T ka akiK ag $a - w$ a ( To o – y)

$e_d$

$su_{c-h}$ $c_{r-u}$ v $e_{o-v}$ erth fied of
$k$ p t v d e d hatthe las $m_s$ nu om b r of an d$k$ s a $l^o$pimeto$^t$6. H en cethe h cpob
es s $nt^i$ aly slve $din^c$ thsc a e – period e i r
e i l o i s

A noth rreaso$^a$lnwea$^t$ r e ntereste $di^o$ nt hscase $s^h$ e r aed$^s$ o Shim u ra ' s u r e $s^e$ obta e din e i foll win g wa y . L et $N$ $r_b$ ea pos iv ef nd a c v i n $t^h$ o i u
$sp^s$ a e $i_S^m$ n $(n\Gamma(N^n d, \chi\ t\ )\chi_o$
d imensio nal $\mathbb{Q}-$ s mple fact r , Shi m s ua [ Shim ] – c o n truc $td^e$ an t a b t la n $A$ define do e r i$\mathbb{Q}$. O ve rt ohe r e a qua d r aic fil $s_d k$ $= \mathbb{Q}($ $\sqrt{}$ e i
v t e

$N), A$ s

$B \times B$, w h – e r e $B$ i san e l – l i p – t i cu v – r ed e – fi ned o ve $k$ and $B\prime$ s h – t e o ju
$B$ . W ecall $B$ S him ura quoteright – s e i – l – l pt i – c cur e – v o – v er $k$. It s – i k no wn that $B$ isiso
to $B\prime$ ov r $k$ parenleft – bracketleft$_{\text{Shi}}$ m ) , and hat $B$ h as ever w he $eg^{o-o}$ d r d c t – i on ve
Ca , [D – R], [ K M ]period – parenright C onve r – s ey , an e l – l ip i ccu ve $E$ ov er areal qu
field $k$ withthe p r o pert essta t – e da b o ve is con e – j c $t^{u-r}$ ed b y P inch [ P 1
iso e n ous ov e r $k$ t oShi m – u r quoteright – a s el l – $ipt^{c-i}$ cur e . F r r e – l at ed o – t $p^{i-c}$ s c on
mo g d $ua^{r-i-t}$ y f l – $e^{l-i}$ ptic cu $v^{s-e}$ ov er nu mberfi e – l s , se [ Ha 1 ] [ HH M ] .
H enc eh e ca se f a re lquad r a ic fi ed ise pe illyi ntr e t n g . cas e , th e o l w ngisk n o $a_{wn:}$ s c

f $l^o$

- S ever $a^e$ x a $m^p e^{sar-e}$ k $n_o w_n$[$C_o$, [$I^{s,[}$ $Se_{t,[}$ S h io , e $t_c$ .
- T he r – e i a met

r o r $u^a$ r $t^i$ $e^l$ s [ $U^m$ ) R c

[ Co − bracketright’[K1]).

ever , as fara s t

l ptccu r v sw ithev ryw$^{a}$ h er goodr e u − c tio ovea r$_{re}$ lq u r − da$^{i-cn}$fi ed

I − ni$_{p}^{e-h-t}$ r ese n t p ap e$_{e}$

r $\surd$ l t a l u v t$^{h}$ e

e r$^{e}$ g o$^{d}$ r e d

2· Th e$^{s}$ p c$_{e}$$S_{(r}^{2}$ $^{0}$($^{3}$7,$\chi^{37}$ i s 2_d$^{m}$ e n$_{s}$o$^{n}$ a a n$^{d}$ $\mathbb{Q}$ s i m p$^{e}$ b$_{y}$ S

[ h$_{m}$] He$^{n}$ c e$^{S}$h$^{i^{m}}$ u a s$^{a}$ be i$_{an}$ivae t i s u n$_{q^{u}}^{i}$$^{e}$ l yd$_{t}^{e}$ er m$^{ni}$ e ( u$_{t}^{p}$ i s g$^{eny}$) a n$_{d}$ w$_{e^{d}}$e$_{n}$o$^{t}$ i t$^{b-y}$$A3$. T he m a$^{t}$ x

$$\sqrt{v}line \quad 0 \quad - \quad 1$$
$$3^{7} \quad 73 \quad 0 \quad \surd \quad vline$$

indu ces a − n u − at − o m o rp i − h$_{s}$ m $\eta$ f $A_{3}7$ de fin e e a $_{3\,7}$ :

$$B37: \quad y^{2} - \varepsilon = \quad xline - plus3 \quad \varepsilon_{x}^{+}2 + \quad line - one_{\text{one-epsilon}} + 1 \quad x \quad \Delta = \varepsilon \quad , \quad j = 2^{1}2$$

2 2 ,

w h ere $\Delta$ sth e d cur v wi h 7e − v re w$^{=}$ed$_{o-g}^{\langle 00\rangle}$ r 5$\mathbb{Z}$.dvo$^{K}$ i e w ti h$^{p}$$j$ d $\mathbb{Z}$ a r e t$_{i}$ o v − e r $kt_{o}$ei$_{th}$re $C_{1:}$ $= B37$ g vi$_{e}$ n a obveo$^{r}$$C^{2}$ :=$_{C^{1}}$ $\langle^{\text{zero-parenleft}}, 0)\rangle^{\text{give}_{n}}$ b y

$$C \quad : \quad y2 - \varepsilon y \quad = x + 3\varepsilon \quad + \quad 1669\varepsilon + \quad 39 \quad x^{2} - \quad 1 \quad x - 7\text{parenleft} - \text{five}449\varepsilon + 451)$$

2 $_{6}$ 2 $_{3}$

W e e a t $\binom{C}{2}$ $k$) $_{r}$ vt $\Delta$ ia (, $j =$ As 33 3 o f [ Sho ] s ee a l

ev eyw h er e$^{r}$ g o od oret$_{\text{dut}}^{h}$ $^{p_{r}}$$_{o}$ nve r $k$w$^{p}$etho uta y$^{t}$rs$^{m}$ric$^{i}$ t o n n$^{l}$ h e $j$− in a

