$e-t$

alra $t-i$ o a numb $r-e$ s h aving (w $r-i$ t

naor . An umber $p/q \in$ $\mathbb{Q}bracketleft-parenleft)$] with q o d d $s-i$ con s

n $a_{te_{db}}y_x$ $0^{\in}$ $\mathbb{Q}$ $2_{)]}$\{ d n e d y

$g0(n-1)$ i f $xn1$ s ve ,

( ) $xn=$ $gone-parenleftxn-1)$ if xn $-$ is od d ,

fo n $\in \mathbb{N}$. T he t uctr o fth s $e-t$ of C $olla$ z cyl s (. ep eriod c C la

eq uences) in $\mathbb{Q}[2)]hs^u$ b e_e n tud edb yLaga r i sin $6.i_O$ co us e i f x o $\in \mathbb{N}$ t s

henthe g $e-n$ er t d s e q u e n cisthe us

C t $a_W r^n a-c-lCllth^{ol_{at}z}_{a^t \; th}$ $qu_C e_{l-o^{ce}_{at\,z}}$ s $h_c nj^{a-v}_{c_t}$ ei_m

(C f_o r $a-l_l x0 \in \mathbb{N}$ th re exi tsa ni $d-n_e$ x n w $i-t^h$ x $n=_1$,

isstllo p en O f c o rse (C) s q uia en ttoth e cn ju n ti ono f (A)

an d B)

i . u i e v l o c (

A) $(_1 2)_{is}$ t he o ny C $o_l la_t z_c$ yc l^{en} $\mathbb{N}$.

(B)E^{v-e} ry Co $a-l$ t seque ncei n $\mathbb{N}i-s$ b oun ded .

sa m a n r e_f e c ea bo u t_h e C $llat_{zp}$ r b em w_e r f_e

r e^c en t r^{es} ts see $[1][^2,4[],[^{8'}[9_{][6]}[3]_a$ nd $[^1 0]$ P_{ar} a r^e s lt a r^e

ee [ ]) . h e C l a_t zc $o_n e_c tu_{re}$ h $o^{ld_s} tu$ e

(E) The ength f aC ola tz y ce in $\mathbb{N}$ w h

le $a-s1707915$(se $[3parenright-bracketright.$

T he i m oft h i s art i le i s t o p r e sen t so men e w t chniqu eswh ic hal l

fin e dan $l-y$ sis of ra $i-t$ o nal (a nd he ncei teg er) Co $l-l$ at zc yce

$period-s_I$ n p_a r i c u l a

we pr o_v

a 0 $\leq_2$ 12 36 603280 $1_{1(t^{hi}}$ s n m b
$o_i gnal$ $r - i_e$ io n wo^u d r_{eq} re o ch e ka lli_n iial v lu e^s u p t o $2_{.9} \cdot$ 1^0
$w_h i^c$ h a b ut 46_{ti} m e_{sl} r g_e).

2 . A ux $\mathbf{i_{lia} y_r}$ e − s u t s . Le tus sta t w t h so men $o_{ta} io$ n $:_{Le}$ t $S_l n$ d eno

t $\bigcup h - l^e$ et o al_l $01_s \bigcup q_u nce^s$ o f $e - l_n$ g h l co n t an n ge $x^{a-c} t_l$ y n $n - o$ e s $,_{Sl} =$

$n = 0Sl, n$ nad $S =$ $l \in \mathbb{N} S^{l_{\mathrm{W}}}$ i_t $e_{ve} r - y$ s $=$ $(s_1 \quad , .._{,sl)} \in Sl_w$ e a s o c ia

thea ffi neunc t_i o n $\phi s :$ $\mathbb{R}$ $\rightarrow$ $\mathbb{R},_\phi$

$$1 \quad ._{,s}^{)} \in \quad Sl such t_{h_a} \quad t$$

$$(^{(}ii)\phi_{gi}^{s(x0}{}_{+(}) = i - parenright_{=}^{0} \in i - x +^{\mathbb{Q}} {}_{1}^{[}(2) or = zero - comma.., l - \quad \overset{1}{.}$$

