\}intobelongs^0_{(c.f.} = [^{t_0}_{5}]). < Int_1 \quad < the_{sense} t_2 \quad < \cdot of_n \cdot\cdot_- < 11_{dimensional}$ and the $shape_{Lebesgue}$ of T_A measure , the smallest tile T_A co $r-r$ esponds to a suffix A which satisfies $\max_{k\geq 1} U_\beta^k(1) \leq A < 1$ by the lexicographical orde $r-i_{ng.}$ The larger the su $fi-f_x$ the stricter the rest $r-i$ ction on the integer parts $...a_3a_2a_1a_0$ by the admissibility condition (1 . 2) . Conversely T_A becomes biggest when $0 \leq A < \min_{U_\beta^k(1)\neq 0} U_\beta^k(1)$, identifying 0 with λ. Especially the central tile T_λ is the biggest tile .

Theorem 4 . 1 *Let* $\beta > 1$ *be a Pisot unit . Set* $\eta = \max_{k\geq 1} U_\beta^k(1)$ *which gives the smallest ti le* T_η. *If*

$$T_\eta \cap (T_\eta - \Phi(\beta^{-1})) \neq \varnothing, \tag{4.1}$$

then each Pisot dual ti le is arcwise connected .

To begin our proof , we recall $g-r$ aph d $r-i$ ected attractors and graph directed iterated function system
(GIFS for short) . Let $G = G(V,E)$ be a strongly connected $g-r$ aph where $V = \{1,...,q\}$ is the set of

$vertices_{definea} unifo^{andEisthe}_{r-m\ ly} setofdirectedcontractivemapedges^{:Let}_{F_e:\mathbb{R}^d} \to \mathbb{R}^{E_d i,j}\dot{} Then^{bethe} setby[of32, edges^{from}_{Theorem}1]ther$

$K_1, ..., K_q$ of compact non - empty sets s atisfying

$$K_i = \bigcup_{j=1}^{q} \bigcup_{e\in E_{i,j}} F_e(K_j). \tag{4.2}$$

The set of contractions $\{F_e \mid e \in E\}$ is called a *graph directed i teratedfunction system* and the sets K_i $a-r$ e called *graph directed attractors* . Connectedness and $r-a$ cwise connectedness criteria for these graph directed attractors $a-r$ e found in [29] as well . We claim that Pisot dual tiles form graph directed self affine attractors . Let G_t be the natural map defined by the following commutative diagram :

$$\begin{array}{ccc} \mathbb{Q}(\beta) & \xrightarrow{\times\beta^t} & \mathbb{Q}(\beta) \\ \Phi\downarrow & & \downarrow\Phi \\ \mathbb{R}^{d-1} & \xrightarrow[G_t]{} & \mathbb{R}^{d-1}. \end{array}$$

 Then G_t is contractive for $t > 0$ since β is a Pisot number . The set equations $a - r$ e given in the following

$$form:$$

$$T_A = \bigcup G_1(T_{i\oplus A}), \quad (4.3)$$

$$i \oplus A$$

where the summation is taken over all possible $i \in [0, \beta) \cap \mathbb{Z}$ such that $i \oplus A$ is admissible (see [5]) . Note that we identify $i \oplus A$ with the co $r - r$ esponding $\beta-$ expansion to realize it as a nonnegative real number . Since there are finitely many tiles up to translation , it is easy to show that they fo $r - m$ graph d $r - i$ ected self affine attractors by using (1 . 2) . This proves the claim .

ProofofTheorem 4 . 1 . To prove that all tiles are connected , it suffices to show that two $neighbo^{r-i}$ ng tiles $T_{(i-1)\oplus A}$ and $T_{i\oplus A}$ have nonempty intersection . Indeed , if this is t $r - u$ e , then for any two points on a tile it
is easy to find an $\varepsilon-$ chain connecting these by repeated applications of (4 . 3) (see [2 1]) . $uSince_{\oplus\eta is}the_{admissible}admissibility_{thenu}condition_{\oplus\kappa isadmissible}(1.2)$ is $described_{foranyadmissible}$ by the $lexicographic_{word\kappa.Hence}order, forawordputting_{w=}ui_{\oplus} \in_A \beta{,}^{-if}_{\mathcal{A}^*\eta}$, we have

$$S_{i\oplus A} \supset S_\eta + wandS_{(i-1)\oplus A} \supset S_\eta + w - \beta^{-1}.$$

This shows that

$$T_{(i-1)\oplus A} \cap T_{i\oplus A} \supset (T_\eta \cap (T_\eta - \Phi(\beta^{-1}))) + \Phi(w).$$

Thus , by the assumption , each tile is connected .

Finally we discuss shortly the local connectedness and $a - r$ cwise connectedness . Recalling the theorem of H $a - h_n$ and Mazurkiewicz , it suffices to show that each tile is a locally connected set having at least two points . Local connectedness is shown easily by (4 . 3) , since each tile is reconst $r - u$ cted as a finite union of sufficiently small connected compact sets . $a50$

From Theorem 4 . 1 on the previous page we immediately get a

Corollary 4 . 1 *Iffor the Pisot unit* $\beta, \exists a_i \in \mathbb{Z}(i = 1, 2, \cdots)$ *such that* $\mid a_i \mid < \lfloor \beta \rfloor$ *and* $\Phi(1) + \sum_{i=1}^{\infty} a_i \Phi(\beta^i) =$
0 *then each Pisot dual ti le is arcwise connected .*

which is akin to the Kirat - Lau criterion . In practice , this Coroll $a - r$ y is quite useful but not enough in some
cases .

Proof : Let $x_i =$ max $(a_i, 0)$ and $yi =$ max $(-a_i, 0)$. Then we have

$$\infty \quad \infty$$

$$\sum x_i \Phi(\beta^{i-1}) + \Phi(\eta) = \sum yi^{\Phi}(\beta^{i-1}) + \Phi(\eta) - \Phi(1/\beta)$$

$$i = 1 \quad i = 1$$

Since the maximal digit $\lfloor \beta \rfloor \in \mathcal{A}_\beta$ does not $appe^{a-r}$ in x_i and yi, both $...x_2x_1x_0 \oplus \eta$ and $...y2y1y0 \oplus \eta$ $a-r$ e admissible by (1 . 2) . Therefore the left hand side belongs to T_η and the right to $T_\eta - \Phi(1/\beta)$. $a50$

For a string of symbols $\varpi = a_1, a_2, \cdots, a_n$ let us w^{r-i} te ϖ^ω for the right periodic expansion

$$a_1, a_2, \cdots, a_n, a_1, a_2, \cdots, a_n, \cdots, a_1, a_2, \cdots, a_n, \cdots$$

and ω_ϖ for the left periodic expansion

$$\cdots, a_1, a_2, \cdots, a_n, a_1, a_2, \cdots, a_n, \cdots, a_1, a_2, \cdots, a_n$$

Here we shall prove Theorem 1 . 2 on page 273 that each tile is connected if $d_\beta(1)$ terminates with 1 .
$Proof_{P(\beta)=}of_0Theorem_{with}1.2 : d \quad d-1^{=x \quad -c_{-1}x}_{By^{the}_{P\ (x)}assumption,1=} d-2^{-c_{-2}x \quad -\cdot \quad with^{c_{-d}} =}_{\sum_{i=1}^{d} c_{-i}\beta^{-i\cdots -c\ -d+1^x}} 1_-, which_{1.}$ gives rise to a relation
$Let^{k}_{x^{d-k})}{}^{be}_{\sum} i the \infty_{=0x} greatest^{integer<}_{di=0 wegetthat}$ than d such that $c_{-k} \quad = \quad \lfloor\beta\rfloor.$ Since β is also a root of $P(x)(1-$

$$\omega(\lfloor\beta\rfloor, c_{-k-1}, \cdots, c_{-d+1}, 0, \lfloor\beta\rfloor, c_{-2}, \cdots, c_{-k+1}), \lfloor\beta\rfloor-1, c_{-k-1}, \cdots, c_{-d+1}.\eta =$$
$$\omega(c_{-1}, c_{-2}, \cdots, c_{-d+1}, 0),_{braceleft-z}\, 0, \cdots, 0.\eta \quad -0.1$$
$$d-k-1$$

is a common point of T_η and $T_\eta - \Phi(\beta^{-1})$, where η is the biggest su $f-fi_x$ in the $\beta-$ expansion of 1. $a50$

We remark here that this Theorem 1 . 2 is a generalization of the same result proved under the finiteness condition (F) (see [2]) .

4 . 1 Connectedness of $sel^{f-hyphen}$ *affi ne tilings generated by a quadratic Pisot unit* .

It is well $k-n_{own}$ that Pisot dual tiles for quadratic Pisot units $r-a$ e nothing but intervals . For the s $a-k_e$ of the completeness , we describe them in this subsection . Let β be a quadratic Pisot unit . Its minimal polynomial
is either $x^2-ax-1(a\geq 1)$ or $x^2-ax+1(a\geq 3)$.

$Case\_x^2-ax-1(a\geq 1)$. In this case $d_\beta(1)=a,1$ which satisfies the condition of Theorem 1 . 2 on page 273 . Therefore T_A is a non empty compact connected set in $\mathbb{R}^1$, that is , a closed interval . One can
obtain their concrete shapes by computing extremal values . Take the conjugate $\beta' = (a-\sqrt{a^2+4}_{)/}2 \in$

$$(-1,0). Then$$

$$T_\lambda = \{\sum_{\infty}^{i=0} a_i(\beta')^i| \quad a_{i+1}, a_i <_{lex} a, 1\}$$
$$= \left[\begin{array}{cc} \infty & \infty \\ \sum a(\beta')^{2k-1}, \sum a(\beta')^{2k} & \\ k=1 & k=0 \end{array}\right]$$
$$= \left[\begin{array}{cc} a\beta' & a \\ \text{-1}_{-(\beta\prime)}{}^2, \text{-1}_{-(\beta\prime)}{}^2 & \end{array}\right] = [-1, \beta]$$

The other tile is

$$T_1 - \frac{1}{\beta'} = \{\sum_{\infty}^{i=0} a_i(\beta')^i \in T_\lambda| \quad a_0 \neq a\}$$
$$= [\frac{a\beta'}{1-(\beta')^2}{}_{,} \frac{a}{1-(\beta')^2} - 1] = [-1, \beta-1].$$

The translation $-1/\beta'$ was perfo $r-m$ ed to make clearer the si $u-t$ ation .

$Case\_x^2-ax+1(a\geq 3)$. We have $d_\beta(1)=(a-1),(a-2)^\omega$ and $\eta = \max_{k\geq 1} U^k_\beta(1) = (a-2) \quad {}^{\omega}_{.}$

T $a-k_e$ the conjugate $\beta' = (a-\sqrt{a^2-4}_{)/}2 \in (0,1)$. By (4 . 3) we have

$$G_{-1}(T_\lambda) = \beta'^{-1} T_\lambda = T_\lambda \cup T_1 \cup \cdots \cup T_{a-2} \cup T_{a-1}$$

$$G_{-1}(T_{a-1}) = T_{0,a-1} \cup T_{1,a-1} ... T_{a-3,a-1} \cup T_{a-2,a-1}.$$

Up to translation , there $a-r$ e only two tiles T_λ and T_η. If $A <_{lex} \eta$ then T_A is con $u-r-g$ ent to T_λ and if $A \geq_{lex} \eta$ then T_A is con $g-r-u$ ent to T_η. Observing the above set of equations , the smaller tile $T_\eta appe^{r-a}$ s only at the las t te $r-m$ s T_{a-1} and $T_{a-2,a-1}$. Therefore in view of the proof of Theorem 4 . 1 on page 287 , to prove the connectedness of tiles , we only need to show a we^{a-k} er condition :

$$T_{a-1} \cap T_{a-2} \neq \varnothing,$$

which is shown by

$$T_{a-1} \ni \frac{a-1}{\beta'} = \frac{a-2}{\beta'} + a - 1 + \sum_{\infty}^{i=2} (a-2)(\beta')^i \in T_{a-2}.$$

As a result , the condition of Theorem 4 . 1 on page 287 is sufficient but not necess $a-r$ y to have connectedness . A simil $a-r$ computation yields :

$$T_\lambda = \quad [0, 1 + \frac{a-2}{1-\beta'}] = [0, \beta] \quad and \quad T_{a-1} - \frac{a-1}{\beta'} = \quad [0, \frac{a-2}{1-\beta'}] \quad = [0, \beta - 1].$$

4 . 2 Connectedness of $sel^{f-hyphen}$ affi ne tilings generated by a cubic Pisot unit .