As a

o f$^{f}$ p $\surd$

T H E O R M . $U$ $p_{t^{o}}$$s_{o}$ $m$ $o$ $r^{hp}$$i_{m}^{s}$ $o$ $v^{r}$ $k$ $= \mathbb{Q}$ ( 3), $C$ $a$ $n$ $d$ $C$ $ab$ $o$

$he$ $o$ $n$ $ye$ $lipt$ $i$ $cu$ $rves$ $w$

$Pnch^{quoteright-s}$ $n - o_{j}$ $ct$ $ur$ $e$ $i - s$ $r - t$ $u$ $efo$ $the$ $fi - ed_{l}period - k$ $e$

Conse ue nly , all uch curves aret he on esa ra d y o ta ie din [ K q t s sio n − a l $\mathbb{Q}^{m}$− sim p l e , an dh$^{(}$ enc m Sh m$^{h}$ u a ’ se l$^{a}$ pticc rve 4over$^{1}$1$_{\mathbb{Q}}$$\surd^{a}$ l s o

n §5 of C o ] (

i − s√r − t$_{ue}$ al s − o fo r $\mathbb{Q}$( 1). W e asl o find that h − t ereare nosu chc ur v − e$_s$ o $\mathbb{Q}$ $(N)$ $(N = 17, 21\ 74397,\ 19173181)$. Note ta t $S$ (Γ ( $N$parenright − comma$\chi N$ ) hasn $2^{\text{hyphen−d}}$ ime s − n$^{\text{i−o}}$ nal $\mathbb{Q}^{\text{s−hyphen}}$ im p l − e,$_{\text{a−fc}}$ tor orth e − s e $N$ a n for $N$ = 5, 13bracketleft − parenleft Ha 2comma − bracketleft$_S$ h $i$ ]
m

H e n c eth ec nje turei st u e ls ofor tese 1 0 d vauesof $N$. ]

h l

**3** N o TAT ON For nu m br fied $F$, e den ote by $\mathcal{O}$ ( esp .$\mathcal{O}\times_)$ i rigo in e e s ( r e s p . i tsg rou p e of u n t s ) If w$_F$ s aq ua dra icfi e l − d a n d $xF \in F$ neden f oteth ec ojuga te of $x$ by $x\prime$. i i t

w T hro g hout h$^n$ s p aper w e en teth e e alqu a ra ic

S − e t $\omega = (_1$ seven − plus$_{)/}$, andet $\pi = (+ \sqrt{}$ 3seven − parenright/2 be pr me l − e$_e$ m − e$_n$ t div

3 in $k$. O bserv$_e$ th at $\pi$ $\pi = 3.\sqrt{}$ W e de n ot e by $\varepsilon$ the f unda men a u n$_t$ large rth n 1 , name ly $\varepsilon = 6+$

$$7^{\cdot} \quad e \quad t \quad t^N \quad k_{/}\mathbb{Q}() = -1.$$

H e e weg ve t$^h$ e o$_u$ t n e o f h epro o$^{\text{f−period}}$L$_e$ t $E$ be a$^n$ l i$^p$ t c c$^u$ rv e e$_v$ yw$_h$e$_r$ e go$_o$ d re$^d$c$^u$o$_n$ o − v$_e$ r $k$ S$_i$c$_e$ t ec$_l$s$_s$ n$_u$ m b$^e$ r o $k^i$ s 1$^{\cdot}E$ a mo$_d$ e l 2 3 2

2 6 $n$ w$^{t^h}$ c o e$^{\text{ffice}}$ n t s $ai \in \mathcal{O}k_(i = 2, 3, ,^6$ a$^n$ d i c$^i$m$_i$ n a$^n$ t $\Delta = \varepsilon \in$

I$^v_n$

< .T$_h$ e d s r$_i$ m i$^n$a$^n\Delta$ a n$^d$ h e$_3$ u n i$_2$ hat $(c\text{a}4, c6)$ is a n $\mathcal{O}$ − ai nt g r a l p o − i$_n$ t t fon eo$^e c$ − fthe elip c cu ves i$^m$

± 2 3 o $n$ l

$$E^n : \quad y \quad = \quad x \pm^1 72^{\text{eight−epsilon}} \quad , \quad -_6 \leq \quad n \quad <_6 .$$

Thu sto d etermine t e e − l$^{\text{i−lpti−c}}$ c$_u$rv$^{\text{s−e}}$ wi h v e − r$_y$ hw e re o − g$_o$ dred u − c t o − i n $k$, w e fi sr t de ermineth e s − e$^{\text{ts}}$

$$E^{\pm}_n(\mathcal{O} \quad ) = \{x, y - parenright \in \quad \mathcal{O}_k\times \quad | \ y2 = x\pm$$

W e need notd ete m in e a lthe s t − s h − t ough , becau seth edi s − c ri mi ant i a c be , aswil be pro v

$$n^E(\mathcal{O}k) \to E \pm_+ (O - k_) \quad (,y) \mapsto (x\varepsilon 2, y\varepsilon_3$$

is a b i − j e tion , an d the ma p $(x, y) \mapsto x\varepsilon 2, y\prime\varepsilon)$ i a lso a b j − i ec io f

$E$ $n - plusminus_{(\mathcal{O}}k)$ to $E \pm_- n^{(\mathcal{O}}k)$( re p . om $E_n \pm_( \mathcal{O}$ $k$ to $E$ e − f$^o$ re t ffi cs t$^{\text{odet}}$ i e t h f l lo$^w$ g t T$_h$ e d t − e$^r$m$^{\text{nato}}$ n wi$_{l-l}$ be $0^{\pm}_{\text{d−oei n§5.}}E$ $3^{+^{\mathcal{O}}_(}k$ ).

t t

$$h - a_s \quad c \;\; u_i \;\; c \;\; d \qquad s_{c^r} imi^n \;\; a \;\; t.$$

N ote th$_{\mathrm{a}}$ t he d i $-$ s$_{\mathrm{criminant}}$ being a ub e or o $-$ n$_{\mathrm{t}}$ is i de p $-$ e$_{\mathrm{nd}}$ T opro veP rop os ton 1 , s u p p$^{\mathrm{o}-\mathrm{s}}$ e that , on the co ntra y $-$ r, the r $-$ e is an e r$_{\mathrm{v}-\mathrm{e}}$ $E1$ w ithe e$_{\mathrm{r}}$ wh r $-$ e e g oo dredu to n over $k$ give nbya global m nima Wei rstra s e quationwh

Lem m a . *L t M bea rea q u a − radicalr − da − t c fi$_{l-e}$. A s − s u m that 3s − i un am ifie in M an d t − h e cl$_a$*

−3) *is prim eto* 3 . *L e t Eb e a n el c u − r$_v$ e wi h eve$^{y-r}$ wh e e g o − o d r d − e u cto n ov e − r Mgi v n − e b y agl bal min equ a t − i o n who se i − d$_{rim}^{s-c}$ inant* $\Delta$ *isn t − o a cub e n M .*

*g o − o dr d$_u$t − c io a ta l prim esofM l i − n$^g$ a v − o e* 3 .