N tcet hatif pq $\in \mathbb{Q}$ w t h $2r_q$ t he n $2r \mid \tilde{wide}$, i f $q\tilde{wide}$ d e o esth ede no m i n a t or o

oi / i | q n t
i f p/q a n d $g(pq). = p/\tilde{wide}q$
$i =$ 0 i p s c
$=$ i p i o d i^n c e l s e p $\not\in$ (2 .
T h e c o n $l^c s_i$ o n f_{this_s} i mp l o b srv a i o n i^s

L EM MA 1 . *T eh s e − t of peud oc yce s cincid s w t − i hthe set o Col*]).^{C} *y sa r ee ther o sti e or n e − gt − a ve .*

w ed fi he f un $c - t$ io n $\varphi : S$ $\rightarrow \mathbb{N}$ re u $i - s$ v e l y b

$$\varphi parenleft - braceleft\}) = 0,$$
$$) \quad \varphi(s0) = \varphi(s$$
$$\varphi(s1) = 3\varphi(s) + \quad 2s),$$

h $e - r$ e s den $e - t_s$ an $a - r_b t - i_{rary}$ $e - l$ m e $t - n$ o S and $l(s)$ t $h - e$ y (s e e a s

$$l\mathbf{summationdisplay - parenleft} s^{)} \quad sj+$$
$$s = \quad j1s_3 \quad 1 \quad + sl$$

A ne a y conse ue ce o fthe d efinii n 2 ist h e f llo ingd e o m p $s - i$ io for m ua s wich wll^n b ec o v e n n t ft e r

$$(\quad \varphi s = 3 \quad \varphi^{(s}_{)} \quad + 2 \quad \varphi_{(} s$$

$or\ a\ rbi\ rar\ y\quad s \in \quad S\ w$

$$\phi s() = 3 \quad 2(s - parenright \quad (\quad)$$

$and\ h \quad e_n\ efo \quad r^{ev_e}\ ry \quad s^{\in} \quad S_t\ h \quad e_r e^s_{xist} \quad a\ u \quad cege \quad n - ea - rt^{e-d} \quad by \quad s.x_0\ isg\ ve$
$n\ by$

$$x0 = l^{-}_{\varphi(s-parenright} \ {}^{3} \quad .$$

Proo f . T hepr o f s byin u ctio n wth re pe t $t - ol(s)$
(i)$l(s = 1$ i seasi l $c - h$ ecked f r o m the de fin to n (i)$(s)) > 1$: i f $s =$ 0
the n

$$) \quad 3)^{(\quad (s}_{x+\varphi}$$

e c a e $s =$1 isa a l go us . $2 \cdot^2 ls \quad 2 \quad s$ h F r s s $s \quad S$ l e t $\sigma(s)$ denotet her
ito f s in S g

e r m u t a t n l

$$k \quad \lambda_s l : (\quad ,^1 \therefore^{sl}) \mapsto (s2, \quad \therefore,^1_s \quad ,s),$$

$$i \quad \{\lambda(\quad k$$

$$M_l n : \quad s \in S^a_{l,} \quad t \in i(s\varphi$$

N $o - w$, su pp
wt h a t

$$(5) \quad \forall n, l \quad < \quad L : \quad M \quad l - comman \quad \leq m$$

h o f a o l laz_t c $y^2 - l_c$ e n $n\mathbb{N}$ w
a t_l ea s L . B e $f - o_r$ e we s_t rt w_{it} h a $de \quad tal^e$ d a a_{ly} s is of t h e cr $u_{cia} l_q$ a n_t
$M \quad l - comman$ n t he_{ne} x t_{sc} t $o - i$ n
$b\ e \quad eq\ u\ e\ c\ s \quad s^c \quad u^t_{gse}{}^{afo}\ ar\ l$

summationdisplay − n $\sum n$

$$c_i \leq \quad di.$$
$$i = 1 \quad i_{=}1$$

$$h_{enor}\ a \quad n \in \mathbb{N} \quad w_e\ h\ ave \quad n \quad n$$

$$\sum c \quad f \quad \leq \quad \sum df.$$
$$equal - one \quad equal - i1$$

$i \quad 1 \quad i = 1 \quad i \quad = 1$

u C e $e - r - t$ y $f - o$ h $f - uc_t o - i$ n .

