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ON A NONLINEAR INTEGRAL EQUATION WITHOUT COMPACTNESS

F . ISAIA

Abstract . The purpose of this paper is to obtain an existence result for the integral equation

$$u(t) = \varphi(t, u(t)) + \int_a^b \psi(t, s, u(s))ds, \quad t \in [a, b]$$

where $\varphi : [a, b] \times \mathbb{R} \to \mathbb{R}$ and $\psi : [a, b] \times [a, b] \times \mathbb{R} \to \mathbb{R}$ are continuous functions which satisfy some special growth conditions . The main idea is to transform the integral equation into a fixed point problem for a condensing map $T : C[a, b] \to C[a, b]$. The " a priori estimate method " (which is a consequence of the invariance under homotopy of the degree defined for $\alpha-$ condensing perturbations of the identity) is used in order to prove the existence of fixed points for T. Note that the assump - tions on functions φ and ψ do not generally assure the compactness of operator T, therefore the Leray - Schauder degree cannot be used (see K . Deimling [**2** , Example 9 . 1 , p . 69]) .

1 . Introduction

The topological methods proved to be a powerful tool in the study of various problems which appear in nonlinear analysis . Particularly , the a priori estimate method (or the method of a priori bounds) has been often used in order to prove the existence of solutions for some boundary value problems for nonlinear differen - tial equations or nonlinear partial differential equations . For example , J . Mawhin uses this method together with the coincidence degree and shows that under ap - propriate assumptions , the boundary value problem

$$\begin{cases} -x''(t) = f(t, x(t), x'(t)), \quad t \in [0, \pi] \\ x(0) = x(\pi) = 0 \end{cases}$$

and the problem

$$\begin{cases} x'(t) = f(t, x(t)), \quad t \in [0, 1] \\ x(0) = x(1) \end{cases}$$

admit solutions (see J . Mawhin [**6** , Sections V . 2 and VI . 2]) . This method is also used (but together with the Leray - Schauder degree) in G . Dinca , P . Jebelean [**3**]

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 and G . Dinca , P . Jebelean , J . Mawhin [**4**] to prove the existence of solutions for
the problem

$$\begin{cases} -\Delta_p u & = f(t,u) \quad \text{in}\Omega \\ u \mid \partial\Omega & = 0. \end{cases}$$

In the present paper , the a priori estimate method is used together with the degree for condensing maps in order to prove the existence of solutions for the integral equation

$$u(t) = \varphi(t, u(t)) + \int_a^b \psi(t, s, u(s))ds, \quad t \in [a, b], \tag{1}$$

under appropriate assumptions on functions φ and ψ. The result presented herein is in relation with a result of F . Isaia [**5**] . The hypothesis imposed on functions φ and ψ are stronger (and considerably simpler) , but the result is stronger as well , namely the solution u of equation (1) is in $C[a, b]$, while in F . Isaia [**5**] , we obtained

$$u \in L^p(a, b).$$

2 . THE TOPOLOGICAL DEGREE FOR CONDENSING MAPS For a minute description of the following notions we refer the reader to K . Deimling

[2].

In the following , X will be a Banach space and $\mathcal{B} \subset \mathcal{P}(X)$ will be the family of all it s bounded sets .

Definition 1 . The function $\alpha : \mathcal{B} \to \mathbb{R}_+$ defined by
$\alpha(B) = \inf \{d > 0 : B$ admits a finite cover by sets of diameter $\leq d\}$, $B \in \mathcal{B}$, is called the (Kuratowski -) measure of noncompactness .

In the whole paper , the letter α will only be used in this context . We state without proof some properties of this measure .

Proposition 1 . *The fo l lowing ass ertions hold :*

(a) $\alpha(B) = 0$ *iff* B *is re latively compact .*

(b) α *is a s eminorm , i . e .*

$$\alpha(\lambda B) =\mid \lambda \mid \alpha(B) \quad \textit{and} \quad \alpha(B_1 + B_2) \leq \alpha(B_1) + \alpha(B_2).$$

(c) $B_1 \subset B_2 implies \alpha(B_1) \leq \alpha(B_2);$

$\alpha(B_1 \cup B_2) = \max\{\alpha(B_1), \alpha(B_2)\}.$

(d) $\alpha(convB) = \alpha(B).$

(e) $\alpha(\overline{}B) = \alpha(B).$

Definition 2 . Consider $\Omega \subset X$ and $F : \Omega \to X$ a continuous bounded map . We say that F is $\alpha-$ Lipschitz if there exists $k \geq 0$ such that

$$\alpha(F(B)) \leq k\alpha(B) \quad (\forall) B \subset \Omega \text{bounded}.$$