P roo f . ( T heesse ti lp$^{\mathrm{r}-\mathrm{t}}$ o f th ep o − r o i d e t K id a [ i 2]period − parenright L b ea p r − i m e d − i eal f $M$ n div i − d i a$_{\mathrm{ng3}}$ , $u$0 f undam en l u nt o f $M$ , and

$$F \quad = M \quad \text{parenleft} - \text{line}\surd \quad \surd \quad \underline{\qquad} \quad \text{a}_{\surd} \quad \underline{\qquad} \quad \text{t} \quad \text{i}$$
$$-\text{three} - \text{parenright} \quad \text{and} K \quad = \quad M(3 \quad \Delta) = M(3$$

o f $M$ e$^{\mathrm{ne}}$ r ted b y t$^{\mathrm{h}}$e$^{\mathrm{c^o}}$o$\mathrm{r}^{\mathrm{d}}$n$_{\mathrm{t}}^{\mathrm{a}}$ s o a l p o$_{\mathrm{in}}$ o o$^{\mathrm{r}}$d$_{\mathrm{e}}$ 3· N o e$_{\mathrm{th}}$ t $K \subset F \quad K \subset L^{(}$[ r , p .$^3$05 a d$_{[}$ i , p 9 8 a n$_{\mathrm{d}}$ h$^{\mathrm{a^t}}$ t h$^{\mathrm{e}}$ e n$^{\mathrm{s}}$ i n $L$

Og$^{\mathrm{a}-\mathrm{m}-\mathrm{i}}$ a e are $i$ $\overset{\mathrm{e}}{(}$ $S$,$^{\mathrm{t}-\mathrm{d}}$ .18$^{\mathrm{r}}$. A s − o$^{\mathrm{ed}}$ n a$_{\mathrm{o}}$ e r$_{\mathrm{th}}$ t $\mathfrak{p}$ b i r a mifi di o$_{\mathrm{n}}$ n o f$K$ an 2 g$^{--}$ S h f v c$_{\mathrm{h}}$ [i$^{\mathrm{l}}$ p 4 ) l t a s e

$\mathfrak{P}F = \mathfrak{p}\mathcal{O}F$. S p ose t$^{\mathrm{h}}$ a $E$ s su p r$^{\mathrm{i}}$ n g$_{\mathrm{a}}^{\mathrm{l}}$ r e$_{\mathrm{d}}$ uc t$_{\mathrm{o}}$ n a$^{\mathrm{t}}\mathfrak{p}$.$^{\mathrm{T}}$h$_{\mathrm{e}}$ d e$_{\mathrm{c}}$ o mp o$_{\mathrm{s}}^{\mathrm{i}}$t$_{\mathrm{o}}$ n g$^{\mathrm{r}}$ o p$_{\mathrm{o}}\mathfrak{p}$i$^{\mathrm{s}_2^{\mathrm{a}}}$−$_{\mathrm{gr}}$ o p ($_{\mathrm{se}\S}$1.1$_{\mathrm{n}}^{\mathrm{a}}$ d$^{\S}$2.2$_{\mathrm{o2}}$[ı/e$_{)}^{]}$.H$^{\mathrm{e}}$ c$^{\mathrm{n}}$ n$_{\mathrm{o^t}\prime}$ be t t a$^{\mathrm{ll}}$ y a i fi$^{\mathrm{e}}$ d n$_K$/ $M$. T$_{\mathrm{r}}^{\mathrm{e}}$ e o r$_{\mathrm{e}}\mathfrak{p}\mathcal{O}^K$ $= \mathfrak{P}_K K$, w r w$^{\mathrm{a}}$ eha v $\mathfrak{p}\mathcal{O}$ r ds = n ($\mathfrak{P}\mathfrak{P}\prime\prime$)2wit$^{\mathrm{o}}$hthree$^{\mathrm{S}}$ di tnct $K_{\mathrm{prm}}^{/}$ e s d e s o$_{\mathfrak{P}}$,$\mathfrak{P}'\mathfrak{P}$ e $F \quad K \quad \mathfrak{P}$ s$^{\mathrm{i}}$ i i a , $FK_{\mathrm{H}}$ ence , l$_{\mathrm{f}}$3 s hem a an $\mathfrak{P}$sp m i in $M$p$_{,\mathrm{th}}$e$_{\mathrm{en}F}K/F$s· annra ifie d xe of egr eet hree$^{\mathrm{i}}$ T hs i s ac$^{\mathrm{i}}$ o tr ad cion . i u m e t

d . i i n i t ı

N$_{\prime \mathrm{x2}}$ o$_{\mathrm{ns}}$ d r h e c$_{\mathrm{a_s}}$ew$^{\surd}$3$^{\mathrm{e_{re}}}$3 dec om

$$(\mathfrak{P}F \quad \mathfrak{P}F).\mathrm{Si}^{\mathrm{nc}}\mathrm{e}FK \quad = F($$
$$u0)\mathrm{i}_{\mathrm{s}}$$

$F$, w esee , by T h orem 119 o [He$_{\mathrm{bracketright-comma}}$ h − t at $\mathfrak{P}F$ s l − i − t$^{\mathrm{s}}$ cm$_{\mathrm{p}}$e$^{\mathrm{t}}$ lyi n $F_K$i$^{\mathrm{f}}$

$$\mathrm{o}_{\mathrm{l}}^{\mathrm{n}}\mathrm{yi}_{\mathrm{f}} \quad \text{thecon} - \mathrm{g}u - r \overset{\mathrm{c}}{\underset{\mathrm{n}}{}} e$$
$$\text{parenleft} - \text{one}) \quad X3 \quad \equiv \quad u0(\mathrm{mod}\mathfrak{P}4)$$