$$s = \quad ., s \quad = ($$

$$mentso - fScomma - ln.I - f \quad \sum_i k_{=1si\leq} \quad \sum$$

c a e o_t B $ceyareitont$ t ee u e_{exs} sa^a sm al et n u^a m b e r k w th o s^e s a = and $u_{ts} = h_1$ a nd a d_s ma les tnum ber t k

ow $l - e$ t s b suc h $h - t$ t s_i = si f r $i \notin \{ \genfrac{}{}{0pt}{}{0}{k} k_{1_\}}$, $sk0 =$ 1 nd $\prime_1$ $= 0$. T he

$\sum ik_= si^{\leq}$ iks'_i $\leq \sum$ $k_i t($ f r a $l - lk \in$ $\{, .., l\})$. k

C o n i_d r the $s - eu - q$ e n e $\bar{s} = (sk - zero$ $+$ ... s $1 - 1)$. F om (2) t sn o har to see that $\varphi(01)$ $>$ $\varphi($ $\bar{s}0)$ a d by 4) we get $\varphi(s)$ $< \varphi_(period - parenright$ If $s_{\prime_{\neq}}$ t w c a n_{r_e} e^a t he s m e p ro_c e ue $w_{th}s$ a w_e h av d_o e w h sa_{n_d} fi^n d

$s - e$ q uen c e $s_\prime$ such that $\varphi()^s >$ $\varphi(\prime\prime$. A $f_{t_e} r_{fi} n_i te_{l_y}$ m a y $e - rpe_t$ $i^t n_s$ we g

$$> \quad \varphi t - parenright^{\cdot} \qquad \varphi($$

Int hen ext e mma we d e_t r mi n^e t h e s eq_{u^e} n $c_e s\tilde{wide}$ f $o - r$ w h i ch $\varphi t - a$ ta nsth v alu $M_{l,}$.

Lemm a 5 . *Let* $n \leq l$ *benat ral* u *be* s . *Let* $\tilde{wide} := \lceil n/l \rceil - (i - 1)nl$ (fr $1 \leq i$ $\leq l$). *Then* $\varphi(\tilde{wides}) = \min_o$ $\in \vec{\sigma})\{\varphi()\} = Mn.$ /

it e (ln) f ort . c o u e d o n n^d n e w o f

$w \quad s$

$\leq$) by a tai c sa e se eFigur t e $1.S_{Assu}$ me e $at_h e^n$ e ex^p ists a k i u c h $(t^1$ h $kkn/l\_$ $\sum k0$ $tr_{>0}$ s (m mal . T h e nf o r t $\lambda_{k_0}(t$

$$0 \quad i =_1 \quad i \quad i_k \quad = \quad l \quad e$$

$$i =$$

y_B c o $n - s_t$ u tio nof $\widetilde{s}$ w ea so hav

$$\sum k \quad \mathbf{summationdisplay - k}$$

$$equal - one \quad i_= 1 \quad \prime$$

$$e^g e$$

$$\varphi(\widetilde{s} \geq \quad \min\{$$

$$\in$$

$m-m-u-period-s-t-T-h-s-h-e-n-o-w-b-y-t-h-c-a-o-t-n-s-phi1-t$

n t ee oe $\varphi(s\tilde{wide} \leq \varphi(\prime)($ b y L em m a $four-parenright.$
h
c