 If , in addition , $k < 1$, then we say that F is a strict $\alpha-$ contraction . We say that F is $\alpha-$ condensing if

$$\alpha(F(B)) < \alpha(B) \quad (\forall)B \subset \Omega \text{bounded with} \alpha(B) > 0.$$

In other words , $\alpha(F(B)) \geq \alpha(B)$ implies $\alpha(B) = 0$. The class of all strict $\alpha-$ contractions $F : \Omega \to X$ is denoted by $SC_\alpha(\Omega)$ and the class of all $\alpha-$ condensing maps $F : \Omega \to X$ is denoted by $C_\alpha(\Omega)$.

We remark that $SC_\alpha(\Omega) \subset C_\alpha(\Omega)$ and every $F \in C_\alpha(\Omega)$ is $\alpha-$ Lipschitz with constant $k = 1$. We also recall that $F : \Omega \to X$ is Lipschitz if there exists $k > 0$ such that

$$\| Fx - Fy \| \leq k \| x - y \| \quad (\forall)x, y \in \Omega$$

and that F is a strict contraction if $k < 1$.

Next , we state without proof some properties of the applications defined above .

Proposition 2 . *If $F, G : \Omega \to X$ are $\alpha-$ Lipschitz maps with constants k, respectively k', then $F + G : \Omega \to X$ is $\alpha-$ Lipschitz with constant $k + k'$.*

Proposition 3 . *If $F : \Omega \to X$ is compact , then F is $\alpha-$ Lipschitz with constant*

$$k = 0.$$

Proposition 4 . *If $F : \Omega \to X$ is Lipschitz with constant k, then F is $\alpha-$ Lipschitz with the same constant k.*

The theorem below asserts the existence and the basic properties of the topo - logical degree for $\alpha-$ condensing perturbations of the identity .

Let

$$\mathcal{T} = \left\{ \begin{array}{ll} (I - F, \Omega, y) : & \Omega \subset X \text{open and bounded}, \\ F \in C_\alpha(\overline{\Omega}), & y \in X \setminus (I - F)(\partial\Omega) \end{array} \right\}$$

be the family of the admissible triplets . There exists one degree function $D : \mathcal{T} \to \mathbb{Z}$ which satisfies the properties :

Theorem 1 . (D 1) $D(I, \Omega, y) = 1$ *for every* $y \in \Omega$(*Normalization*) .

(D 2) *For every disjoint , open s e ts* $\Omega_1, \Omega_2 \subset \Omega$ *and every* $y element - slash (I - F) line - parenleftbig \Omega \setminus (\Omega_1 \cup \Omega_2))$ *we have*

$$D(I - F, \Omega, y) = D(I - F, \Omega_1, y) + D(I - F, \Omega_2, y)$$

(Additivity on domain) . (D 3) $D(I - H(t, \cdot), \Omega, y(t))$ *is independent of* $t \in [0, 1]$ *for every continuous , bounded map* $H : [0, 1] \times \overline{\Omega} \to X$ *which satisfies*

$$\alpha(H([0, 1] \times B)) < \alpha(B) \quad (\forall)B \subset \overline{\Omega} \text{ with } \alpha(B) > 0$$

and every continuous function $y : [0, 1] \to X$ *which satisfies*

$$y(t) \neq x - H(t, x) \quad (\forall)t \in [0, 1], (\forall)x \in \partial\Omega$$

(Invariance under homotopy) . (D 4) $D(I-F,\Omega,y)\neq 0$ *implies* $y\in(I-F)(\Omega)$ (*Existence) .*

(D 5) $D(I-F,\Omega,y) = D(I-F,\Omega_1,y)$ *for every open s e t* $\Omega_1 \subset \Omega$ *and every*

$$y element-slash(I-F)(\overline{\Omega}\setminus\Omega_1) \quad (Excision).$$

Having in hand a degree function defined on $\mathcal{T}$, we study the usability of the " a priori estimate method " by means of this degree .