Gl ($M/\mathbb{Q}$ ). Ap ply i have a s − ou$_{\mathrm{o}}$ n $N(u)X\sigma^{\mathrm{o}}$ thecon gr ence

$$Y3 \quad uminus - zero1 \quad (\mathrm{mod}\mathfrak{P}\prime).$$

ad m it a ny 3

L E M M A 2. $L$ $e-t^{M}$ $a_{nd}$ $E$ $b$ $e$ $a^{sinL}$ $emma_{\surd}$ $n-a_{d}$ $le$ $u0$ $be$ $f$ $u$ $n^{d^{ma}}$ $e^{n}$ $u$ $n^{i}o-f$ $M$ . $I_{f}$ $t$ $eh$ $c$ $a^{s}s_{n}$ $u$ $mb$ $er$ $o$ $f$ $K$ a − l po $n_{t}$ sof r − o er 3 a nd et $F = M(-vline$ . W em ay eg a rd $G$ = G a ($L$ as as ub r o p ofG L (𝔽) Si n $L$ cn a − i − $n^{3)}$ s $M($ $3\surd vline$ r l g u i n a $o_{r}$ e s $b_{g}ou_{p}$ o f G $L_{(}{}_{\mathbb{F}}3)$o − r t o n t $a_{i}$ n $S_{L}2_{\mathbb{F}}^{(}3)$ b yP r $o_{po}i^{t}$ o $f_{[}S^{e}r\cdot$ $T^{h}$ e $f^{o}$ r $m_{e}$ r $c$ $\begin{matrix}s\\e\end{matrix}$ $i-se^{i-u-q}$ v $a_{l}$ n − t t $t_{h}$ e a $s^{s}r-ei-o$ n $h_{a}$ t $E$ ad $3-_{s}$ o ge n $y^{d}efi_{n}$ ed $o_{v}$ r $M$. S $up^{p}$ o $s_{et}h-a$ t $E$ d $e^{s}n_{o}t^{ad}$ m t n y $3-i_{s}$ d e $n^{e}o_{v}er_{M}$ . T h $e^{n}G\supset$ S $L_{2}(\mathbb{F}3,_{\times}h_{i}$ $hi_{s}e^{q}u^{i}$ v l − $en_{t}to^{t}$ h e a s − s t h $a^{t}$ $G=$ GL 2($^{\mathbb{F}}$parenright − comma$_{i}^{s}$ nc e $d^{e}:G\rightarrow\mathbb{F}_{3}$ s − $i^{s}$ $u^{r-j}$ e c $i_{v}b_{y}t^{h}$ e co m m u

$d_{i}$ $ra_{m}$

$G$ G2(𝔽)

$d47-d47$ 3

R s

$\sim$ $d47-d47$

a 2 - p o

$t_{h}^{e}gr^{o}p3oo^{n}$

$G^{a-l}($ $L/$ $K=\langle\sigma,_{angbracketright-tau}$ $whe^{r-e}$ $\sigma$ = 1 0 $,\tau=11$ − 11 .

ing $abve$ $\begin{matrix},\\3\end{matrix}$ $we$ c $a_{n}$ $p^{p-l}$ y t a $g_{r}m-e$ $\begin{matrix}e&t\\fi&d\end{matrix}$ $n$ t e $r_{p}o-angbracketleft$ $\begin{matrix}f&Pr\\,&i\end{matrix}$ $o-s$ p

q $u_{d}r_{t^{ce}x}$ $e_{n}$ i − $on_{of}K$.

$\surd$ $\surd$ $vline$ $\surd$ $vline$

T h e cl $a_{s}$ s $n^{u}$ m $b^{e}s_{o}$ f on Spar $cs^{t-a}$ to n SS $4b^{u-y}$ s $i_{n}$ g KASH V e − r s on . 7 ) T $e_{r}$ efo e − r $E1$ adm

$3^{hyphen-i-s}$ ogen y d fin edo ver $k$. Wesho w hatth slead s to a $c^{n-o}$ t a dic t − i on .

pre isely , w epr ve

Pr oposi

$$= \quad ( \quad + \quad 2)$$

T hen , b y P nc h [ Pi 2 ] , t h e $j$ iva r at $j(Ei)$ o f $Ei$ $i = \ ,2)$ c an be r a s n ( 1 w

n o − f $\mathrm{h_t} j_{\mathrm{u-t-n-a-i-r-a-v-n-i}} j-) \quad = J \quad 2$ s e d

$\mathrm{w_h h_{is}}$ g $\mathrm{v^e}$ n b $\mathrm{y_r^F}$ c ke $[\mathrm{F^{r]}})\cdot$ M $\mathrm{o^r}$ o $\mathrm{e^r l_e}$ $c46$ b e $\mathrm{th^e u_s u^{lq}}$ u a

$$(E1_{)} =^{c} 4 \quad (\tau \quad 27)(\tau + \quad 3 \quad 3$$
$$\Delta \quad = \quad 1 + \quad \tau 1 \quad ,$$
$$_{2} \quad _{2} \quad _{1} \quad _{2}$$

$$j(E1 \quad - 1728 =^{c} 6_{=} \quad line - tau - one + 1$$

$E2$ ha v e ev − e rywe $\Delta$ d r $\tau$ o a nd a s u a r − e, $\mathrm{r_e s_e p^c}$ $\mathrm{c_t y^{-1}_{vel^e}}$.Thus$i$)$\mathrm{ca^n}$

$$1 \quad =_{\pi} \quad \pi \quad u, \quad 2 \quad = \pi \quad a\pi - \quad u \quad , \quad a' = \quad 0,^{6} , \quad u^{\in}\mathcal{O} \quad k\cdot$$