Let β be a Pisot unit of de $g-r$ ee 3 defined by the monic polynomial $p(x) \in \mathbb{Z}[x]$. In this subsection we prove that the dual tiling generated by β is connected , i . e . each tile is connected . To make explicit the
cubic case of Coroll $a-r$ y 2 . 2 on page 278 , we have
Theorem 4 . 2 *A monic polynomial*

$$p(x) = x^3 - ax^2 - bx - c \in \mathbb{Z}[x]$$

is a Pisot polynomial if and only if three inequalities

$$1 < a + b + c, \quad | b - 1 | < a + c \quad and \quad (c^2 - b) < \operatorname{sgn}(c)(1 + ac)$$

hold . The following Theorem due to Akiyama [4] and Bassino [1 1] gives the $\beta-$ expansion of 1 for the cubic
Pisot units . Note that [1 1] also dealt with non unit Pisot cases .
Theorem 4 . 3 *Let β be a cubic Pisot unit and le t*

$$p(x) = x^3 - ax^2 - bx - c$$

with $c = \pm 1$ be its minimal polynomial . Then the $\beta-$ expansion of 1 is given by thefollowing table .

Table ignored!

From now on , for simplicity we denote $\beta i = \Phi(\beta^i)$. Theorem 4 . 4 *Let* β *be a Pisot unit ofdegree 3 . Then each tile is arcwise connected .*

Proof :

We only need to prove this theorem for the cases when the $\beta-$ expansion of 1 is infinite because the other cases are shown by Theorem 1 . 2 on page 273 (c . f . [4]) . We use Coroll $a-r$ y 4 . 1 on page 288 to prove the connectedness of each tile .

$$c = 1 and - a + 1 \leq b \leq -2. \qquad Case\underline{1}.$$

Here $d_\beta(1) = .a-1, a+b-1, (a+b)^\omega$, $\lfloor \beta \rfloor = a-1$ and the smallest tile in this case is T_η for $\eta = (a+b)^\omega$.

Since every conjugate of β is also a root of $p(x)(x^3-1)(1+x)(1+x^6+\cdots+x^{6n}+\cdots)$, we have

$$1 + (b+1)\beta 1^+ P^\infty_{\text{equal}-\text{zero}i}((a+b)\beta 2^+(a-2)\beta 3 - (b+2)\beta 4 - (a+b)\beta 5 - (a-2)\beta 6^+(b+2)\beta 7)\beta 6i^{=0}$$

and all the coefficients have absolute value less than $\lfloor \beta \rfloor = a-1$.

$$c = -1 and - a + 3 \leq b \leq 0. \qquad Case\underline{2}.$$

Here $d_\beta(1) = .a-1, a+b-1, (a+b-2)^\omega$, $\lfloor \beta \rfloor = a-1$ and the smallest tile in this case is T_η for

$$\eta = a+b-1, (a+b-2)^\omega.$$

$$\blacklozenge Supposethatb \leq -1.$$

Since every conjugate of β is a root of $p(x)\sum_{i=0}^\infty x^i$, we have

$$1 + (1-b)\beta 1^+(1-a-b)\beta 2^+(2-a-b)P^\infty_{i=3}\beta i = 0$$

and all the coefficients have absolute value less than $a-1$.

$\blacklozenge$ Suppose that $b = 0$. Since every conjugate of β is a root of $p(x)\sum_{i=0}^\infty 2^i_x$, we have $\omega(1, 1-a), 0, 1. = 0$ and $\omega(1,0), 0.1 =$
$\omega(a-1,0).1 \quad -0.1$. Adding $.(a-2)^\omega$ we get that a common point of T_η and $T_\eta - \Phi(\beta^{-1})$ is

$$\omega(1,0), 0.\eta = \omega(a-1,0).\eta \quad - \quad 0.1$$

According to (1 . 2) , both expansions are admissible .

$$Case3. \quad c = -1 and 1 \leq b \leq a-1.$$

Here $d_\beta(1) = .a, (b-1, a-1)^\omega$ and the smallest tile in this case is T_η for $\eta = (a-1, b-1)^\omega$. Since every conjugate of β is also a root of $p(x)\sum_{i=0}^\infty x^{2i}$, we have

$$1 - b\beta 1^{+} P_{i=0}^{\infty}((1-a)\beta 2^{+}(1-b)\beta 3)\beta 2i = 0.$$

and all the coefficients have absolute value less than $\lfloor\beta\rfloor = a$. $a50$

4 . 3 Connectedness of $sel^{f-hyphen}$ affi ne tilings generated by a quartic Pisot unit .

Let β be a Pisot unit of degree 4 defined by the monic polynomial $p(x) = x^4 - ax^3 - bx^2 - cx - 1 \in \mathbb{Z}[x]$. We prove that the dual tiling generated by β is connected , i . e . each tile is connected , if $p(0) = 1$. We also

prove that if $p(0) = -1$ then $a+c-2\lfloor\beta\rfloor \leq 1$ and that each tile is connected if and only if $a+c-2\lfloor\beta\rfloor = 1$. If $p(0) = -1$ and $a + c - 2[\beta] = 1$, we prove the existence of a disconnected tile . As a byproduct , we give a complete clas sification of the $\beta-$ expansion of 1 for qua^{r-t} ic Pisot units . Let us start with a

Proposition 4 . 1 *A monic polynomial*

$$p(x) = x^4 - ax^3 - bx^2 - cx - d$$

with $d = \pm 1$ *is a Pisot polynomial if and only if*

$$\{^{|b-2|}_{a\ -c>0}{}^{<a}_{,} + c, \quad ford = -1; \quad \{^{|b|}_{a}{}_{2} < +a4b^{+}c^{,}_{-c^2} > 0, \quad ford = 1.$$

which is just an explicit form of the qu^{a-r} tic case of Coroll $r-a$ y 2 . 2 on page 278 . In the Theorem 4 . 5 and Theorem 4 . 7 on page 301 , we frequently use P $a-r-r$ y ' s conditions (1 . 1) and (1 . 2) on admissible words .

Theorem 4 . 5 *Let* β *be a Pisot unit ofdegree* 4 *with minimal polynomial* $p(x) = x^4 - ax^3 - bx^2 - cx + 1$. *Then each tile is arcwise connected .*

Proof : To prove this and the following Theorem we use Theorem 4 . 1 , Coroll $a-r$ y 4 . 1 on page 288 . If β

is a Pisot unit of de $g-r$ ee 4 then , according to the $r-P$ oposition 4 . 1 , we have that the coe $f-fi$ cients s atisfy the

system of inequalities :

$$\jmath\, |^{b}_{a} -^{-} 2^{|}_{c} \leq \geq 1^{a}_{,} + c - 1, \quad \Longrightarrow colon-less-eight \quad \begin{array}{l} a \geq 1, \\ 3^1 -^{-} a^a \leq_{-} c^c \leq \leq ab - \leq 1^{,}_{a} + c + 1. \end{array}$$

Case 1 . If $1 - a \leq c \leq -1$ then $4 - a \leq b \leq a$.

- If $2 \leq b \leq a$, we have $a \geq 2$. Here , $\lfloor\beta\rfloor = a$,

$$d_\beta(1) = .a, b-1, (a+c, b-2)^\omega,$$

and the smallest tile is T_η for

$$\eta = \jmath \quad \begin{array}{ll} (a+c, b-2)^\omega, & ifb - 1 < a + c; \\ b-1, (b-1, b-2)^\omega, & ifb - 1 = a + c. \end{array}$$

Since every conjugate of β is also a root of $p(x)\sum_{i=0}^{\infty} x^{2i}$, we have

$$1 - c\beta 1 + (1-b)\beta 2 + P_{i=0}^{\infty}(-(a+c)\beta 3 + (2-b)\beta 4)\beta 2i = 0.$$

Here , all the coefficients have absolute value less than $\lfloor\beta\rfloor$, so according to Coroll $a-r$ y 4 . 1 on page 288 , each tile is $a-r$ cwise connected . • If $b=1$ then $a \geq 3, c \geq 2-a$. In this case $\lfloor\beta\rfloor = a$,

$$d_\beta(1) = .a, 0, a+c-1, a-1, a+c, (a+c-1) \quad {}^{\omega}_{,}$$

and the smallest tile is T_η for $\eta = a-1, a+c, (a+c-1)^\omega$. Since every conjugate of β is also a root of $p(x)\sum_{i=0}^{\infty} x^{3i}$, we have

$$1 - c\beta 1^+ P_{i=0}^{\infty}(-\beta 2^+(1-a)\beta 3^+(1-c)\beta 4)\beta 3i = 0,$$

and all the coefficients have absolute value less than $\lfloor\beta\rfloor$. • If $4-a \leq b \leq 0$ then $a \geq 4$ and $c \geq 3-a$. Here $\lfloor\beta\rfloor = a-1$,

$$d_\beta(1) = .a-1, a+b-1, a+b+c-1, (a+b+c-2) \quad {}^{\omega}_{,}$$

and the smallest tile is T_η for $\eta = a+b-1, a+b+c-1, (a+b+c-2)^\omega$. Since every conjugate of β is also a root of $p(x)\sum_{i=0}^{\infty} x^{3i}$, we have

$$1 - c\beta 1^+ P_{i=0}^{\infty}(^{minus-b}\beta 2^+(1-a)\beta 3 + (1-c)\beta 4)\beta 3i = 0,$$

and all the coefficients except $1-a$ have absolute value less than $\lfloor\beta\rfloor$. A common point of T_η and

$$T_\eta - \Phi(\beta^{-1})is$$

$$\omega(1-c, 0, -b), -c.\eta = \omega(a-1, 0, 0).\eta \quad - \quad 0.1.$$

Case 2 . If $0 \leq c \leq a-1$ then $4-2a \leq b \leq 2a$.

• If $4-2a \leq b \leq -a$, we have $a \geq 4, c \geq 3, 2 \leq 2a+b-2 \leq a-2$ and $\lfloor\beta\rfloor = a-2$.

∗ If $b \leq -a-1$, then $c \geq 4$ and $a \geq 5$.