$e\ e\ r\ \ l_y a-n\ d\ \ n\ \ l\ we\ h\ av\ e\ \ r\ \ \ v$

$$M \sum_{=} \ l\ (jn/\rceil - \lceil j-1 \quad n$$

$$j=1$$

U s^{ng} t_{h^s} e p_{ii} f or m u a w n o w de i_{vd}
$t-i$ O N $1o-F^r a^{l-l}$ $l\ a\ d$ $n \leq n(parenright-l := \lfloor l$ o g $32\rfloor$ we h

$$2l\_3n \leq l \cdot 2 - 3n/l.$$

$$=_1$$

$$= \quad (ceilingleft-parenleftl- \quad +1)n/l\rceil - \quad \lceil$$

$$i$$

(w $e_{/l^a}e_vh-a^n$ ge d th e r de r o fsum ma t i n) O b p o

$$\sum \quad \sum k$$

$$(\lceil$$

$$i =_1 \quad i = 1$$

A p p ly_i n g L e m m^a3 w_e $h_{encec}on_c$ u d^e

$$-$$

$$i1 \quad l \quad l \quad 2- \qquad /$$

si

p p ew a s e ie_p $c \genfrac{}{}{0pt}{}{e}{s}$ o f t $h-ea$ $u-qm$ $=-l^n_{ce\tilde{wide}\ln}$

$$M \quad n$$

$$) \quad n \quad \leq \quad 1_x \quad + \quad 2 \quad x1/m-1$$

$$f-o\ ra\ l \quad l>6 \quad an\ d \quad n=n(l).\ Here \ ,$$

$$\alpha \quad = 2n+3l \quad a \quad d \quad \alpha =^6 n+l.$$

.,.. 59, t h e s $s-eihc^2_{t-r}{}^2_{nan}{}_be^m$ in L a e m m a 5) s c mo p a o s e dof t

be r $\tilde{wide}^{Zi}_{a_i\in A}$ hen fo $thl-periodowsf^I_{o-r}$ m

$$5a \quad + \quad 3z = l-comma \quad 3a \quad +2z = n$$

t a_t $m=$ a $+_z=2n$ $-$. T h e n () ( )

$$ns-tilde-parenright^m \sum \quad 8 \quad \varepsilon Zk_) \quad 32 \quad \varepsilon A()$$

$$\varphi(_s) \quad = 3 \quad 9 \quad 27 \qquad (eight-parenright$$

) d n te h e mb r so Z

$$(\quad \textbf{braceleftbtbraceex}23 \quad if ak \quad =$$

T he orm ula (8) i s e a s ly p o ve d byi

$$A - asteriskmath) = 3()^{(}_{*} \quad \varphi_{n(}(*_{)})3_{2}27$$

$$\varphi(_{Z}) \quad = 3Z_{*}) \quad 3n(*)) \cdot \quad 9 + \quad 9.$$

$$2\varepsilon Z(k) + (^{3}_{\varepsilon e} \quad ^{A}_{c}k)^{\geq}_{Fr}n$$

$$+ \quad \begin{matrix} 5\varepsilon & A &) & l \\ n & l & & \end{matrix}$$

$$\varepsilon Z_{(}k) \geq^{\varepsilon A(k)} \quad l - 3$$

a n o m $\varepsilon A(k \quad + \varepsilon Z(k \quad =_{-} k \; s_{)}$ o l o $w - s_{-} :_{+})(\quad)\varepsilon(m$

$$\varphi(w\tilde{i}de_{n)}) = \sum \quad 8 \quad 3^{2} \quad \chi(m \quad - \quad i \quad + 1) \tag{1}$$

$$(_{8})two - five -_{3} l - n3)(ml_{7}^{-)^{n-}}{}_{2-n}$$

$$\leq \quad 9$$

$$= m\sum 1 \; \frac{1}{x} \quad - \quad i \quad + 1$$

$e_{O}s - t - e - h_{n}$ u $n - t \; \frac{c}{h} \; e - e - rx1 - 1,^{/m}$ f o s sit v e k a $\in$ d e,. a s