Consi de i g t − h e d a − $\mathrm{u^l}$ isogen y $\hat{f}$: $E \quad 2 \quad \to E \quad 1$ an dt he c n j ugat e $f :, E \quad \prime \to$ w e m ays u $\mathrm{ppo^{s-e}t_h}$ at $(a, b) \quad =_{(} \quad ,0)$ o r ( , 6 ) W e $\mathrm{i_p c^i c^u r^e d_e}$ $\mathrm{e_d v^o e^{r_a q}}$ $\mathrm{ua_d r_{a^t}}$ c d w $\mathrm{h^t} j = \quad 0$ h a t l

$$u_1 = \quad \Delta \text{ an d } u2 \quad = \quad \Delta/u, \text{ w e ob a − i} \quad t \quad = \quad \tau$$
$$\mathrm{t} \quad _{3}$$

$$(2) \quad X \quad = u1 \quad + \quad \begin{matrix} 7 \\ u \\ 2 \end{matrix}$$

I n c − a se $(a, ) = (0, \quad 6_{\mathrm{comma-parenright}}$ i − f we $\mathrm{p_u}$ t $X \quad = c4\pi \quad /(1{+}3), \mathrm{one - u} =$ $\Delta$ a $\mathrm{n_d} u2 =$ w eob $\mathrm{a_i}$ n

$$X \quad = \pi 3 u_{1+}$$

S n c e $u \overset{1}{,} \ u2 \in \mathcal{O} \quad \times k$, we hav e $X \in \mathcal{O}k$ in b o h c a − $\mathrm{s_e s}\cdot$

L

$$\mathbb{Z}/9_{\mathbb{Z}}.In \quad atcu_{la}$$
$$c_u \quad e \quad md^{o} \quad \pi.$$

L M $ma_4$ $E$ $q$ $a$ $o$ $s^{(}$ $)$ $n$ ( 3 ) $a$ $e$ $n$ $o$ $so$ $l$ $t^{o}$ $ns$ .

Pr of . W e $\mathrm{pr^{o-v}}$ e he ass erion n $\mathrm{y_l}$ f requ ti o two − parenleft) s nc e a s

proofw or k

S p $\mathrm{o^a_{seth}}$ tt $\mathrm{h^e}$ r ex i t $X \in_{\mathcal{O}} \quad k$ an d $u^{u}_{1,} \quad 2 \in$

L e 1 c b w h e

$$+1 = \quad \pi 3 - a\pi \qquad , b$$

$$\mathrm{w_h} \quad \mathrm{n\ c}$$

$$\pi \quad 2^b 3 + 3\pi^{\pi} u3 \quad + \quad 3 = - \quad \prime 3$$
$$\mathrm{Wi-t}$$

$$0, \quad \mathrm{a_n^{d}}a \quad = \quad \mathrm{i_m^m e_{i^t}\ l\ l^{e_{ad}} s^{to}} \quad \mathrm{a_c} \quad \mathrm{r_d} \quad \mathrm{c_t\ o\ .^T} \quad \mathrm{e\ r\ m\ a\ n\ g\ sid}$$

$$\mathbb{Q}(_4) = 33\mathrm{T} - \mathrm{r}_k/\mathbb{Q}^{(u}3) + \quad (_3 + 35_N k/\mathbb{Q}_{(u3}$$
$$+ \quad (36 + 1 + 33Nk\mathbb{Q}^{(}u3)$$

For l p s ilesin s o f theno r m , Tr ( $u$) can t b erto nal , acon d c io n . $\mathrm{o_s}$ b g s $k/_{\mathbb{Q}}$ n o a i

$$\mathrm{i^t}$$

p i on t a t $E1$ admi ts a 3 a $\mathrm{c^{on}r^ad^ict_o}$ .n T $\mathrm{i^s}$ $c - \mathrm{o_m}$ p $\mathrm{l^{ee}}$ s $\mathrm{h_{epr}}$ o o f f Pr o p $\mathrm{o_i}$ i o $\mathrm{n_{2,}}$ $\mathrm{a_n}$ d $\mathrm{he^n}$ c e
$\mathrm{P_o^r}$ p s

**5 .** W e n ow $\mathrm{d^{e-t}}$ ermin e $E_n \pm \mathcal{O}_k$period − parenright

$$+ \qquad \sim$$
$$P^r \quad o_{p^o} s$$

P ro o f − period We $\mathrm{fir^{t-s}}$ c a − $\mathrm{lu_a}$ l tet $\mathrm{h_e}$ $\mathrm{fi_{n^{edo}}}$ ve $\mathbb{Q}$ h $\mathrm{nt_hr_{an}}$ k o f $E_(\mathbb{Q}$ ( *mvline* is ac u t ed f r t h e r

$$\surd$$

$$\mathrm{r\ an^k} \quad E_{(\mathbb{Q}(} \quad m_) \quad = \mathrm{r} \quad \mathrm{n_k} E^{(\mathbb{Q})} \quad + \quad \mathrm{a\ n\ k}\ E \quad (\mathbb{Q}),$$

w h re $Em$ is t e q uad$^{\mathrm{r-a}}$i − $\mathrm{t^c tw^{i-s}}$ tby $m$ ( or a pr o f − o see R o parenright − period e − L t $E$ $E0$ o r $\mathrm{i^t s^{t-w}}$ i s t $(E0)^3$) a n l t $L(E^/ \quad \mathbb{Q},)$
$\mathrm{f^{nct}}$ o o S $\mathrm{c^e}$ $E$ h a c $\mathrm{l^e}$ i $\mathrm{n^b}$ y $\mathbb{Z}$ (+
a d {1214 ... i $E$ $= E+$ ,

$$L - parenleftE/\mathbb{Q}1) = \quad 394^3 1... \quad \mathrm{if^f} \quad E = (0+)(\mathrm{seven} - \mathrm{three}$$

$$\mathrm{W\ ies\ [\ C\ l\ W\ ,a - byk - nS}_E\mathrm{IMA}_{(\mathbb{Q})} = \mathrm{T} \quad .^{\mathrm{V}} \quad \mathrm{T^r r^{o_i^e}} 9^{\mathrm{eo^{e-r}}}_{\mathrm{n^{three-period}}} \quad \mathrm{r\ an}$$
$\mathrm{N^e x,}$ w e $\mathrm{o_m put^t}$ he $\mathrm{t_r}$ s $\mathrm{i^{on}}$ s + b g o p L e $p$ b a p $\mathrm{r^i}$ e d
$\mathrm{a\ b\ o_{ve}a^{p_r}}$ i m $\mathrm{e^{nu}}$ m $\mathrm{b^e}$ $p$ n l e $(E^0$ $)p$ b e $\mathrm{t_r^e}$ e d u t