F $i-r$ st , let us find the $\beta-$ expansion of 1 . S ince $1 \leq a+b+c-2 < a-2$, there exists an integer $2 \leq k \leq a-2$ with $\frac{a-2}{k} \leq a+b+c-2 < \frac{a-2}{k-1}$, which implies that $(k-1)(a+b+c-2) <$

$$a-2 \leq k(a+b+c-2).$$

◇ If $(k-1)(a+b+c-2) \geq c-2$ we get $k \geq 3$. Let m be the integer defined by $m = \inf\{i : (i+1)(a+b+c-2) \geq c-2\}$. Since $b < -a$, we have $m \geq 1$. By the definition

$$m \leq k-2and(m+1)(a+b+c-2) \leq a-3.$$

⧫ If $(m+1)(a+b+c-2) < a-3$ let us show that the $\beta-$ expansion of 1 is even - $t-u$ ally periodic with period 1 and preperiod $m+3$, so let us write it as $d_\beta(1) \quad =$

$$.d_1, d_2, \cdots, d_{m+3}, d_{m+4}^{\omega}.$$

$$W - h_{en}m = 1, since$$

$$p(x)(1+x) = 5_{x-}(a-1)4_{x-}(a+b)3_{x-}(b+c)2_{x-}(c-1)x+1,$$

 we get that

$$1 = .a-2, 2a+b-2, 2a+2b+c-2, 2a+2b+2c-3, (2a+2b+2c-4)^{\omega}.$$

$Here d_5 = d_{m+4}, d_4 = d_{m+3}, d_3 = d_{m+2}.$

$W - h_{en} m = 2, since$

$$p(x)(1+x+x^2) = 6_{x-}(a-1)5_{x-}(a+b-1)4_{x-}(a+b+c)3_{x-}(b+c-1)2_{x-}(c-1)x+1,$$

we get that

$$1 = .a-2, 2a+b-3, 3a+2b+c-3, 3a+3b+2c-4, 3(a+b+c)-5, 3(a+b+c-2)^{\omega}.$$

Here $d_6 = d_{m+4}, d_5 = d_{m+3}, d_4 = d_{m+2}, d_3 = d_{m+1}$, where the formulas of d_i will be given later .

$W - h_{en} m \geq 3, since$

$$p(x)P_{i=0}^{m} x^i = xm_2^{\text{plus−four}-\ (a-1)x^{m+3}-(a+b-1)x^{m+2}-(a+b+c-1)x^{m+1}-}_{)xi-\ (a+b+c-1)x^3-\ (b+c-1\ )2_{x-}(c-1)x+1} P_{i=4}^{m}(a+b+c-$$

(where the terms $P_{i=4}^{m}(a+b+c-2)x^i$ do not appear for $m = 3$), we have that

$$d_1 = a-2, \quad d_2 = 2a+b-3, \quad d_3 = 3a+2b+c-4,$$
$$d_i = d_{i-1} + (a+b+c-2) \quad fori \in \{4, 5, \cdots m\},$$
$$d_{m+4} = (m+1)(a+b+c-2), \quad d_{m+3} = d_{m+4}+1, \quad (**)$$
$$d_{m+2} = d_{m+3} - (c-1), \quad d_{m+1} = d_{m+2} - (b+c-1).$$

We now verify that the conditions of lexico $g-r$ aphic order on $d_\beta(1)r-a$ e satisfied . Since $a+b+c-1 \geq 2$, we have that $d_2 < d_3 \quad < \cdots < d_m < d_{m+1}$. Here we get that $d_2 \geq b+2a-3 \geq 1$ and $d_{m+1} =\leq a-2$. Since $d_{m+1} > d_{m+2}$ and $d_{m+2} \geq 0$ we need to check only the case when $d_{m+1} = a-2$, which implies that $d_2 - d_{m+2} = a-c > 0$. So the conditions of lexicographic order $a-r$ e satisfied . ♦ If $(m+1)(a+b+c-2) = a-3$ then $m = k+2$. As a result we have

$$m(a+b+c-2) < c-2 \leq (m+1)(a+b+c-2) = a-3 < a-2 \leq (m+2)(a+b+c-2),$$

which implies that $b+2c-2 \geq 0$. $FormSince_{p\ (x)(x}^{=1} weget +_{1)(x} a_2 +_{+} 2_x b_{+} +_{1}^{2c})_{=} -1x_7 =_{-} P^{5}{}_{0.}^{i=1} d_i x^{7-i} -$ $(c+2)x+1$ is equal to

$$7_{x-}(a-2)6_{x-}(b+a-2)5_{x-}(2a+2b+c-1)4_{x-}(b+2c-2)2_{x-}(c+2)x+1,$$

and $d_1 = a-2 > d_2 > d_3 > d_4 = 0, 0 \leq d_5 \leq c-3 \leq a-4$, we get that

$$d_\beta(1) = .a-2, (b+2a-2, 2a+2b+c-1, 0, 2c+b-2, c-3, a-3)^{\omega}.$$

For $m \geq 2$ we will show that ▼ If $b+2c-2 > 0$, the $\beta-$ expansion of 1 is $even^{t-u}$ ally pe^{i-r} odic with period $2m+4$

and preperiod 1 . So

$$d_\beta(1) = .a-2, (d_2, \cdots, d_{2m+3}, c-3, a-3)^{\omega}.$$

▼ If $b+2c-2=0$, the $\beta-$ expansion of 1 is $even^{t-u}$ ally periodic with period 1 and preperiod $2m+4$. So

$$d_\beta(1) = .a-2, d_2, \cdots, d_{2m+1}, d_{2m+2}-1, a-2, a+b+c-3, (a+b+c-2)^{\omega}.$$

In both cases , d_i ' s satisfy

$$p(x)P_{i=0}^{m} x^i P_{i=0}^{m+1} x^i = x^{2m+5} - P_{i=1}^{2m+3} d_i x^{2m+^{\text{five-minus-i}}} - (c-2)x + 1.$$

Since $ma+(m+1)b+(m+1)c-2m+1=0$, we have

$$d_1 = a-2, \quad d_2 = 2a+b-3,$$

$d_i = ia + (i-1)b + (i-2)c - 2(i-1)$ for $3 \le i \le m$, (these terms do not appear for $m = 2$)

$$d_{m+1} = a - b - 2c, \quad d_{m+2} = a - c, \quad d_{m+3} = 0, \quad d_m \text{plus} - \text{four} = a - minus - b,$$

$$d_{2m+3-i} = ia + (i+1)b + (i+2)c - 2(i+1) \quad \text{for } 1 \le i \le m-2, (\text{ these te } r - m \text{ s do not}$$

$$appear form = 2)$$

$$d_{2m+3} = b + 2c - 3$$

Since $a + b + c - 2 > 0$, we have $d_i < d_{i+1}$ and $d_{m+2+i} > d_{m+3+i}$ for $2 \le i \le m$. We also have that $d_2 > 0, 0 \le d_{m+2} < d_{m+1}, d_{m+4} \le a-4$ and $d_{2m+2} < a-2$. Since $d_{m+1} \le a-3$ for $b+2c-3 \ge 0$, for $2 \le i \le 2m+3$ we have that

$$0 \le d_i \le a - 3.$$

For $b+2c-2 = 0$ we have that $m(a+b+c-2) = c-3, d_{m-plus1} = a-2$ and $d_3 = 3a-3c$ for $m \ge 3$. Since $a+b+c-3 = a-c-1, d_{m+2} = a-c$ and $d_2 - (a-c) = a-c-1 \ge 0$,

for $c = a-1$ we need to $comp^{a-r}$ e d_3 with $d_{m+3} = 0$. Since $d_3 > 0$, the conditions of lexico $g - r$ aphic order on $d_\beta(1)r - a$ e satisfied in this case also . • If $(k-1)(a+b+c-2) < c-2$, let us show that the $\beta-$ expansion of 1 is eventually periodic with $prepe^{r-i}$ od 1 and period $2k+2$. So

$$d_\beta(1) = .a-2, (d_2, \cdots, d_{2k+1}, c-3, a-3)^{\omega},$$

where d_i ' s $a - r$ e as follows : $p(x)P_{i=0}^{k-1}x^iP_{i=0}^{k}x^i = x^{2k+3} - P_{i=1}^{2k+1}d_ix^{2k+^{\text{three}-\text{minus}-\text{i}}} - (c-2)x + 1$. So we have

$$d_1 = a-2, \quad d_2 = 2a + b - 3,$$

$$d_i = ia + (i-1)b + (i-2)c - 2(i-1) \quad \text{for } 3 \le i \le k-1, (\text{ these terms do not } appe^{a-r} \text{ for}$$

$$k = 3)$$

$$d_k = ka + (k-1)b + (k-2)c - 2k + 3, \quad d_{k+1} = ka + kb + (k-1)c - 2k + 3,$$

$$d_{k+2} = (k-1)a + kb + kc - 2k + 3, \quad d_{k+3} = (k-2)a + (k-1)b + kc - 2k + 3,$$

$$d_{2k+1-i} = ia + (i+1)b + (i+2)c - 2(i+1) \quad \text{for } 1 \le i \le k-3, (\text{ these terms do not } appe^{a-r}$$

$$fork = 3)$$

$$d_{2k+1} = b + 2c - 3.$$

So we have that $d_1 > d_2, d_i < d_{i+1}$ for $2 \le i \le k-1, d_k > d_{k+1} > d_{k+2}, d_{k+2} < d_{k+3}$, $d_{k+i} > d_{k+1+i}$ for $3 \le i \le k$. F $i - r$ st we notice that $d_2 \ge 1, d_k \le a-2, d_{k+2} \ge 1$, $d_{k+3} < d_k$ and $d_{2k+1} \ge d_{k+1} - 1$. So all the d_i ' s $a - r$ e nonnegative and smaller than d_1, only d_k can be equal to d_1. But , if $(k-1)(a+b+c-2) = c-3$ we have that $d_3 > d_2 \ge d_{k+1} > d_{k+2}$. So conditions of lexicographic order are satisfied .

Second we find the common point of T_η and $T_\eta - \Phi(\beta^{-1})$. Since every conjugate of β is also a root of $p(x)(x^2+x+1)(x+1)\sum_{i-equal-zero}^{\infty} 6_x^i$, we have

$$1 + (2-c)\beta 1^{+} P_{i=0}^{\infty}((2-2c-b)\beta 2^{+}(1-a-2b-minus_{2c})\beta 3^{+}(1-2a-2minus-b_c)\beta 4$$
$$\text{plus} - \text{parenleft}2 - 2a - b)\beta 5^{+}(^{a-minus-three})\beta 6^{+}(^{\text{three}-\text{minus}}c)\beta 7)\beta 6i = 0$$

and all the coe $f-fi$ cients have absolute value less than $\lfloor\beta\rfloor$. $*$ If $b=-a$, we have $5-a \leq 2c-a-1 \leq a-3$. We get that

$$\begin{aligned} d_\beta(1) = .a-2,(a-2,c-1,2c-a-1,2c-a-2,c-3,a-3)^\omega \\ for 1 \leq 2c-a-1 \leq a-3, while \\ d_\beta(1) = .a-2,a-2,c-2,2c-3,(2c-4)^\omega \end{aligned}$$

$$for 5-a \leq 2c-a-1 \leq 0.$$

$\diamond$ If $3 \leq c \leq a-2$, since every conjugate of β is also root of $p(x)(x^2+x+1)P^{\infty}_{\text{equal}-\text{zero}i}x^{3i}$, we have

$$1+(1-c)\beta 1+(a+1-c)\beta 2+(1-c)\beta 3+(2-c)\beta 4P^{\infty}_{i=0}\beta i=0.$$

For $4 \leq c \leq a-2$ all the coefficients have absolute value less than $\lfloor \beta \rfloor$, so according to Corollary 4 . 1 on page 288 , each tile is $a-r$ cwise connected . For $c=3$ we get that

$$a-2,0.1=\omega_1,2,0,2.0$$

which shows that $a-2,0.\eta=\omega 1,2,0,2.\eta-0.1$ is a common point of T_η and $T_\eta-\Phi(\beta^{-1})$ for $a \geq 5$. $\diamond$ If $c=a-1$, since eve $r-y$ conjugate of β is also a root of

$$p(x)(x^3-1)(x+1)^2\sum_{i=0}^{\infty}x^{6i}=0, then$$

$$1+(3-a)\beta 1^{+}(3-a)\beta 2^{+}(-2\beta 4-\beta 5^{+}\beta 6^{+2}\beta 7^{+}\beta 8-\beta 9)P^{\infty}_{i=0}\beta 6i=0$$

and for $c \geq$ 4 all the coe $f-fi$ cients have absolute value less than $\lfloor \beta \rfloor$, so , according to Corollary 4 . 1 on page 288 , each tile is $a-r$ cwise connected . If $b=-a, c=3$ and $a=4$ we have that

$$d_\beta(1)=.2,(2,2,1,0,0,1)\quad \begin{matrix}\omega\\ ,\end{matrix}$$

and for $\eta=(2,2,1,0,0,1)^\omega$ we get that

$$\omega(1,2,1,0,0,0),0,0.\eta=\omega(1,0,0,0,1,2),0,1,1.\eta \quad -0.1$$

is a common point of T_η and $T_\eta-\Phi(\beta^{-1})$.