$$\chi(m - i + 1 = \quad 7(1_{5} \quad \begin{matrix} k \\ \\ = 1 \end{matrix}$$

w her $E_{Z}(\quad k)$ a nd $EA \quad (k$ d e n ot t h enu mbe of Z a nd Ar_{e} p ec $t - i \quad e \quad \frac{y}{i}$

$$(a - k + 1, .. \quad , a).From$$

$$3EZk)3_{+}5EA(k) \quad \geq \quad l$$

T

$$\chi(\quad m - \quad i+$$

1 and $di \quad = \alpha 2$ fo r $i \geq 2$ cAp p i a no_i

$$a_{nd}$$

$$s - i \quad o$$

$$= nlparenright - comma \quad e$$

$$20 \leq \quad Mn_3 n \quad \leq_{10}$$

3 $S_n c_e$ w w a n tt o i m r v e

$$m_3 at_{la} \quad ll_1$$

1 $n\ d \quad > a\ d \quad \alpha 1 \alpha 2 \quad a\ nd \quad m(\quad a\ inPr\ o \quad p^{siti}\ n \quad 2 \quad w \quad e_h\ a\ v \quad)$

$$fparenleft - x) := \quad one - alphax1 - 1m + \quad \alpha_2 x \quad 1m \quad - \quad 1 \quad 3(x \quad - 1) \leq \alpha.$$

$$x \quad - \quad 1$$

povd a t o im . W $f() = l_0 .82...$ fo r l r g e nou gh . l o f c l d b

$$im \; \overset{4}{r} \quad e \quad l \quad z \quad \to 1 \quad z \quad l^{\;a}$$

$$2 \quad g_1^{l_l og^3} 2^1 o \leq n \leq_{+l}^{l} g \quad 3$$

$$\alpha \leq \quad 9 \quad 2 \quad \text{g 2 lo g} \quad 3 - \quad 2_{\text{ g }} 3 \quad \to 9 \quad \cdot 2 \log \quad 2 \quad - \text{og } 3 \quad = 1. \quad 8..,$$

$$- \quad l$$

$$\alpha 2 \leq \quad 27 \quad \cdot 2 \text{ o } 2 - \quad \text{l g} \quad 3 - \quad 2_{\text{ o g }} 3 \quad \to 272l - og_2 \quad - \text{lo g} \quad = 0_{.67} 6..$$

$n - c_e f(x) \leq_{ed_{e \to} efnt_{ri} y}^{imz \quad 1(z \quad m^{li} \quad z} \to (_f' z)_{l>}^{(x \quad -1.C_{al} u}{}_{1))fcc}$ at n g th e s e v

m $f(_z =) \quad 1(\alpha 1 + \alpha 2_{(m -} \quad \to)) \quad 2parenleft - two - \quad =_{log_2 3} \quad {}_{0eight-period4} \cdot\cdot$ a s l

$$imf\prime_z) = 3 \quad \alpha \quad + \alpha m \quad - 1)) + \quad 3 \quad \alpha \quad (\quad m - \quad 2) \quad - 2 \quad \alpha m \quad - 1 \quad)$$

$$(\quad 2n2 \quad 1 \quad 2(\quad 2mn \quad 2(\quad z \to 1$$

a n t h e c $m_i f_o$ l w_s e a i f r m $t_{h_e} s^e$ f c_{ts}

et hat $n \quad = n\prime \quad +1.$ L $\quad , n \prime_r$ s t_{pec} i $0^1_{el}.Fro^e$ m $\quad the^a$ s o nstruct $onitfolo^m w^m a_s$
tht $t\prime \quad = 1s\prime_\in \quad Sn$ n
$,_{dom}$ i n $eatestv$=0$s \in S+1, n$ i nt esen s e

$$k \quad \sum k$$

$$ti \leq \quad i\prime \quad f^o r^a lk_\in \{1..\cdot l+1\}.$$

$$i=1 \quad i=1$$

He nce , Le mm a 4 imp est $h-a$ t $\varphi(parenright - t > \quad \varphi t).$ N ow ( )
im pl $e_s \quad \varphi_{parenleft-s}$) $< \varphi_{(}s_{)-}$ $23^n_\prime$ a nd t s_h p o es t h cl im .