Theorem 2 . *Let* $F : X \to X$ *be* $\alpha-$ *condensing and*

$$S = \{x \in X : (\exists)\lambda \in [0,1] \quad such that x = \lambda Fx\}.$$

If S *is a bounded s e t in* X, *s o there exists* $r > 0$ *such that* $S \subset B_r(0)$, *th en*

$$D(I-\lambda F, B_r(0), 0) = 1 \quad (\forall)\lambda \in [0,1].$$

Consequently , F *has at least one fixed po int and the se t of th e fixed points of* F *lies in* $B_r(0)$.

Proof . First , we remark that every affine homotopy of $\alpha-$ condensing maps is an admissible homotopy . To see this , let us consider a bounded open set $\Omega \subset X$, the maps $F_1, F_2 \in C_\alpha(\overline{\Omega})$ and let $H : [0,1]\times\overline{\Omega} \to X$ be defined by

$$H(t,x) = (1-t)F_1x + tF_2x.$$

For every $B \subset \overline{\Omega}$ with $\alpha(B) > 0$ we have

$$H([0,1]\times B) \subset \operatorname{conv}(F_1(B)\cup F_2(B))$$

and , using Proposition 1 ,

$$\begin{aligned} \alpha(H([0,1]\times B)) &\leq \alpha(\operatorname{conv}(F_1(B)\cup F_2(B))) \\ &= \alpha(F_1(B)\cup F_2(B)) \\ &= \max\{\alpha(F_1(B)), \alpha(F_2(B))\} < \alpha(B). \end{aligned}$$

Next , we fix $\lambda \in [0,1]$ and we consider the affine homotopy between the $\alpha-$

$$condensing maps \lambda F, 0 \in C_\alpha(X)$$

$$H : [0,1]\times X \to X, \quad H(t,x) = (1-t)0x + t\lambda Fx = t\lambda Fx.$$

By the previous argument ,

$$\alpha(H([0,1]\times B)) < \alpha(B) \quad (\forall)B \subset X bounded with \alpha(B) > 0.$$

If $x \in X$ and $t \in [0,1]$ verify $x - H(t,x) = 0$, then $x \in S \subset B_r(0)$. Thus , we can use the properties (D 3) , (D 1) of the degree and we obtain

$$\begin{aligned} D(I-\lambda F, B_r(0), 0) &= D(I-H(1,\cdot), B_r(0), 0) \\ &= D(I-H(0,\cdot), B_r(0), 0) \\ &= D(I, B_r(0), 0) = 1. \end{aligned}$$

Finally , the property (D 4) of the degree is used . □

 3 .
THE EXISTENCE RESULT
Consider equation (1)

$$u(t) = \varphi(t, u(t)) + \int_a^b \psi(t, s, u(s))ds, \quad t \in [a, b],$$

where $\varphi : [a, b] \times \mathbb{R} \to \mathbb{R}$ and $\psi : [a, b] \times [a, b] \times \mathbb{R} \to \mathbb{R}$ are continuous functions which satisfy the following conditions :

(a) There exist $C_1, M_1 \geq 0, \quad q1 \in [0, 1)$ such that

$$| \varphi(t, x) | \leq C_1 \mid x |^{q1} + M_1$$
$$\text{forevery}(t, x) \in [a, b] \times \mathbb{R}.$$

(b) There exists $K_1 \in [0, 1)$ such that

$$| \varphi(t, x) - \varphi(t, y) | \leq K_1 \mid x - y |$$
$$\text{forevery}(t, x), (t, y) \in [a, b] \times \mathbb{R}.$$

(c) There exist $C_2, M_2 \geq 0, \quad q2 \in [0, 1)$ such that

$$| \psi(t, s, x) | \leq C_2 \mid x |^{q2} + M_2$$
$$\text{forevery}(t, s, x) \in [a, b] \times [a, b] \times \mathbb{R}.$$

Under these assumptions , we will show that equation (1) has at least one solu -

$$\text{tion} u \in C[a, b].$$

Define operators

$$F \quad : \quad C[a, b] \to C[a, b], \quad (Fu)(t) = \varphi(t, u(t)), \quad t \in [a, b],$$
$$G \quad : \quad C[a, b] \to C[a, b], \quad (Gu)(t) = \int_a^b \psi(t, s, u(s))ds, \quad t \in [a, b],$$
$$T \quad : \quad C[a, b] \to C[a, b], \quad Tu = Fu + Gu.$$

Then , equation (1) can be written as

$$u = Tu. \tag{2}$$

Thus , the existence of a solution for equation (1) is equivalent to the existence of a fixed point for operator T.