• If $-a+1 \leq b \leq -1$, we have $3-a \leq b+c \leq a-2$. $\diamond$ If $b+c \geq 0$ and $c \geq 2$, we have $a \geq 3$ and

$$d_\beta(1)=.a-1,(b+a,c+b,c-2,a-2)\quad \begin{matrix}\omega\\ .\end{matrix}$$

Since every conjugate of β is also a root of $p(x)(x+1)\sum_{i=0}^{\infty}x^{4i}=0$, we have

$$1+(1-c)\beta 1^{+}(^{minus-b}-c)\beta 2-(a+b)\beta 3^{+}(2-a)\beta 4^{+}(2-c)\beta 5)P^{\infty}_{i=0}\beta 4i=0$$

and for $b \quad \leq \quad -2$ all the coefficients have absolute value less than $\lfloor \beta \rfloor$. So , according to Coroll $a-r$ y 4 . 1 on page 288 , each tile is $r-a$ cwise connected . For $b=-1$, since $d_\beta(1)=.a-1,(a-1,c-1,c-2,a-2)^\omega$, the smallest tile is T_η for $\eta=(a-1,c-1,c-2,a-2)^\omega$. Thus we get that

$$.\eta=\omega(c-2,a-2,a-1,c-1),c-1.\eta \quad -0.1$$

is a common point of T_η and $T_\eta-\Phi(\beta^{-1})$. $\diamond$ If $b+c \geq 0$ and $c=1$, we have $b=-1, a \geq 3$ and

$$d_\beta(1)=.a-1,a-2,a-1,(a-2)\quad \begin{matrix}\omega\\ .\end{matrix}$$

Hence the smallest tile is T_η for $\eta=a-1,(a-2)^\omega$. S ince every conjugate of β is also a root of $p(x)(1+x^3)\sum_{i=0}^{\infty}x^{6i}=0$, we have

$$1-\beta 1^{+}(\beta 2^{+}(1-a)\beta 3)P_{i=0}^{\infty}\beta 3i=0.$$

So a common point of T_η and $T_\eta-\Phi(\beta^{-1})$ is

$$\omega(0,1,0).\eta=\omega(0,a-1,0),1.\eta- \quad 0.1.$$

$\diamond$ If $b+c\leq -1$, we have $a\geq 4$ and

$$d_\beta(1)=.a-1,a+b-1,a+b+c-1,(a+b+c-2)^{\omega}.$$

So the smallest tile in this case is T_η for

$$\eta=\begin{cases}.a+b+c-1,(a+b+c-2)^{\omega}, & forc\geq 1;\\ .a+b-1,a+b-1,(a+b-2)^{\omega}, & forc=0.\end{cases}$$

Since every conjugate of β is also a root of $p(x)(x+1)\sum_{i=0}^{\infty}x^{4i}=0$, we have

$$1+(1-c)\beta 1^{+}(^{minus-b}-c)\beta 2-(a+b)\beta 3^{+}(2-a)\beta 4^{+}(2-c)\beta 5)P_{i=0}^{\infty}\beta 4i=0$$

and for $b \quad\leq\quad -2$ all the coefficients have absolute value less than $\lfloor\beta\rfloor$. So , according to Coroll $a-r$ y 4 . 1 on page 288 , each tile is $r-a$ cwise connected . For $b=-1$ we have that $c=0$ and $\eta=a-2,a-2,(a-3)^{\omega}$. So

$$\omega(2,0,0,1),1.\eta=\omega(a-2,a-1,0,0).\eta \quad -0.1$$

is a common point of the smallest tile T_η and $T_\eta-\Phi(\beta^{-1})$. • If $0\leq b\leq a$, we have $a\geq 2$ and

$$d_\beta(1)=greater{-}greater{-}less{-}greater{-}greater{-}colon \quad \therefore a_a^{'}, a_{a-0}^{'} \quad (^{b,c-1,a}_{b\ -\ 1,(a,} \ {-1)}^{\omega}, \ {b-2)}^{\omega}, \ ,a1^{-1,}_{,a-1,}\ (^{a,}_{a-1}{}^{0,a}_{,} \ {-2)}^{\omega}, \ {(a-2)}^{\omega}, \quad if^{if^{if}_{if}}\ a^{a^{a}_{a}} \geq^{\geq\geq}_{\geq\geq} 3^{2,2,}_{,\ 2and}\ c1_{=}\ \substack{\leq 0,\\ c_c==0_0} ar$$

So $\lfloor\beta\rfloor=\jmath\quad a^{a-1}_{,},\quad if^{b=c=0;}_{ot-herwise;}$ and the smallest tile is T_η for

$$\eta=eight{-}greater{-}greater{-}greater{-}greater{-}less{-}greater{-}greater{-}greater{-}greater{-}colona\ (^{a^{a}}_{((}\ ^{a^{a}}\ {}^{,-}_{,-,}{}_{0}c-1,b_{1,}{}^{a}a-2^{1b}_{-}$$

Since every conjugate of β is also a root of $p(x)\sum_{i=0}^{\infty} x^{3i} = 0$, we have

$$1 - c\beta 1 - (b\beta 2^{+}(a-1)\beta 3^{+}(c-1)\beta 4) X \beta 3i = 0. \quad \underset{i=0}{\overset{\infty}{}} \tag{4.4}$$

$*$ For $b \leq a-1$, with the exception of the case where $b = c = 0$, all the coe $f - fi$ cients have absolute value less than $\lfloor \beta \rfloor = a$. So , according to Coroll $a - r$ y 4 . 1 on page 288 , each tile is $a - r$ cwise connected .

For $b = c = 0$, from (4 . 4) , we get that

$$\omega(1,0,0),0.\eta = \omega(a-1,0,0).\eta \quad -0.1$$

is a common point of the smallest tile T_η and $T_\eta - \Phi(\beta^{-1})$, where $\eta = a-1, a-1, (a-2) \overset{\omega}{.}$ $*$ For $b = a$ and $c \geq 1$, from (4 . 4) , we get that

$$.\eta = \omega(c-1,a-1,a), c.\eta - \quad 0.1$$

is a common point of the smallest tile T_η and $T_\eta - \Phi(\beta^{-1})$ where $\eta = (a,c-1,a-1) \overset{\omega}{.}$ $*$ For $b = a$ and $c = 0$, from (4 . 4) , we get that

$$\omega(1,0,0),0.\eta = (a-1,a,0).\eta - \quad 0.1$$

is a common point of the smallest tile T_η and $T_\eta - \Phi(\beta^{-1})$ where $\eta = (a,a-2) \overset{\omega}{.}$ • If $a+1 \leq b \leq 2a$, we have $c-b+a \geq -1$.

$*$ If $c-b+a \geq 0$, we have $\lfloor\beta\rfloor = a+1$ and

$$d_\beta(1) = .a+1,(b-a-1,c+a-b,b-c-1,c,a) \overset{\omega}{.}$$

Since every conjugate of β is also a root of $p(x)(x-1)(4_{x-}1)P_{i=0}^{\infty}x^{8i} = 0$, we have

$$1-(c+1)\beta 1^{+}((c-b)\beta 2^{+}(b-a)\beta 3^{+a}\beta 4^{+c}\beta 5)(1-x^4)P_{i=0}^{\infty}\beta 8i = 0$$

and all the coe $f-fi$ cients have absolute value less than $\lfloor\beta\rfloor$. $*$ If $c-b+a = -1$ and $b \geq a+2$, we have $\lfloor\beta\rfloor = a+1$ and

$$d_\beta(1) = .a+1,(b-a-2,a+1,b-a-2,0,a-1,b-a,a-1,b-a-1,a) \overset{\omega}{.}$$

Since every conjugate of β is also a root of $p(x)(2_{x-}x+1)P_{i=0}^{\infty}x^{5i} = 0$, we have

$$1-(c+1)\beta 1-(a\beta 2-\beta 3^{+c}\beta 4^{+a}\beta 5^{+c}\beta 6)P_{i=0}^{\infty}\beta 5i = 0$$

and all the coe $f-fi$ cients have absolute value less than $\lfloor\beta\rfloor$. $*$ If $c-b+a = -1$ and $b = a+1$, we have $c = 0$, $\lfloor\beta\rfloor = a$ and

$$d_\beta(1) = .a,a,(a,a-1) \overset{\omega}{.}$$

Since every conjugate of β is also a root of the $p(x)(x^2+1)P_{i=0}^{\infty}\beta 4i_{,}^{=0}$ we have

$$.\eta = \omega(a-1,a),a,0.\eta \quad -0.1$$

is a common point of the smallest tile T_η and $T_\eta - \Phi(\beta^{-1})$.

$a50$

From the proof of the Theorem 4 . 5 on page 292 we get also the following theorem which gives the $\beta-$ expansion of 1 for any Pisot unit of degree four with minimal polynomial $x^4-ax^3-bx^2-cx+1=0$.
Theorem 4 . 6 *Let* β *be a Pisot unit of degree four with minimal polynomial* $p(x)=x^4-ax^3-bx^2-c-x+1=0$. *The* $\beta-$ *expansion of* 1 *is :*

$$- \quad When 1-a\leq c\leq -1,$$

$\star and 4-a \leq b \leq 0, \quad d_\beta(1) = .a-1, a+b-1, a+b+c-1, (a+b+c-2)^\omega$

$\star and b=1, d_\beta(1) = .a, 0, a+c-1, a-1, a+c, (a+c-1)^\omega$

$\star and 2 \leq b \leq a, d_\beta(1) = .a, b-1, (a+c, b-2)^\omega$

$- \quad When 0 \leq c \leq a-1$

$\star$ *and* $4-2a \leq b \leq -a-1$, *let* k *be the integer of* $\{2,3,\cdots,a-2\}$ *with* $(k-1)(a+b+c-2) <$

$$a-2 \leq k(a+b+c-2).$$

$*$ *If* $(k-1)(a+b+c-2) \geq c-2$ *le t*

$m = \inf\{i \in \mathbb{N}$ *such that* $(i+1)(a+b+c-2) \geq c-2\}$. *We have* $1 \leq m \leq k-2$.