s − o n f − od − iff e t − n c r it e i − a

l l ' s roi n

$$ofCo \quad l - a_{tzy-cc} Lem^e_m \quad {}_a \quad {}_{8(Cran} si - sdal^{due} toC^{an}_{.} \ {}^{da}_{Ifm}{>} l - l1] :$$

$$a \quad d \quad q \quad s \quad \in S_{\ln} \quad i \quad n_\leq \quad n$$

$$m < \quad (\quad 2 + l - /3mn \quad .$$

$r - a_{ndl-a'}$ t h^{e-o} rem . el l n o b d c o n l

$$p^{kqk}_{(} \quad e^c \quad n$$

$$n \quad > \mathrm{m\ i\ n} \quad qk_{,qk} + \quad qk + \quad 1$$

wt h m a nd n asin

Uingte f cthat the C $o - l_{la}$ zconj ct rewas verifi d or a l i
talv $l - a$ ue

$$x \leq$$

$l \geq 031 \quad bracketright - nine4.$ ho u m pr $v - o$

E s c .

T HE ORE M 2 (Ela hou . *I f* $k(m)$ *d enotes he sm ales tin e g er k su ch tha*

i) (*t*) *t*

$$k \quad \leq$$

$$n_(k) \quad m$$

$$t^n_{h_e} \quad f\ o \quad e_v \ r\ yC \ \ o^l \ l\ a\ z \ \ ycc_l e_C$$

$$|\ C\ | \quad \geq_{k(} \quad m_{in} C).$$

$o\ r \quad v\ e\ r\ y \quad p\ o\ s\ t\ ve \quad Co \quad a\ t\ z\ cy\ c\ e\ \ C \quad i\ n$

$$| \, C \, | \geq k \quad m \quad inC \quad ,$$

$$\alpha$$

$$rv_d d \, |^C \quad | \quad >^1 \quad 60.$$

P ro $o^{.L}$ e C b ea p s^{ii} v e C $o - l_l$ a t z_c c l $e_g n - e$ $eat - e$ d b y o m_{es}

$$C \quad \leq M \ln_n \quad M$$

wh erew e u d t h e

P r o p $o - s \quad i - t_{on} 2a - n_d$ L em m a 6 w e hav e

$$Mnl)$$

h o $f L^e_x$ mm a $.^{-}\underset{r}{\leq} \alpha_{om}$ t final yc oncl de

$$| \, C \, | \quad ($$

e o r mis or e d $.C \, |_)$ m n C

T $h - e$ o em 3 $e - t_l$ sin p a $r - t$ cu artat E l aho $quoteright - u \quad e - s_t$ mat es o $h - t$ e l e n g ho cycle s ($e - period$ g . the

$c^{eo^{i-r} gi}_{u^{r-e} n}$ an ly $e_d s - m$ a n $b^{edva}_h e - u - k$

4 . . *Optim lc* $r - i$ *terion* T

a 5 isopt m a , si cn e f^{o-r} e v $r - e$ yg ven yclelengt ht h $e - r$ ee x $t - s$ s acy le , nam el

th ec yc $l - e$ gen e $a - r$ t edby $\widetilde{s}$ i n Le mm a 5 , wi theq ual $t - i$ yin (12) be low) hasth dis adv ntag et hati tis $r - e$ l $t - a$ ive y co_{s-t} ly $o - t$ heck i $t - period$ Ho we ve $comma - r$ w e $wi_{l-l} n - i$ clud a n ex ample b e $l - o$ w whi h s how sh owto han le $h - ti_s$