Proposition 5 . *The operator* $F : C[a, b] \to C[a, b]$ *is Lipschitz with constant* K_1. *Consequently* F *is* $\alpha-$ *Lipschitz with the same constant* K_1.

Proof . From (b) , we have

$$\begin{aligned} \| Fu - Fv \|_{C[a,b]} &= \sup_{\in_t [a,b]} | (Fu)(t) - (Fv)(t) | \\ &= \sup \quad | \varphi(t, u(t)) - \varphi(t, v(t)) | \\ & \qquad t \in [a, b] \\ &\leq K_1 \sup_{\in_t [a,b]} | u(t) - v(t) | = K_1 \quad \| u - v \|_{C[a,b]}, \end{aligned}$$

for every $u, v \in C[a, b]$. By Proposition 4, F is $\alpha-$ Lipschitz with constant K_1.

Moreover , F satisfies the following growth condition :

$$\| Fu \|_{C[a,b]} \leq C_1 \| u \|^{q1}_{C[a,b]} + M_1, \tag{3}$$

for every $u \in C[a,b]$. Relation (3) is a simple consequence of condition (a). $\square$

Proposition 6 . *The operator* $G : C[a,b] \to C[a,b]$ *is compact . Consequently* G *is* $\alpha-$ *Lipschitz with zero constant .*

Proof . First , we prove the continuity of G. Let $(u_n) \subset C[a,b], u \in C[a,b]$ be such that $\| u_n - u \| C[a,b] \to 0$. We have to show that $\| Gu_n - Gu \| C[a,b] \to 0$. Fix $\varepsilon > 0$. There exists a constant $K \geq 0$ such that

$$\| u_n \|_{C[a,b]} \leq K \quad (\forall) n \in \mathbb{N}^*,$$
$$\| u \|_{C[a,b]} \leq K.$$

Using the uniform continuity of ψ on $[a,b] \times [a,b] \times [-K,K]$, we derive that there exists $\delta = \delta(\varepsilon) > 0$ such that

$$| \psi(t_1, s_1, x_1) - \psi(t_2, s_2, x_2) | \leq \frac{\varepsilon}{b-a},$$

for every $(t_1, s_1, x_1), (t_2, s_2, x_2) \in [a,b] \times [a,b] \times [-K,K]$ such that $| t_1 - t_2 | + | s_1 - s_2 | + | x_1 - x_2 | < \delta$. From $\| u_n - u \| C[a,b] \to 0$, it follows that there exists

$$N = N(\varepsilon) \in \mathbb{N}^* \text{suchthat}$$
$$\sup_{t \in [a,b]} | u_n(t) - u(t) | < \delta,$$

for every $n \geq N$. Consequently ,

$$\begin{aligned} \| Gu_n - Gu \|_{C[a,b]} &= \sup_{\in_t [a,b]} |\int_a^b \psi(t,s,u_n(s))ds - \int_a^b \psi(t,s,u(s))ds| \\ &\leq \sup_{\in_t [a,b]} \int_a^b | \psi(t,s,u_n(s)) - \psi(t,s,u(s)) | ds < \varepsilon, \end{aligned}$$

for every $n \geq N$. The continuity of G is proved .

Moreover , G satisfies the following growth condition :

$$\| Gu \|_{C[a,b]} \leq C_2(b-a) \| u \| q2C_{[a,b]} + (b-a)M_2, \tag{4}$$

for every $u \in C[a,b]$. Relation (4) is a simple consequence of condition (c) .

In order to prove the compactness of G, we consider a bounded set $M \subset C[a,b]$ and we will show that $G(M)$ is relatively compact in $C[a,b]$ with the help of Arzela - Ascoli theorem . Let $\text{-}K \geq 0$ be such that

$$\| u \|_{C[a,b]} \leq \text{-}K,$$

for every $u \in M$. By (4) , we have

$$\| Gu \| C[a,b] \leq (b-a)[C_2 \overline{\quad} K^{q2} + M_2],$$

for every $u \in M$, so $G(M)$ is bounded in $C[a,b]$. Fix $\varepsilon > 0$. Using the uniform continuity of ψ on $[a,b] \times [a,b] \times [\overline{\ \ \ \ }K, vlineK$, we derive that there exists $\delta = \delta(\varepsilon) > 0$ such that