$\diamond$ *If* $(m+1)(a+b+c-2) < a-3$, *the* $\beta-$ *expansion of* 1 *is eventually periodic with period 1 andpreperiod* $m+3$.

$$m=1 \Rightarrow d_\beta(1) = .a-2, \mathrm{two}-\mathrm{a}_+ b-2, a-two_+^{\mathrm{two-b}}{}_+ c-2, a-two_+^{\mathrm{two-b}}{}_+ c-two_-3, ({}^{\mathrm{two-a}}{}_+{}^{b-}$$

$$m=2 \Rightarrow d_\beta(1) = .a-2, \mathrm{two}-\mathrm{a}_+ b-3, a-three_+^{\mathrm{two-b}}{}_+ c-3, a-three_+^{\mathrm{two-b}}{}_+ c-two_-4, 3(a+b+c)-5, ({}^{\mathrm{three-a}}{}_+{}^{b-thr}$$

$$m \geq 3 \Rightarrow d_\beta(1) = .a-2, \mathrm{two}-\mathrm{a}_+ b-3, a-three_+^{\mathrm{two-b}}{}_+ c-4, d_4, \cdots$$

with $d_i = d_{i-1} + a+b+c-2$ *for* $4 \leq i \leq m($ *these terms do not appearfor* $m=3)$

$$andbraceex-braceex-braceleftmid-braceex-braceex-braceleftbt \quad \begin{array}{l} d_{m+1} = d_{m+2} - b - c + 1, \\ d_{m+2} = d_{m+3} - c + 1, \\ d_{m+3} = d_{m+4} + 1, \\ d_{m+4} = (m+1)(a+b+c-2). \end{array}$$

$\diamond If (m+1)(a+b+c-2) = a-3 then$

$$m=1 \Rightarrow d_\beta(1) = .a-2, ({}^{\mathrm{two-a}}+b-2, a-two_+^{\mathrm{two-b}}{}_+ c-1, 0, c-two_+ b-2, c-3, a-3)^\omega$$

If $m \geq 2$ *and* $b+2c \geq 3$, *the* $\beta-$ *expansion of* 1 *is eventually periodic with preperiod 1 and period* $2m+4$. *So*

$$d_\beta(1) = .a-2, ({}^{\mathrm{two-a}}+b-3, d_3, \cdots, d_{2m+3}, c-3, a-3)^{\omega}.$$

If $m \geq 2$ *and* $b+2c=2$, *the* $\beta-$ *expansion of* 1 *is eventually periodic with period 1 andpreperiod* $2m+4$. *So*

$$d_\beta(1) = .a-2, {}^{\mathrm{two-a}}+b-3, d_3, \cdots, d_{2m+1}, d_{2m+2}-1, a-2, a+b+c-3, (a+b+c-2)^\omega where$$

$$d_i = ia + (i-1)b + (i-2)c - 2(i-1) \quad for \quad 3 \leq i \leq m, (\text{ these terms do not appearfor } m=2)$$

$$d_{m+1} = a-b-2c, \quad d_{m+2} = a-c, \quad d_{m+3} = 0, \quad d_m \mathrm{plus}-\mathrm{four} = a-minus-b,$$

$$d_{2m+3-i} = ia + (i+1)b + (i+2)c - 2(i+1) \quad \textit{for} \quad 1 \leq i \leq m-2, (\textit{ these terms do not appear} form = 2)$$

$$d_{2m+3} = b + 2c - 3.$$

$*$ *If* $(k-1)(a+b+c-2) \leq c-3$, *the* $\beta-$ *expansion of* 1 *is eventually periodic with preperiod 1 andperiod of length* $2k+2$. *So*

$$d_\beta(1) = .a-2, (2a+b-3, d_3, \cdots, d_{2k+1}, c-3, a-3)^\omega$$

$$\textit{with} \ \ d_i = ia + (i-1)b + (i-2)c - 2(i-1) \quad \textit{for} \quad 3 \leq i \leq k-1, (\textit{ these terms do not appear} fork = 3)$$

$$d_k = ka + (k-1)b + (k-2)c - 2k + 3, \quad d_{k+1} = ka + kb + (k-1)c - 2k + 3,$$

$$d_{k+2} = (k-1)a + kb + kc - 2k + 3, \quad d_{k+3} = (k-2)a + (k-1)b + kc - 2k + 3,$$

$$d_{2k+1-i} = ia + (i+1)b + (i+2)c - 2(i+1) \quad \textit{for} \quad 1 \leq i \leq k-3, (\textit{ these terms do not appear} fork = 3)$$

$$d_{2k+1} = b + 2c - 3.$$

$\star and b=-a, d_\beta(1)=\jmath^{.a-2,(a-2,c-1,}_{.a-2,\ a-2,c}\ {}^{2c-a-1,}_{-2,2}\ {}^{2c-a-2,}_{c-3,(2c-}\ {}_{4)}\ \omega\ c-3,a-3)^\omega, if^{if}{}^{2c-a}_{2c-a}\geq 2\leq^{;}_{1;}$

,

$\star and -a+1\leq b\leq -1,$

$* \ for\ \ b+c\geq 0\ we\ have\ \ d_\beta(1)=\jmath$ $.a-1,(^{b-plus-a},cb-plus,c-2,a-2)^\omega,\quad if(b,c)\neq(-1,1);$
$.a-1,a-2,a-1,(a-2)^\omega,\quad if(b,c)=(-1,1);$

$* \quad if b+c\leq -1, d_\beta(1)=.a-1,a+b-1,a+b+c-1,(a+b+c-2)$ ω

,

$\star and 0\leq b\leq a,$

$* \ for\ \ c\geq 1\ we\ have\ \ d_\beta(1)=.a,(b,c-1,a-1)$ ω , $* \ for\ \ c=0\ we\ have\ \ d_\beta(1)=less-colon,0,a-$
$1,(a,0,a-2)^{\omega},{}^{b-1,(a,b-2)^\omega}_{.a:^{a}_{a}{}^{-1},a-1,a-1,(a-2)^\omega}\quad if^{if}_{if}b^b_b\geq^{==}1;^{2}_{0;}$

$\star anda+1\leq b\leq 2a,$

$* \ for\ \ a-b+c\geq 0\ we\ have\ \ d_\beta(1)=.a+1,(b-a-1,a-b+c,b-c-1,c,a)^\omega$ $* \ for\ \ a-b+c=-1$ *we have*

$d_\beta(1)=\jmath$ $.a,a,(a,a-1)^\omega,\quad if^{b-equal}a+1;$
$.a+1,(^{minus-a}-2,a+1,minus-b_{a-2},0,a-1,b-minus_a,a-1,a-minus-b_{-1},a)^\omega,\quad if b\geq a+2.$

$polynomial^{Example2}x^{4-}_{Here}we_{ax^3}want_{-bx^2}to\_show_{cx+}that_{1=}from_{0,we}$ $a\ class^{of Pisot}_{canobtain}\ a\ \beta units^{of degree}_{-expansion}$ *of* $14 which with_{an} are roots of arbitrarily_{long}$ *the preperiod . For* $n\geq 5, a=n+2, b=4-2a=-2n$ *and* $c=a-1=n+1$ *we have that* $\lfloor\beta\rfloor=a-2$ *and the* $\beta-$ *expansion of* 1 *is*

$$d_\beta(1)=.n,1,_{braceleft-z}3,4,5,\cdots,n-3,n-2,n,1,0,n-2,_{z-braceleft}n-4,n-5,\cdots,3,2,0,n,0,1^\omega.$$

$n-4$ e lements $\quad n-5$ e l ements *Therefore the length of the preperiod is* $2n.$
Lemma 3 *If* β *is a Pisot unit ofdegree four with minimal polynomial* $p(x)=x^4-ax^3-bx^2-cx-1,$ *and if the negative root* γ *of the polynomial* $x^2-\lfloor\beta\rfloor x-1$ *has the property*

$$p(\gamma)>0,$$

then at least one of the ti les is not connected .
Proof : Let $d_\beta(1)=.d_{-1},d_{-2},\cdots$ and $\xi=\xi_-1\beta^{-1}+\xi_-2\beta^{-2}+\cdots$ be a $\beta-$ expansion with $d_\beta(1)>.\xi_-1,\xi_-2,\cdots\geq .d_{-2},d_{-3},\cdots$. Since $p(-1)>0$ and $p(0)=-1$, the polynomial $p(x)$ has at least one root in the inte $r-v$ al $(-1,0)$. Let $\theta\in(-1,0)$ be the biggest among such roots . First we want to show that

$$T_\xi\cap(T_\lambda+\Phi(\xi-m\beta^{-1}))=\varnothing \tag{4.5}$$

for $m\in\{1,2,\cdots\}$ such that $\xi_-1\geq m.$ If we suppose the contr $a-r$ y , then there exists an expansion $\cdots,c_1,c_0.m$ with $c_i\in[-\lfloor\beta\rfloor,\lfloor\beta\rfloor]\cap\mathbb{Z}$ for $i=0,1,\cdots$ and $c_0\leq\lfloor\beta\rfloor-1$, which implies that

$m\theta^{-1} + c_0 + \sum_{i=1}^{\infty} c_i\theta^i = 0$. The as sumptions of the lemma show that $\gamma < \theta$. So θ is between two roots of the polynomial $x^2 - \lfloor\beta\rfloor x - 1$ and we have that

$$\frac{1}{\theta} + \lfloor\beta\rfloor - 1 + \frac{\lfloor\beta\rfloor\theta^2}{1-\theta^2} < \frac{1}{\theta} + \lfloor\beta\rfloor - 1 - \frac{\lfloor\beta\rfloor\theta}{1+\theta} = \frac{\theta^2 - \lfloor\beta\rfloor\theta - 1}{-\theta(1+\theta)} < 0,$$

 which implies that $m\theta^{-1} + c_0 + \sum_{i=1}^{\infty} c_i\theta^i < 0$. Second , we prove the existence of a disconnected tile .

$$Since. d_{-2}, d_{-3}, \cdots < d_\beta(1), let$$

$l = \min \{k \in \mathbb{N} \mid .d_{-2}, d_{-3}, \cdots, (d_{-k}+1)$ is admissible $\}$. For $l = 2$ we have by (4 . 3) ,

$$G_{-1}(T_\lambda) = T_0 \cup T_1 \cup \cdots \cup T_{\lfloor\beta\rfloor} - 1 \cup T_{\lfloor\beta\rfloor} \quad and \quad \lfloor\beta\rfloor \geq (d_{-2}+1) \geq d_{-2}, d_{-3} \cdots$$

By using (4 . 5) with $\xi = \lfloor\beta\rfloor/\beta$ and

$$T_{\lfloor\beta\rfloor - m} \subset T_\lambda + \Phi(\frac{\lfloor\beta\rfloor - m}{\beta}),$$

we deduce

$$T_{\lfloor\beta\rfloor} \cap T_{\lfloor\beta\rfloor} - m = \varnothing$$

for $m = 1, 2, \cdots, \lfloor\beta\rfloor$. Therefore the central tile T_λ is disconnected . For $l \geq 3$ let $\epsilon = d_{-3}, \cdots, (d_{-l}+1)$. Then we have

$$G_{-1}(T_\epsilon) = T_{0,\epsilon} \cup T_{1.\epsilon} \cup \cdots \cup T_{d_{-2},\epsilon}$$

with

$$d_{-2}, \epsilon >_{lex} d_{-2}, d_{-3}, \cdots = U_\beta(1).$$

Therefore the tile T_ϵ is disconnected in the same way by using (4.5). a50 Theorem 4 . 7 *Let* β *be a Pisot unit ofdegree* 4 *with minimal polynomial* $p(x) = x^4 - ax^3 - bx^2 - cx - 1$.
Each ti le is arcwise connected exceptfor the following cases :

$$eight{-}greater{-}less{-}greater{-}colon^{a_c} \frac{a^{3a}_{5\geq = 5,}}{2} \, {}^{3,}_{b} \leq a{-}minus, \quad a{-}\frac{\leq}{eight{-}greater{-}less{-}greater{-}colon^{a_c} \frac{1 \geq = a}{2}} {}^{3,} 1b' \leq -1, \quad a+$$

Proof : We only need to prove this theorem for the cases when the $\beta-$ expansion of 1 is infinite because the other cases $a - r$ e shown in Theorem 1 . 2 on page 273 . According to $r - P$ oposition 4 . 1 on page 292 , the coefficients satisfy the following system of inequalities :

$$\jmath \, |^{b|}_{a} \, 2 \leq +a4b^{+} c^{-}_{-c^2} 1^{`}_{\geq} 1.$$

Here we have the following bounds for the coe $f - fi$ cients :

$$eight - greater - less - greater - colon \quad \begin{array}{l} a \geq 1, \\ 2^1 -^{-} a2_a \leq c \leq b \leq a \leq 2^{+} a +^{3,} 2. \end{array}$$