T_h EOREM 4 . *If* $L(m_)$ *den otes thes m al lesti n* $t - e$ *ger* L *su ch* $h - t$ $a\ t\ f \quad o - rn =$

$$nL),$$

$$\sum L_j = parenleft - ceilingleft_j n/L - \lceil (-1)n/L \rceil 2j - 13n - jnL \rceil)$$

$$1 \quad \rceil \quad L \quad n \quad \geq m$$

$$2 \quad -_3$$

$$2) \quad | \, C \, | \geq L(mn \quad C).$$

$$(^1$$

$e^0_{s^m_{x0 \leq} lem = u2p \quad 366^h \quad 2807211^{ll^{at}} h^{c^o} n j_e ctuu}$ t h tc om pu tatio n , w ec 915 S e_c o n s t_e p : A si m pl $M\ a\ t\ h^e\ m\ a\ t_c^i\ a$ p_r o c u r^e a_c hec k t_h a

$$n_(l). \quad 1 \quad < \quad m$$
$$l \quad 2 \quad - \quad l$$

o $a_t e^v_{rnatel}$ t ha t (l)

$$2^{1n} parenleft - l - parenright < \quad m \quad 3n() - 1$$
$$\ldots^{+}_{,} p8^{:k=\backslash 3,4}_{\}} 1 \cup$$

1 7 0 8791 5 and $pone - eight$

$1^)_n$ f o^a r h e $m - e - ecint^e$ o n d $(parenright - e - l - s$

T hird st e p Us i_n g C oro a y 1 wec a nc h ckb y $d - i_r c_t$ c o m p u a t i t h a t

$$M comma - n \quad < \quad m$$
$$2l_{-3n}$$

f r $l = p6$ an d $n = n()$.

O b s r e t h $a - t \quad six - one_{lo} \quad g - three2 = q \quad + \varepsilon$ o so men um be $q \in \mathbb{N}$ a n d $\varepsilon \in]0, 1/10$
and

$$d^{encw\tilde{i}de(}_{nLe} n)$$

fr o m $parenleft - four^)$ tfo o $\frac{w}{s}$ t h a t

$$M \quad q16(k - q6) = \quad M1comma - sixn - parenleftq_6) \quad < m$$
$$2 \quad q_{one-six} - 3n \quad k^(\quad 6 \quad 2q \quad 6\_ \quad 3 \quad q^(\quad)$$

for $k = 1, 2..., 0$. T $h - i$ s ru es outnp a tic l ra the w h o^{e-l} s et $A1$ withou furth rc o mputat on . Th e case $l \in A$ 2 c a n r beha ndlede the r iw th a $s - i$ mi $a - l$ a $r - g$ um e nt or byd i $e - r_{ct}$

$$pti^{:From \quad thefi \quad s \quad eestepsw}_{r \quad t \quad e} f$$
$$2l-_{,3} parenleft - l) < \quad m$$
$$f^{oall\in} \quad \{1.\dot{,}_p 18 - 1^\}._B \quad y^t \quad h$$
$$Ml - comma_n$$
$$ld^{2l}_l \quad -_n 3_{\leq} \quad (.$$

r s A l es t i at e o $t_{hel^{en}}$ g t h o_{fC} a_{ly} fo i_{nt} ge Co la z_c y les B u_t si n_{ce} w h a e s e e^n t_h a $t_{he}m_i n^{im}$ u

$$o_f \quad raonalcy_cl_egr^o \quad w_s \quad a_tle^as_tl_{in}$$

a c^e_{to} cheve f_u $t-h$ e r $p^r_{gr}e_{ss}$ n h a_t q^{us} t o n w o u_l d h e_{nc} e inv lv n u m er t h e_{or} ti c^a_{lar} g u me n_s e $.g._o$ f t h_{efo} l $w_{ng}k_{ind}$

I $f_{\varrho l}d_{en_{otes}}$ h e_{ri_g} t s h_{it} e_{rm} u_a i n c_{tng} on $l_{,ie}$.