$$\mathrm{weav}\#_{,\mathrm{e\,y}}(_{\mathrm{T}}E0^{)}\mathfrak{ph}^{(}_{\mathfrak{ce}}$$
$$\mathcal{O}k^{/}_{\mathrm{m1f}}{}^{\mathfrak{p-seven}}) = \frac{2^{2^{,}}}{[}{}_{\mathrm{MSZ,}}\#(_E E^{0)_{\mathfrak{p}}1(k/\mathfrak{p}4\mathrm{parenright-one}=2}_{+(k)\mathrm{tors}\leq 2} \quad .\mathrm{This} \cdot 3 \cdot 7\mathrm{om,pe}$$

o o .

4 r $t_{oL-fu}$

hesam epap$^{r-e}$, we kno$^{combiningthis\ w-ithte}_{oma\ r-e^{s}_{u}\ tin[Sa]wi}$wthatthe$^{hyphen-three}$prima − rypat o$^{resulto-f[R_{ucomma-bracketright}inwl}_{mai^{houtu}sig\ the\ L^{f-hyphen}unct}$ v a lue o − f th

$L$ hyphen − f unc t i nap pea s , wes ee that t h $e_o$ rder of X

Lemma $._L$ *et* $u1, u2$ *tand fo un tsin k a nd A for an* $n^i$ *e er in k Th n*

( a ) *T hee q u io n* $64u1 + u2 = A_2$ *h as no ou t n*

b ) *T e so uti*$^a$ *o nso th e q ut o n* $8u_1 + u =^l A2$ *are*

$e \quad a_2 \quad 2 \quad 2 \quad \times$

$$A = w \quad 2, \quad 3 \quad ) \quad w \in \mathcal{O}k$$

(

d ) $T \quad u^t \quad s^o \quad e \quad 23^o n_{2\prime}3 \quad 2 \quad = \quad A$

$$(u \quad , u_2 A \quad ) \quad = \quad (w \quad - w_{,0}\,, (w \quad \varepsilon \quad , w_\varepsilon \quad , \pm_4 w^2 \text{parenright} - \text{comma} \quad w\varepsilon$$

P oo $._(\,$ a $)^i$ s a $p_{e_c}$ a $l_a$ e $f^L$ m m a 2.1 $o^f_{lh}$

s p $of_i$ st h $a^t$ 4 i s d $v_i i_b$ e $by_4$. e $c_e()^c$ n ) ( d ) I *As* $\overset{0}{\neq}$ , t h en

P r o p o s − $it^n_{o-i}$ n $2^8$ o f $1_{[}$C − o] im $s_p$ *estha* $\overset{t \quad 2}{t}$

$$x^2 x \in \mathbb{Z} \quad .$$

W m a s p p o s t h a − t $u1^i$ sp o ti v , anh e n c e $u0 = \varepsilon n$ fo s me $n \in \mathbb{Z}$. B T h − eore$^y$ m 1 u of [ KT ], r − $Tk/\mathbb{Q}(\varepsilon^n)i_{=}2$ oldso nl y or $n = 3, x = \pm 2$.

Pr oposition 4 .

$$E + (\mathcal{O}k) = \{(-12\varepsilon, 0), 12588 - \varepsilon^{-}3, \pm 3042196 + -3_{))} \quad .$$

Proo f . F ac o r

3 $\sqrt{\quad\sqrt{\quad}}$ , w e

$$+^{=} \quad 43 \quad ) \quad - 2^4$$

m $E3 \quad \mathcal{O}$ , w u h e o l w

a) $\mathcal{O}L = \mathcal{O}k \quad \oplus \mathcal{O}_k vline \quad \surd$

$N \quad (1_{)} = . \quad$ uf

$L$ ( c )$\varepsilon_2$, an d $\pi\prime_{deco}$ mp oea s (2 = $\mathfrak{P} \quad \pi( \quad ) = \mathfrak{P}$two − $a_n \quad (\prime) = \mathfrak{P}\prime 2$ ( d ) T e l f $L \quad 2\cdot$

$$y \pm \quad 24\varepsilon \quad 3\varepsilon) \qquad \text{can}$$

$$\mathfrak{A} \quad = \quad \mathfrak{P}2^2 3\mathfrak{P} \quad 3 \quad , \quad \leq \quad a_{2,}3,3 \quad <_3 .$$

Si n − c − equal − $e_{\mathfrak{B}}$ a d ( c ) wes e − e th a − t $\mathfrak{A} = \mathfrak{B}$. Moreover , since $\mathfrak{AB}$ sa c be , w h a e $a2 = a3 = a3 = 0$. H ence u

$$\begin{array}{c} \text{v} \quad \surd \\ + \quad \surd \end{array}$$

$$\text{B}y2 \underline{\qquad} \surd \quad \text{d} ( \quad ) \quad \surd \quad vline_{3\text{r}}^{\text{w}} \text{t} \quad = \quad a_{\times} \quad 3 \quad )^{\text{w}} \quad \text{it h}_a \quad \in \quad \mathcal{O}l_{,m} \quad \text{n}$$

$$+4_3\varepsilon =^{\eta} \quad (+ \quad 3 \quad 3\varepsilon$$
$$\text{el s } 1 \quad \text{n} \quad \text{o}$$

$$x = \varepsilon\text{two} - \text{l}a - parenleft\{ \quad +b \quad line - three - epsilon - parenright(a \quad - b^{\surd} \quad 3\varepsilon\} \quad , \qquad )$$

$$\text{w h e}^{\text{nc}} \quad \text{e } l = 0 \quad \text{a}^{\text{n}} \text{ d} \quad \surd$$

$$y + \quad 2 \quad 4$$
$$\text{n} \quad \text{k n} \quad \text{c on u gtion} \quad \text{i l s } 3$$