$Case^1$. For $minus - a + 1 \leq c \leq -1$ we have $a \geq 2, 1 - a - c \leq b \leq -1 + a + c$, hence $2 - a \leq b \leq a - 2$.
• For $b \leq 0$ we have $\lfloor\beta\rfloor = a - 1$ and

$$d_\beta(1) = \jmath \quad \begin{array}{l} .a-1, a+b-1, a+b+c-1, (a+b+c)^\omega, \quad if(c,b) \neq (-1,0); \\ .a-1, a-1, a-1, 0, 0, 1, \quad if(c,b) = (-1,0). \end{array}$$

Therefore the smallest tile in this case is T_η for

$$\eta = less - colon \quad \begin{array}{l} a+b-1, a+b+c-1, (a+b+c)^\omega, \quad forc \leq -2; \\ (a+b-1)^\omega, \quad forc = -1andb \neq 0; \\ a-1, a-1, 0, 0, 1, \quad for(c,b) = (-1,0). \end{array}$$

$*$ For $c \geq 2-a$, since every conjugate of β is also a root of $p(x)(x^3-1)P_{i=0}^{\infty} x^{6i} = 0$, we have

$$1 + c\beta 1^+(b\beta 2^+(a-1)\beta 3 - (c+1)\beta 4 - b\beta 5 - (a-1)\beta 6^+(c+1)\beta 7)P_{i=0}^{\infty}\beta 6i^{=0}$$

and

$$\omega_{(0,}minus - b_, c - minus - 1, a-1, 0, 0).\eta = \quad \omega_{(minus-c-1,}a-1, 0, 0, 0, b - minus), minus - c_.\eta \quad -0.1$$

is a common point of the smallest tile T_η and $T_\eta - \Phi(\beta^{-1})$. $*$ For $c = 1-a$ we have $b = 0$. Since every conjugate of β is also a root of

$$p(x)(x^2$$

$$1 - (a-2)\beta \text{one} - \text{minus}(a-2)\beta \text{plus} - \text{two}\beta \text{three} - \text{plus}(a-1)\beta \text{plus} - \text{four}(a-2)\beta \text{five} - \text{plus}((a-2)\beta 6 - (a-1)\beta 8 -$$

0

and

$$\omega(a-1, 0, 0, 0, 0, a-2), a-2, a-1, 1, 0, 0.\eta =$$
$$\omega(a-2, a-1, 0, 0, 0, 0), 0, a-2, a-2.\eta \quad -0.1$$

is a common point of the smallest tile T_η and $T_\eta - \Phi(\beta^{-1})$. • For $b \geq 1$ we have $a \geq 3, 2-a \leq c \leq -1$ and $1 \leq b \leq a+c-1$. Here $\lfloor \beta \rfloor = a$ and

$$d_\beta(1) = .a, b-1, (c+a, b) \quad \begin{array}{c} \omega \\ , \end{array}$$

therefore the smallest tile in this case is T_η for $\eta = (a+c, b)^\omega$. Since every conjugate of β is also a root of $p(x)(x^3-1)\sum_{i=0}^{\infty} x^{6i} = 0$, then

$$1 + c\beta 1^+(b\beta 2^+(a-1)\beta 3 - (c+1)\beta 4 - b\beta 5 - (a-1)\beta 6^+(c+1)\beta 7)P_{i=0}^{\infty}\beta 6i = 0$$

and all the coefficients have absolute value less than $\lfloor \beta \rfloor$. $Case^2_-$. For $0 \leq c \leq a-1$ we have $-2a+2 \leq b \leq 2a-2$.

• If $-2a+2 \leq b \leq -a$ then $a \geq 5$ and $1 \leq c \leq a-3$.

$*$ If $1 \leq c \leq a-4$ then $1-a-c \leq b \leq minus - a$ and $\lfloor \beta \rfloor = a-2$.

F $i-r$ st , let us find the $\beta-$ expansion of 1 . Since $1 \leq a+b+c < a-2$, there exists an integer $k \in \{2, 3, \cdots, a-2\}$ with $\frac{a-2}{k} \leq a+b+c < \frac{a-2}{k-1}$, which implies that $(k-1)(a+b+c) <$

$$a-2 \leq k(a+b+c).$$

$\diamond$ If $(k-1)(a+b+c) \geq c+2$ we get $k \geq 3$ and $c < a-4$. Let m be the integer defined by $m = \inf\{i : (i+1)(a+b+c) \geq c+2\}$. By definition , $m \leq k-2$ and , since $b \leq -a$, we get $m \geq 1$. Let us show that the $\beta-$ expansion of 1 is eventually periodic with period 1 and that the length of the $prepe^{r-i}$ od is $m+3$. So let us write it as

$$d_\beta(1) = .a-2, d_2, \cdots, d_{m+3}, d_{m+4}^{\omega}.$$

$$W - h_{en} m = 1, since$$

$$p(x)(1+x) = 5_{x-}(a-1)4_{x-}(a+b)3_{x-}(b+c)2_{x-}(c+1)x-1,$$

we get that

$$1 = .a-2, 2a+b-2, 2a+2b+c-2, 2a+2b+2c-1, (2a+2b+2c)^{\omega}.$$

$$Here d_5 = d_{m+4}, d_4 = d_{m+3}, d_3 = d_{m+2}.$$

$$W - h_{en} m = 2, since$$

$$p(x)(1+x+x^2) = 6_{x-}(a-1)5_{x-}(a+b-1)4_{x-}(a+b+c)3_{x-}(b+c+1)2_{x-}(c+1)x-1,$$

we get that

$$1 = .a-2, 2a+b-3, 3a+2b+c-3, 3a+3b+2c-2, 3(a+b+c)-1, (3a+3b+3c)^{\omega}.$$

Here $d_6 = d_{m+4}, d_5 = d_{m+3}, d_4 = d_{m+2}, d_3 = d_{m+1}$, where the formulas of d_i will be given later .

$$W - h_{en} m \geq 3, since$$

$$p(x) P_{i=0}^{m} x^i = m - plus - four_x \quad {}_-(a - ({}^{1)x^{m+3}-(a+b-1)x^{m+2}-(a+b+c-1)x^{m+1}-}_{a+b+c+1)x^3 - (b+c+1)x^2 - (c+1)x-1} P_{i=4}^{m}(a+b+c)i_{x-}$$

(where the terms $\sum_{i=4}^{m}(a+b+c)x^i$ do not $appe^{a-r}$ for $m = 3$), we have that

$$d_2 = 2a+b-3, \quad d_3 = 3a+2b+c-4,$$
$$d_i = d_{i-1} + (a+b+c) \quad fori \in \{4, 5, \cdots m\}$$

(these terms do not $appe^{a-r}$ for $m = 3$), $\quad (*)$

$$d_{m+4} = (m+1)(a+b+c), \quad d_{m+3} = d_{m+4} - 1,$$
$$d_{m+2} = d_{m+3} - (c+1), \quad d_{m+1} = d_{m+2} - (b+c+1).$$

We now verify that the conditions of lexicographic order on $d_\beta(1)$ are satisfied . Since $a+b+c+1 \geq 0$, we have that $d_2 \leq d_3 < \cdots < d_m < d_{m+1}$. Here we get that

$$d_2 \geq b+2a-3 \geq 2 and d_{m+1} = m(a+b+c) + a - c - 3 \leq c+1+a-c-3 \leq a-2.$$

From definition of m, we have that $d_{m+2} = (m+1)(a+b+c) - c - 2 \geq 0$ and , since $m \leq k-2$, we have that $(m+1)(a+b+c) < a-2$. Since $d_{m+2} < d_{m+3} < d_{m+4}$, till now we showed that all d_i ' s are nonnegative and $d_i \leq a-2$.

We now s $t-u$ dy the cases where d_i is not strictly smaller than d_1. For $m=1$ only $d_2=b+2a-2$ may be equal to $a-2$. For $m\geq 2$ only d_{m+1} may be equal to $a-2$, which means that $m(a+b+c)=c+1$ and that $d_2-d_{m+2}=a-c-2$ is a positive integer . So we showed that the above expansions of 1 defined by $(*)a-r$ e $\beta-$ expansions of 1 . $\diamond$ If $(k-1)(a+b+c)\leq c+1$ and $k(a+b+c)=a-2$, let us show that the $\beta-$ expansion of 1 is finite with length $2k+4$. Let us write it as $d_\beta(1)=a-2,d_2,\cdots,d_{2k+3},1$, where

$$p(x)W-h_{en}^{\sum}k -_{i=0^2}^{+c)=} {}_{i^{k}{}_{=0=2}^{-1}}{}^{x^i}\,{}_{,since}^{\sum}\,{}^{k+1}(\,{}^{x^i}_{a+b}{}^{=x2k+4}\sum_a -i2^{k+}_{=12,we}{}^{3}\,{}^{d}i^{x^{2k+4}}get^{-i}-1.$$

$$d_\beta(1)=.a-2,2a+b-2,2a+2b+c-2,a-2,0,2c+b+2,c+2,1.$$

We now verify that the conditions of lexicographic order on $d_\beta(1)$ are satisfied . Since $1-a-c\leq b\leq minus-a_,$ we have $3\leq a-c-1\leq b+2a-2\leq a-2$. Since $d_3=d_2+(b+c)=d_4-(c+2)$, we have $0\leq d_3<a-2$. Since $d_6=-(a+b)$, we have $0\leq d_6\leq a-5$.

 Since $d_2 \geq 3$ and $d_8 = 1$, the conditions of lexicographic order $a - r$ e satisfied .

$W - h_{en}k \geq 3$, since $k(a+b+c) = a-2$, we get

$$d_2 = 2a + b - 3,$$

$$d_i = ia + (i-1)b + (i-2)c - 4 \text{ for } 3 \leq i \leq k-1, (\text{ these terms do not appear for } k = 3)$$

$$d_k = ka + (k-1)b + (k-2)c - 3, \quad d_{k+1} = ka + kb + (k-1)c - 2,$$
$$d_{k+2} = a - 2, \quad d_{k+3} = 0, \quad d_{k+4} = 1 - b - a,$$

$$d_{2k+2minus-i} = ia + (i+1)b + (i+2)c + 4 \text{ for } 1 \leq i \leq k-3, (\text{ these terms do not } appe^{r-a} \text{ for } k = 3)$$

$$d_{2k+2} = 2c + b + 3, \quad d_{2k+3} = c + 2, \quad d_{2k+4} = 1.$$

We now verify that the conditions of lexicographic order on $d_\beta(1)a - r$ e satisfied . Here we have that $d_2 \leq d_3 < \cdots < d_k, d_2 \geq 2$ and $d_k = (k-1)(a+b+c) + a - c - 3 \leq a - 2$. Since $d_{k+1} + c + 2 = a - 2$, we have $0 \leq d_{k+1} \leq a - 5$. We also have that $d_{k+4} \geq d_{k+5} > \cdots >$

$$d_{2k+2}, d_{k+4} = 1 - b - a \leq c \leq a - 4 and d_{2k+2} \geq b + c + 2 + (k-1)(a+b+c) = 0.So$$

we showed that all $d_i a - r$ e nonnegative and not greater than d_1. Since $d_2 \geq 3$ we have that $a - 2$ may be followed by 1 or 2 . If $d_k = a - 2$ we have that $d_2 - d_{k+1} = b + a + c + 1 \geq 2$. So the conditions of lexico $g - r$ aphic order are satisfied . $\diamond$ If $(k-1)(a+b+c) \leq c+1$ and $k(a+b+c) > a-2$, let us show that the $\beta-$ expansion

$$of^1_{p(x)} is P_i finite_{k=0-1x^i} with P_i length k_{=0x} i = 2^{k+3}_{x^{2k+3}} \dot{\_} Let P_i =^{us+_k 2}_2 {}_{1d}^{write} {}_{i^x} it^{as}_{3+_{k_2}} {}_{-i-1.}^{d\ \beta(1)} = .d_1, d_2, \cdots, d_{2k+2}, 1, where$$