$$| \psi(t_1, s_1, x_1) - \psi(t_2, s_2, x_2) | \leq \frac{\varepsilon}{b-a},$$

for every $(t_1, s_1, x_1), (t_2, s_2, x_2) \in [a,b] \times [a,b] \times [\overline{\ \ \ \ }K, \underline{\ \ \ \ }_K]$ such that $| t_1 - t_2 | + | s_1 - s_2 | + | x_1 - x_2 | < \delta$. If $t_1, t_2 \in [a,b]$ satisfy $| t_1 - t_2 | < \delta$, then

$$| (Gu)(t_1) - (Gu)(t_2) | \leq \int_a^b | \psi(t_1, s, u(s)) - \psi(t_2, s, u(s)) | \, ds < \varepsilon,$$

for every $u \in M$. The set $G(M) \subset C[a,b]$ satisfies the hypothesis of Arzela - Ascoli theorem , so $G(M)$ is relatively compact in $C[a,b]$.

By Proposition 3, G is $\alpha-$ Lipschitz with zero constant . □

Now , we have the possibility to prove the main result of this paper .

Theorem 3 . *If the functions* $\varphi : [a,b] \times \mathbb{R} \to \mathbb{R}$ *and* $\psi : [a,b] \times [a,b] \times \mathbb{R} \to \mathbb{R}$ *satisfy the conditions (a) , (b) , (c) , then the integral equation*

$$u(t) = \varphi(t, u(t)) + \int_a^b \psi(t, s, u(s)) ds, \quad t \in [a,b],$$

has at least one s o lution $u \in C[a,b]$ *and th e s e t of the s o lutions of equation* (1) *is*

$$boundedinC[a,b].$$

Proof . Let $F, G, T : C[a,b] \to C[a,b]$ be the operators defined in the beginning of this section . They are continuous and bounded . Moreover , F is $\alpha-$ Lipschitz with constant $K_1 \in [0,1)$ and G is $\alpha-$ Lipschitz with zero constant (see Propositions 5 and 6) . Proposition 2 shows us that T is a strict $\alpha-$ contraction with constant K_1. Set

$$S = \{u \in C[a,b] : (\exists)\lambda \in [0,1] \quad such that u = \lambda Tu\}.$$

Next , we prove that S is bounded in $C[a,b]$. Consider $u \in S$ and $\lambda \in [0,1]$ such that $u = \lambda Tu$. It follows from (3) and (4) that

$$\begin{aligned} \| u \| C[a,b] &= \lambda \| Tu \|_{C[a,b]} \leq \lambda(\| Fu \|_{C[a,b]} + \| Gu \|_{C[a,b]}) \\ &\leq \lambda[C_1 \| u \|_{C[a,b]}^{q1} + C_2(b-a) \| u \| q2C_{[a,b]} + M_1 + (b-a)M_2]. \end{aligned}$$

This inequality , together with $q1 < 1, \quad q2 < 1$, shows us that S is bounded in

$$C[a,b].$$

Consequently , by Theorem 2 we deduce that T has at least one fixed point and the set of the fixed points of T is bounded in $C[a,b]$. □

Remark 1 .

(i) if the growth condition (a) is formulated for $q1 = 1$, then the conclusions of Theorem 3 remain valid provided that $C_1 < 1$;

(ii) if the growth condition (c) is formulated for $q2 = 1$, then the conclusions of

Theorem 3 remain valid provided that $(b - a)C_2 < 1$;

(i ii) if the growth conditions (a) and (c) are formulated for $q1 = 1$ and $q2 = 1$,

then the conclusions of Theorem 3 remain valid provided that

$$C_1 + (b - a)C_2 < 1.$$

Remark 2 . The conclusions of Theorem 3 remain valid provided that equation (1) is replaced by

$$u(t) = \varphi(t, u(t)) + \int_a^t \psi(t, s, u(s))ds, \quad t \in [a, b].$$

Only slight modifications in the proof of Proposition 6 are needed .

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F . Isaia , “ Transilvania ” University of Brasov , Faculty of Mathematics and Computer Science , Department of Equations , Iuliu Maniu 50 , 50009 1 Brasov , Romania ,

e - mail : isaiaflorin@yahoo.com