$$- \quad 3\varepsilon = \quad \varepsilon^1 \quad 3)$$
$$\text{r} \quad \text{f}^{\text{o}} \quad \text{t} \quad \text{s} \quad \sqrt{\surd} \quad \text{n} \quad \text{o} \quad \surd^{\text{e}} \quad \text{t h}$$

$$3\varepsilon = \varepsilon_1 \text{parenleft} - \text{a} + b \quad \varepsilon), \quad comma - a_{b\surd}^{\text{i}}$$
$$\text{C} \quad a_s \text{ E} \quad 1 :^m = \quad 1 \text{ E} \quad \text{qu}_{\text{t}}^{\text{a}} \text{ i n}_{\text{t}}^{\text{g}} \text{h}_{\text{e}} \text{c}_{\text{o}_{\text{e}}}^{\text{ffi}} \quad \text{c}^{\text{e}}$$

$$2^a 3 \quad + \varepsilon ab \quad + \quad 1 \quad a_b + \quad 3b \quad = \quad \begin{array}{c} 4 \\ \varepsilon \end{array}$$

W e s$^{\text{e}}$ t h$^{\text{a}}$t$_a$ i − s$_{\text{d}^{\text{ivi}}}^{\text{s}}$ b l e$_{\text{y}}^{\text{b}}$3, w$_{\text{h}}$e$_{\text{n}^{\text{c}}\text{e}}$ $\varepsilon^b minus - equivalence1(\text{m}_{\text{o}}\ m =^{\text{a}}$ . Eq a n gthe e ffi e st yie l

$$() \quad 8\varepsilon = ba^2 + \varepsilon b2, \quad y \quad = (a2 + \quad 9\varepsilon 2).$$

$$\text{F}_{\text{ro}} \text{ m} \quad \text{t fi}^{\text{e}} \text{r}_{\text{s}} \quad \text{e}_{\text{q}^{\text{uato}}} \text{ i} \quad \text{n o}_{(}^{\text{f}} 5) \text{w} - \text{eh}_{\text{a}} \quad \text{v}_{\text{e}} b^{=u,}$$
$$\text{os lo u ton s y i}_{\text{Le}}^{\text{s}} \quad \text{a } 5 \text{ ( a . If } b = u,' \text{ t h}$$

tha t $u3 = \quad -1$, wh ch c o n ra i$_{\text{c}-\text{t}}$ s $u \quad > .\text{f} - \text{I}_b = 4u$, th en $a2 = -16\varepsilon u2 +_2$ wh c hhas noso u it o ns b − y Le m m 5c − parenleft.$_)$ I $b \quad = 2u \; th$ n e

$$(a$$
$$()_6 \quad )2$$
$$vline \quad = \quad \varepsilon u - 1 - \varepsilon u2.$$

e − e h − t t ( 6 ) o d so n coesp n di n g va ues of $y$ r e 8 $, \pm^{\varepsilon} 30 \quad 41^{\text{B}} 96 +$ − ), r e p$^{\text{q}}$ c t vel . ($^5$,

r r o l a 0 2 ( e y

$$3 \quad ), \quad (\text{five} - \text{seven2}$$

Proo f $\quad \mathrm{Lt}_{L} \quad = k_{)}^{3}$. T $\quad$ o $\quad$ provet $\mathrm{h_e} \quad \mathrm{p} - \mathrm{r\ o\ p_o i} - \mathrm{s\ t\ i\ o^w_{n,}} \quad \mathrm{e\ u\ s\ t\ h\ efo\ l^w_o\ a}$
) $\quad \mathcal{O} \quad = \mathcal{O}\ \mathrm{a} \oplus \quad \mathcal{O} \quad \zeta$, w $\quad \mathrm{h\ r} \quad \mathrm{e}\ \zeta := (1 \quad + \quad \surd$

$$\mathrm{bparenright} - \mathrm{O} \times = \langle \varepsilon \rangle \times \zeta \rangle =\sim \mathbb{Z} \quad \oplus \quad \mathbb{Z}/6\mathrm{period} - \mathrm{Z} \quad -)/^{2\cdot}$$

$$(\mathrm{c}_{)}2, \quad \pi \mathrm{a^n\ d}\ \pi^{prime-d}\ \mathrm{e\ c_o m_p\ s\ e_a\ s}$$

$(_{)} \quad = \mathfrak{P} \quad 2^{\mathfrak{P}} \quad 2^{(} \quad \mathrm{two} - \mathrm{P} \quad \neq \quad \mathfrak{P}2), (\pi) = \quad \mathfrak{P}$

$$(prime - parenright = \mathfrak{P}\prime 3.$$

$$)^{\mathrm{d}}\ \mathrm{T} \quad \mathrm{h\ d_e a_l c^a s^s\ g\ r\ u\ p\ i\ a\ c} - \mathrm{y} - \mathrm{c\ li\ cg_r}$$

$$\mathrm{l} - \mathrm{a^s o} \quad \mathrm{four} - \mathrm{P2}$$

$$\mathrm{e}) \quad \mathfrak{P} \quad = \mathrm{parenleft} - \mathrm{one} \quad +^{\omega} \quad -3\zeta.$$

A

oso ve

$$\mathrm{f\ o} \quad \mathrm{r}_{a2})^{\cdot} = (0,0), (, 1\mathrm{parenright} - \mathrm{comma} \quad y \in \mathcal{O} \quad k \text{ a ndan} \quad \text{in egra i}_{\mathrm{d}}\mathrm{a} - \mathrm{e\ l} \quad \mathfrak{C} \text{ o f } L.$$