$W - h_{en}k = 2$, we get

$$d_\beta(1) = .a - 2, 2a + b - 2, 2a + 2b + c - 1, a + 2b + 2c + 1, 2c + b + 2, c + 2, 1$$

We now verify that the conditions of lexicographic order on $d_\beta(1)$ are satisfied . Since $1 - a - c \leq b \leq \quad minus - a_,$ then $3 \leq b + 2a - 2 \leq a - 2$. S ince $2(a+b+c) \quad > a - 2$ then $2 \leq$

$$2a + 2b + c - 1 \leq c - 1 \leq a - 5, 0 \leq a + 2b + 2c + 1 \leq a - 5. \qquad Also b + 2c + 2 \leq a - 6 and$$

$b + 2c + 3 = 2(b+c+a) - b - 2a + 3 > 1 - b - a \geq 1$. Only d_2 or d_6 can be equal to d_1. Since $1 < d_2$ the conditions of lexicographic order $a - r$ e satisfied .

$$W - h_{en}k \geq 3 we get$$
$$d_1 = a - 2, \quad d_2 = 2a + b - 3,$$

$$d_i = ia + (i-1)b + (i-2)c - 4 \text{ for } 3 \leq i \leq k-1, (\text{ these terms do not appear for } k = 3)$$

$$d_k = ka + (k-1)b + (k-2)c - 3, \quad d_{k+1} = ka + kb + (k-1)c - 1,$$
$$d_{k+2} = (k-1)a + kb + kc + 1, \quad d_{k+3} = (k-2)a + (k-1)b + kc + 3,$$

$$d_{2k+1minus-i} = ia + (i+1)b + (i+2)c + 4 \text{ for } 1 \leq i \leq k-3, (\text{ these terms do not } appe^{r-a} \text{ for } k = 3)$$

$$d_{2k+1} = 2c + b + 3, \quad d_{2k+2} = c + 2.$$

We now verify that the conditions of lexicographic order on $d_\beta(1)a - r$ e satisfied .

Here we have that $2 \leq d_2 \leq d_3 < \cdots < d_k, d_k > d_{k+1} > d_{k+2}, d_{k+2} < d_{k+3}$ and $d_{k+3} \geq d_{k+4} > \cdots > d_{2k+1}$. The condition $(k-1)(a+b+c) \leq c+1 < a-2$ implies that $d_k = (k-1)(a+b+c)+a-c-3 \leq a-2$ and $d_{k+3} \leq a-4$. Also , since $k(a+b+c) > a-2$, then $d_{k+2} = k(a+b+c)+1-a > -1$. Since $c+1 \geq (k-1)(a+b+c)$ then $d_{2k+1} > 0$. So we showed that all d_i ' s satisfy $0 \leq d_i \leq d_1$. S ince $d_2 \geq 3$ we have that $a-2$ may be followed by 1 or 2 . If $d_k = a-2$, which means that $(k-1)a+(k-1)b+(k-2)c-1=0$, then $d_2 - d_{k+1} = a-c-1 \geq 1$. So the conditions of lexicographic order are satisfied .

Second , let us find the common point of the smallest tile T_η and $T_\eta - \Phi(\beta^{-1})$. Since every conjugate of β is also a root of $p(x)(5_{x-}1)(2_{x-}x+1)P_{i=0}^{\infty}x^{10i} = 0$, then

$$1+(c+1)\beta 1^{+}((b+c+1)\beta 2^{+}(a+b+c)\beta 3^{+}(a+b-1)\beta 4^{+}(a-2)\beta 5-(c+2)\beta 6$$

$$-(b+c+1)\beta 7-(a+b+c)\beta 8-(a+b-1)\beta 9-(a-2)\beta 10^{+}(c+2)\beta 11)P^{\infty}_{\text{equal}-\text{zero}i}\beta 10i=0$$

and

$$\omega_{(c+2,}0,1-a-b,0,minus-b_{-c-1,}0,a-2,0,a+b+c,0),c+1.\eta=$$
$$\omega(a-2,0,a+b+c,0,c+2,0,1-a-b,0,-1-b-c,0).\eta \quad -0.1$$

is a common point of the smallest tile T_η and $T_\eta-\Phi(\beta^{-1})$. $*$ If $c=a-3$, we have $\frac{5-3a}{2}\le b\le -a$, $\lfloor\beta\rfloor=a-2$ and

$$d_\beta(1)=.a-2,b+^{\text{two}-\text{a}}-2,(^{a-three}+^{\text{two}-\text{b}}-4,a-three^{\text{two}-\text{b}}_{+}{}_{-}5,a-two_{+}b-3,0,1-a-b,2-a-b,0,b+^{\text{two}-\text{a}}-3)$$

To show that one of the tiles is not connected , according to Lemma 3 on page 300 , it is enough to prove that $p(\gamma)>0$. Since $\gamma^2-(a-2)\gamma-1=0$, we have

$$p(\gamma)\ge\gamma^4-\gamma^3_a+\gamma^2_a-(a-3)\gamma-1=\gamma^2_-(\gamma-2)>0.$$

• If $-a+1\le b\le -1$, we have $a\ge 3, 0\le c\le a-1$. $*$ If $0\le c\le a-3$, we have $a\ge 4$ and

$$d_\beta(1)=eight-less-colon.a^{\cdot a}_{\cdot a}-^{-}_{-}{}_{1}1^{1,}_{,,}a^{a}_{a}+^{+}_{+}a-1,0,c+1,1,^{+b+c-1,(a+b+}_{b-1,^{b-1,c}_{b,b+a},c+1,1,}c)^{\omega},\quad if^{if}_{if}b+^{b+}_{b+}c^{c}_{c}\le^{\ge=0;}_{-2.}{}^{-1;}$$

$\diamond$ If $c$plus $-$ b ≤ -2 and $a-plus-b\ge 2$, s ince every conjugate of β is also a root of $p(x)(x^3-1)P^{\infty}_{i=0}x^{6i}=0$, we have

$$1+c\beta 1^{+}\text{parenleft}-\text{b}\beta 2^{+}(a-1)\beta 3-(c+1)\beta 4-b\beta 5-(a-1)\beta 6^{+}(c+1)\beta 7)P^{\infty}_{\text{zero}-\text{equal}i}\beta 6i=0$$

and

$$\omega(c+1,0,minus-b_,0,a-1,0),c.\eta=\omega(a-1,0,c+1,0,b-minus_,0).\eta \quad -0.1$$

is a common point of the smallest tile T_η and $T_\eta-\Phi(\beta^{-1})$. $\diamond$ If $b=minus-a+1$ and $c\ge 1$, we have $c+b\le -2$ and the smallest tile is T_η for $\eta=(c+1)\overset{\omega}{.}$

Since every conjugate of β is also a root of $p(x)(4_x-1)(x+1)P^{\infty}_{i=0}x^{8i}=0$, then

$$1+(c+1)\beta 1^{+}((c+1-a)\beta 2^{+}\beta 3^{+}(a-2)\beta 4$$
$$-(c+2)\beta 5^{+}(a-c-1)\beta 6-\beta 7-(a-2)\beta 8^{+}(c+2)\beta 9)P^{\infty}_{\text{equal}-\text{zero}i}\beta 8i=0$$

and

$$\omega(c+2,0,0,a-c-1,0,a-2,1,0),c+1.\eta=\omega(a-2,1,0,c+2,0,0,a-c-1,0).\eta \qquad -0.1$$

is a common point of the smallest tile T_η and $T_\eta-\Phi(\beta^{-1})$.

$\diamond every^{Ifb=} minus - a + 1 and conjugate c = of^{0,}_{\beta is}{}^{we} have^{c+}_{alsoaroot} of^{b} \leq^{-2}_{p(x)} {}_{(x} and^{the\ smallest}_{2+x+1)} {}_{P^{\infty}_{i=0}x} tile^{is}_{3i=0,} {}^{T_\eta for}_{then} \eta = 1^{\omega}. Sinc$

$1 + \beta 1 - (a-2)\beta 2^{+2}\beta 3^{+} P^{\infty}_{i=4}\beta i = ($

and all the coefficients have absolute value les s than $\lfloor \beta \rfloor$. $*$ If $c = a - 2$, we have $-a + 2 \leq b \leq -1, \lfloor \beta \rfloor = a - 1$ and

$$d_\beta(1) = .a - 1, a + b, a + b - 2, a - 1, 1.$$

 To show that one of the tiles is not connected , according to Lemma 3 on page 300 , it is enough

to prove that $p(\gamma) > 0$. Since $\gamma^2 - (a-1)\gamma - 1 = 0$, we have

$$p(\gamma) \geq \gamma^4 - \gamma_a^3 + \gamma^2 - (a-1)\gamma - 1 = \gamma^2(1-\gamma) > 0.$$

• If $0 \leq b \leq a$, we have $\lfloor \beta \rfloor = a$ and

$$d_\beta(1) = .a, b, c, 1.$$

• If $a+1 \leq b \leq 2a-2$, we have $a \geq 3, 2 \leq c \leq a-1$ and $1+a \leq b \leq a+c-1. \lfloor \beta \rfloor = a+1$ and

$$d_\beta(1) = .a+1, minus - a_{-1,} ca - plus - b, b - plus 1 - c, c-1, 1.$$

$Case^3_-$. If $a \leq c \leq a+3$, we have $a \geq 1$ and $\frac{1-a^2+c^2}{4} \leq b \leq a+c-1$. • If $c = a$, we get $1 \leq b \leq 2a-1$ and

$$d_\beta(1) = \jmath \quad \begin{array}{l} .a, b, 1, \quad if b \leq a; \\ .a+1, b - minus_{a-1,} 2a - b, minus - b_{a+1,} a-1, 1, \quad if b > a. \end{array}$$

• If $c = a+1$, we get $\frac{a+1}{2} \leq b \leq 2a$ and

$$d_\beta(1) = less - colon1) \quad \begin{array}{l} \omega^{1,} \\ , \; .a{+}1{,}0{,}0{,}(0{,}a{,}0{,}0{,}a{,}a{,}^{.a+1,b-a-1,2a-b+1,b-a,a,0}_{.a,b+1,(0,a-b,b,b,a-b+1,}{,}b)^\omega \end{array} , \quad if^{if}_{if} b^b_b \geq^{\leq =} a;^{a+\,1;}_{a-\,1.}$$

$*Forbpage \leq 300,^a it^-$ is $1, toshowenough$ to $prove^{that1}$ that $of^{the}_{p(\gamma)} tiles^{is}_{>0.Since}{}^{not} connected, 2_\gamma - a\gamma - 1 according^{to}_{=0 we have}$ Lemma 3 on

$$(\gamma) \geq \gamma^4 - a\gamma^3 - (a-1)\gamma^2 - (a+1)\gamma - 1 = \gamma^2(1-\gamma) > 0.$$