$$\varrho : s_1, ..., sl) \mapsto \quad (s, s1.., s2parenright - comma$$

the nw e o tan ro m (4) the olowin g f rm ua s wh ih d ecr ie t eeff et o ϱ_l o n φ : i o s b c

$$3) \quad \varphi(_ls^0 \quad)_= \quad \varphi(_0s^) = 2\varphi_(s^0 \quad)$$

$$1)) = \quad \varphi \quad = 3\varphi(\quad 1_) \quad)$$

T_h ew e_h væ t he o^l o w_ng_l em $m_{a.}$

L E M

$$(1\ 4 \quad \text{g c d } (\varphi(\bar{s}1\varphi(^{\varrho}l(s)_) = \text{g c d } (\varphi s^{1),2l} \quad - \quad 3n_($$

$$a_{nd}ina_{ric}ul - a_r$$

$$) \quad \text{g c d } \varphi \quad) \quad t \in (^s \quad \text{g c} \quad \varphi((\quad)2(\quad) \quad - 3ns^{))}.$$

P $\varphi($ $3^{(2}x$ $+$ $3n$ $- 2))^{an}$ d h e n^e

$$g^{cd}x, 3_{(2}$$

N o $t^{he}fi$ r s^t i n a b o v e e^wu^{se} d t h e f^{act} t $a^{t1}3(2$

m t a x ′ a mu t o 3 ′ v n $th_{\prime}$ h ′

$$g_cd^(\varphi s0^)\varphi(^{\varrho}ls0))_= \quad gc$$

$$g^{et}15 \quad _{\cdot)}$$

U s_{ng} L e m_m a $9w^e$ s e $t_{hat}\neg$(A) i s e q_uv_{ale} n t_{to} (A) Th e e xis s ano n - pe io d_cs $\in Sl - commal >$ 3, s u $h-c$ t

$$g - cd \quad \{\varphi(t) : t \in \sigma(s)\} = 2l \quad s\_3^n \quad (s.$$

A c k n o w a^R_s o g_{rsfr} t er use u h n san d c o m m $e_{nts}a_b$ ea^a r grate *sulto* a h eefere $Veom_{rs}$ v r al b i i o g aphi alh g

[1] R E. Crandll, *On the* $3x+1$ *prob em* M ath. C omp .32(97eight − parenright, 128 one − endash 292. [2] J. M. D o la n − comma A. F . G ilm a n and S. Man i ck am , *A g*

endash − five 6.

c 1 24) (1 98) 7 91 – – 9 6

[. *L a g ar* a *T* *e* $3x$ +1− *ro b m* *a di s* *g ne* *i t ions* A m *e .M r* a h Mo *nt l h*

92(195), 3 2 3.

[— *T* *h se* *o* *r a t o* *n c* *y ces* *fo* *h e* 3 + 1 *p* *r obe m,* Ac a A *r i t* .*h*56(19 9 0), 3 3 – – G . *L e a e* n a n d M . V e *m e* u

[J W .*S an* *e*, *O nth e* (3*N* + 1− *c* *o j n e c* *u r*, *A* c a Ai *ht*.55(1 9 90), 24 – 8

B G . S *e ifer* t *O* *n t* *hea* [1 . W i r c D *ih* i *c n*, *An* *a mp* *r* *oved* 9 *esi mae* – – 2 *co* *ncr nn g* 3 *n* + 1 *ped e ssor* *ts* , Act G A r *t h* 63(1993), 20 5 – 2 10.

M *h e m* − *a t* k D *p ar t m e* nt ET *H Z r c* h *U nv* e *s i−t* y of Mi n esot 0892 Z ¨ r *c* − *h*, S w *i z r* − *e l an d* E − *a* − *i l a* − *h b l* e i @ m *a t period* − *e t* *z*.*h* 206 C hur hS *t r e t* − *eS*.

mn e

a 2 86 2

1 9 (