$*$ For $b = a$, since eve $r - y$ conjugate of β is also a root of $p(x)(x-1)P^\infty_{i=0} x^{3i} = 0$, we have

$$1 + a\beta 1 - \beta 2^+ \text{parenleft} - \text{beta3} - \beta 4) P^\infty_{i=0} \beta 3i = 0$$

and all the coe $f - fi$ cients have absolute value less than $\lfloor \beta \rfloor = a+1$. • If $c = a+2$, we get $a+2 \leq b \leq 2a+1, \lfloor \beta \rfloor = a+1$ and

$$d_\beta(1) = .a+1, b-a-1, 2a-b+2, b-a-1, a+1, 1.$$

• If $c = a+3$, we get $a+2+\frac{a+1}{2} \leq b \leq 2a+2, \lfloor \beta \rfloor = a+1$ and

$$d_\beta(1) = .a+1, b-a-1, (2a-b-plus3, b-a-1, 0, 2a - \text{plus} - \text{b}3, 2b-3a-5, 4a-2b-plus6, 2b-3a-4, 2a - \text{plus}$$

To show that one of the tiles is not connected , according to Lemma 3 on page 300 , it is enough to prove that $p(\gamma) > 0$. Since $\gamma^2 - (a+1)\gamma - 1 = 0$ we have that

$$p(\gamma) \geq \gamma^4 - a\gamma^3 - (2a+2)\gamma^2 - (a+3)\gamma - 1 = -\gamma^3 > 0.$$

$a50$

From the proof of this theorem we can easily see that $a+c-2\lfloor\beta\rfloor=1$ for the cases when at least one of the tiles is disconnected and $a+c-2\lfloor\beta\rfloor\leq 0$ for the cases when each tile is connected . So , the above theorem can be written in the following equivalent way :

Theorem 4 . 8 *Let* β *be a Pisot unit ofdegree* 4 *with minimal polynomial* $p(x)=x^4-ax^3-bx^2-cx-1.$ *Then* $a+c-2\lfloor\beta\rfloor\leq 1,$ *and each tile is arcwise connected if and only if* $a+c-2\lfloor\beta\rfloor\leq 0.$

In [14] , Canterini gave an interesting example of GIFS substi $t-u$ tive tiles that the union $\bigcup_i K_i$ in (4 . 2)
is connected although each K_i is disconnected . In our setting , $\bigcup_i K_i$ co $r-r$ esponds to the central tile T_λ.
As the proof of disconnectedness relies on Lemma 3 on page 300 , the readers see that T_λ is disconnected provided there exists a disconnected tile and $d_{-1}>d_{-2}$. After submission of this paper , we could fu $r-t$ her show that *all* the tiles $a-r$ e disconnected , provided there *exists* a disconnected tile . As this paper is already of this length , this fact will be published elsewhere . Therefore we can not find examples like Canterini ' s among quartic Pisot dual tiles .

Finally from the proof of Theorem 4 . 7 on page 301 we extract the following theorem which gives the $\beta-$
expansion of 1 for any Pisot unit of degree four with the minimal polynomial $x^4-ax^3-bx^2-cx-1=0.$

Theorem 4 . 9 *Let* β *be a Pisot unit of degree four with minimal polynomial* $p(x)=x^4-ax^3-bx^2-c-x-1=0.$ *Then the* $\beta-$ *expansion of* 1 *is :*

$$- \quad When-a+1\leq c\leq -1,$$

$$\diamond\ for\ \ b\leq 0\ we\ have$$

$$d_\beta(1)=\begin{cases} .a-1,a+b-1,a+b+c-1,(a+b+c)^\omega, & for(c,b)\neq(-1,0);\\ .a-1,a-1,a-1,0,0,1, \quad for(c,b)=(-1,0); & \end{cases}$$

$$\diamond\ for\ \ b\geq 1\ we\ have\ \ d_\beta(1)=.a,b-1,(a+c,b)\ \ \overset{\omega}{.}$$

$$- \quad When 0\leq c\leq a,$$

$\diamond$ *for* $b\leq minus-a$ *and* $c\leq a-4,$ *le t k be the integer of* $\{2,3,\cdots,a-2\}$ *with* $(k-1)(a+b+c)<$

$$a-2\leq k(a+b+c).$$

$$*\ If\ \ (k-1)(a+b+c)\geq c+2,\ let\ \ m=\inf\{i\in\mathbb{N}\ such\ that\ \ (i+1)(a+b+c)\geq c+2\}.$$

$$m=1\Rightarrow d_\beta(1)=.a-2,\mathrm{two}-\mathrm{a}_+b-2,a-two_+^{\mathrm{two-b}}{}_+c-2,a-two_+^{\mathrm{two-b}}{}_+c-tu$$
$$m=2\Rightarrow d_\beta(1)=.a-2,\mathrm{two}-\mathrm{a}_+b-3,a-three_+^{\mathrm{two-b}}{}_+c-3,a-three_+^{\mathrm{three-b}}{}_+c-two_-2,\mathrm{three}-\mathrm{a}_+^{b-three}{}_+\mathrm{three}-\mathrm{c}_-1$$
$$m\geq 3\Rightarrow d_\beta(1)=.a-2,\mathrm{two}-\mathrm{a}_+b-3,a-three_+^{\mathrm{two-b}}{}_+$$

$$with \;\; d_i = d_{i-1} + a + b + c \; for \;\; 4 \le i \le m \;\; and$$

$$braceex - braceex - braceleftmid - braceex - braceex - braceleftbt \quad \begin{array}{l} d_{m+1} = d_m + a + b + c + 1, \\ d_{m+2} = d_{m+1} + b + c + 1, \\ d_{m+3} = d_{m+2} + c + 1, \\ d_{m+4} = (m+1)(a+b+c). \end{array} \quad *$$

$$If \;\; (k-1)(a+b+c) \le c + 1 \;\; and \;\; k(a+b+c) = a - 2 \;\; we \; have$$

$$k = 2 \Rightarrow d_\beta(1) = .a-2, 2a+b-2, 2a+2b+c-2, a-2, 0, 2c+b+2, c+2, 1,$$

$$k \ge 3 \Rightarrow d_\beta(1) = .a-2, 2a+b-3, d_3, \cdots, d_{2k+1}, 2c+b+3, c+2, 1 \qquad such that d_i =$$

 $ia+(i-1)b+(i-2)c-4$ *for* $3 \leq i \leq k-1$, (*these terms do not appearfor* $k=3$)

$$d_k = ka+(k-1)b+(k-2)c-3, \quad d_{k+1} = ka+kb+(k-1)c-2,$$
$$d_{k+2} = a-2, \quad d_{k+3} = 0, \quad d_{k+4} = 1-b-a,$$

$d_{2k+2} minus - i = ia+(i+1)b+(i+2)c+4$ *for* $1 \leq i \leq k-3$, (*these terms do not appearfor* $k=3$).
$*$ *If* $(k-1)(a+b+c) \leq c+1$ *and* $k(a+b+c) > a-2$ *we have*

$$k=2 \Rightarrow d_\beta(1) = .a-2, 2a+b-2, 2a+2b+c-1, a+2b+2c+1, 2c+b+2, c+2, 1, k \geq 3 \Rightarrow$$
$$d_\beta(1) = .a-2, 2a+b-3, d_3, \cdots, d_{2k}, 2c+b+3, c+2, 1 such that$$

$$d_i = ia+(i-1)b+(i-2)c-4 \text{ for } 3 \leq i \leq k-1, \quad (\text{ these terms do not appearfor } k=3)$$

$$d_k = ka+(k-1)b+(k-2)c-3, \quad d_{k+1} = ka+kb+(k-1)c-1,$$
$$d_{k+2} = (k-1)a+kb+kc+1, \quad d_{k+3} = (k-2)a+(k-1)b+kc+3,$$

$d_{2k+1 minus - i} = ia+(i+1)b+(i+2)c+4$ *for* $1 \leq i \leq k-3$, (*these terms do not appearfor* $k=3$),
$\diamond$ *for* $b \leq -a$ *and* $c = a-3$ *we have*

$$d_\beta(1) = .a-2, \mathrm{two-a}_+ b-2, (^{a-three} +^{\mathrm{two-b}} -4, a - three^{\mathrm{two-b}}_{+} \_5, a-two_+ b-3, 0, 1-a-b, 2-a-b, 0, \mathrm{two-a}_+ b-$$

$\diamond$ *for* $-a < b \leq -1$ *and* $c \leq a-3$ *we have*

$$d_\beta(1) = less-colon.a^{\cdot a}_{\cdot a} -^{-}_{-} 1^{1}_{1},, a +^{a+}_{a+} a-1, 0, c+1, 1,^{+b+c-1,(a+}_{b-1,^{b-1,}_{b,b+}{}^{c}_{a},c+1,1,} b+c)^\omega \quad for^{for}_{for} b +^{b+}_{b+} c^c_c \leq^{\geq =} -1;^{-2;}_{0;}$$

$\diamond$ *for* $-a < b \leq -1$ *and* $c = a-2$ *we have* $d_\beta(1) = .a-1, a+b, a+b-2, a-1, 1,$

$\diamond$ *for* $minus{-}a < b \leq -1$ *and* $c = a{-}1$ *we have* $d_\beta(1) = .a-1, a+b, (a+b, 0,^{b-minus}, 0, a+b-1)$ ω ,

$\diamond$ *for* $0 \leq b \leq a$ *we get* $d_\beta(1) = .a, b, c, 1,$
$\diamond$ *for* $b \geq a+1$ *we get* $d_\beta(1) = .a+1, b-a-1, c+a-b, b-c+1, c-1, 1.$
\- *When* $a+1 \leq c \leq a+3$ *we have*

$$\omega$$
$$\diamond forc = a{+}1, d_\beta(1) = a{+}1, 0, 0, (0, a, 0, 0, a, a, 1)^{\omega+1, minus-b_a-1, a-two\_b-plus1, minus-b_a, a, 1,}_{\cdot\{\cdot{}^{a}_{a}, b+1, (0, aminus-b, b, b, a-b-plus1, 0\quad ,,} {}^{b)}, \quad for^{for}_{for} b^b_b \geq^{=} a;^{a+1}_{\leq a-1};;$$

$\diamond$ *for* $c = a+2$ *we have* $d_\beta(1) = .a+1, b-a-1, 2a-b+2, b-a-1, a+1, 1,$ $\diamond$ *for* $c = a+3$ *we have*

$$d_\beta(1) = .a+1, b - minus_{a-1,}(^{a-two} - b - plus3, b-a-1, 0, a - two\_b - plus3, b - two_{-\mathrm{three-a\_5},} a - four\_\mathrm{two-b-p}$$

Example 3 *Here we want to show that , from a class of Pisot units of degree* 4 *which are roots of the polynomial* $x^4 - ax^3 - bx^2 - cx - 1 = 0$, *we can obtain an arbitrarily long* $\beta-$ *expansion of* 1 . *For* $n \geq 3$, $a = n+2$, *and* $c = a-4 = n-2b = 1-a-c = 1-2n$ *we have that* $\lfloor \beta \rfloor = a-2 = n$ *and the* $\beta-$ *expansion of* 1 *is*

$$d_\beta(1) = .n, 2,_{braceleft-z} 2, 3, \cdots, n-3, n-2, n, 0, n, 0, n-2,_{z-braceleft} n-2, n-3, \cdots, 3, 2, 0, n, 0, 1.$$

$n-3$ e l ements $n-3$ e lements *Hence the length of the* $\beta-$ *expansion of* 1 *is* $2n+4.$

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this paper . On the way of revision , we completely reorganized Section 2 . It becomes shorter and $r-i$ cher as
a result of the $i-r$ const $u-r$ ctive criticism .

 Fig .2 : $\beta-$ expansion of 1 for $x^4-ax^3-bx^2-cx+1=0.$
The $leng^{t-h}$ i s not fixed in the shaded box .

 Fig .3 : $\beta-$ expansion of 1 for $x^4 - ax^3 - bx^2 - cx - 1 = 0$. $T - h_e$ length i s not fixed in $h - t_e$ shaded box . Four
di sconnected cases are indicated .

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