CA

2

$$L^{\mathrm{i}-\mathrm{s}} \text{ p r me o} - \mathrm{t3, w\ e}_{\mathrm{s}}^{y}\mathrm{e}^{+}\mathrm{eh}^{4}\mathrm{a} - \mathrm{t}$$

$$\mathrm{b} \quad \mathrm{y}()\mathrm{an_d} \quad (_{\mathrm{b}}), \quad \pm y \quad + 2 - 3 = \varepsilon m \zeta n_{(a+} \quad b\zeta 3), a, b \in \quad O - k,$$

$$n_{=} \quad 0,^{\pm} \quad 1. \quad \mathrm{aT_k} i \overset{\mathrm{g}}{\underset{\mathrm{n}}{}} th \quad \mathrm{n_o} \quad \mathrm{m_{fro}\ m}\ L \quad \mathrm{t} - \mathrm{o} \quad k\mathrm{f} - \mathrm{o^b\ o} \quad \mathrm{h\ id\ s,w_e o_b\ t\ an} \quad m \quad =$$
$$\mathrm{a\ d} \quad \mathrm{c\ o_s\ i\ d^e i} - \mathrm{ng_t\ h\ eo\ n\ j_g a_t, w\ e\ m\ ay} \quad \mathrm{s\ p\ po} \quad \mathrm{e\ h\ a_t} n \quad = 0\mathrm{o_r} \quad 1$$
$$\mathrm{I^f} n = 0, \quad \mathrm{e_q\ a\ i} - \mathrm{n\ g\ h} - \mathrm{t^e o^c e^{ffi-c}\ ie\ n\ t\ g\ ve\ s}$$
$$1$$

$$(\mathrm{seven} - \mathrm{parenright} \quad \pm y_{=} \quad a - b\mathrm{parenleft} - \mathrm{parenright2}a \quad + b)(a + 2 \quad \overset{,}{\underset{b}{}}$$

2

$$\mathrm{i} - \mathrm{n}$$

$$(u28u - 1\mathrm{parenright} - \mathrm{comma}(4$$

f r $\quad$ s me unit $u$

$$\mathrm{o\ t_h\ e\ q} \quad \mathrm{a^u\ r\ a_t\ i\ c} \quad \mathrm{p\ l\ y_n} \quad \mathrm{o\ m_{i_a}\ l}$$

$$X \quad - 4_u X \quad + 4u \quad -^{\cdot}$$

T e $\quad$ dscrimi nantofthe

$$u, -u \quad ) = (_{w}^{;} -$$

$$\mathrm{e\ con^T dcasel^h\ ea\ dst\ o}\ w^{\mathrm{d}}2 \quad = u\varepsilon,^{=} \text{ a c o n tra i c} - \mathrm{t\ o\ ,_{n.}\ If^w}(a + b, a\mathrm{parenright} - \mathrm{b} \quad = (2u, 8u$$
$$\mathrm{t\ h\ en\ t} - \mathrm{h_{equa}\ r\ at\ i\ c\ pol\ y\ no} \quad \mathrm{i\ a\ as\ t\ s} - \mathrm{i_{fi}\ edb} \quad y \ a \text{ an d } b \text{ i s}$$
$$2 \quad - \quad 1$$

$$X \quad -2u \quad X \quad + \quad ,$$

a e

re p ecti v − e ly , no n eof w i − $h_{ch}$ i s asqu re b Lem m 5parenleft − acomma − parenright$_{(c.}$ I $n$ = 1 t h e n ew o b ain a y a

f , 3 2 3

a$^{t}a$ $\equiv_b$ ( 3d$^{3)}$. Le$^{ig}_{t}$ = 3 $A + b^{\cdot A}$

o u o t$^{a}$ n $b$ d ) h c o r a

$$Ca \quad e^2 : a2$$

$$2^{1})\mathrm{M} \quad \mathrm{u_l} \quad \mathrm{l^y nb_o}$$

$$\mathrm{e} \quad vline \quad \surd \quad 4vline \quad 3 \quad vline \quad 3$$

$$( \quad 4)(\pm^{y} + - \quad 3) \quad = \mathfrak{P}^2( \quad 2 \quad \mathfrak{C}_{)} \quad = ( \quad 1 + \quad \omega \quad - \quad \zeta_{)(\mathfrak{P}} \quad \mathfrak{C}$$

whe$^{\mathrm{n}}$ c, yb( ,

$$\surd$$

$$4(\pm y + \quad 2 - 4$$

= 0, h e eq u − a − $t_{\mathrm{in}}$ gt e − h c o − $e_{\mathrm{ffi}}$ c nt s y i d

) 9

10) ± four − y − 96 = $(\omega + 1a \quad + 9ab -$ 3$(\omega \quad 2)a2 - (\omega \quad + 1b_3$.

As w e w i lsee l − a ter , h − t e sol tio$^{\mathrm{n-s}}$ of ( 9 are t

$$(- \quad , 0),$$

$$\surd \quad vline \quad \surd \quad \surd \quad vline \quad \surd \quad vline \quad \surd \quad vline \quad \surd$$

$$($$

$$37, -_{\surd}2_{\surd}37),(-^{2}_{\surd} \quad 3 \quad 7 \quad 3 + \quad \surd^{3}7, (_3 + \quad 37^{-}_{\surd}$$

$$( \quad -40-_3 \quad , \surd \quad 37_{)}$$

(−2 3three − plus$comma - parenright$(− 1 − $\surd 37, 2)(3 + \quad \surd \quad 3comma - minus1 \quad - \quad 7),$

$$(-\mathrm{plus} - \mathrm{three7,2parenleft} - \mathrm{comma1} - \quad \surd \quad , 37 - 3 + \quad \surd 37_{),} \quad (2, 1 - \surd 3_7,$$

$$-9 - 3 \quad , 6 \quad + \surd_3, \quad (1 \quad 6 + 2\surd 3_7, 3 + \surd 37_{),}$$

$$(-163_{\mathrm{plus-seven}} \quad 29 - radical - three_3), (\mathrm{three} - \mathrm{plus}\surd \quad 37, 1\mathrm{plus} - \mathrm{six}_2$$

Subst tu n gthe m i n ( 0 , we g e a lthe v l esof $y$ e x e p t . fI $n = 1$ or $n = -1$, hen w o t a − i n

$$192 \quad = \quad (-2 + \quad \omega)a3 \quad + 3(1 \quad + \omega)ab \quad + 9ab +$$

$$1 \quad 92 = (+ \quad \omega)a3 +$$

c ti le y . T e y a r s h o w n o b$^{b}$ e m pos$^{+}$ l es m l a r$^{-}$ y

C a *se* .

1

R E M R

. $y_{2-e_{s^e}}$

i$_{s}$ th eonly xa mpe i nt he i$^{t-e}$ r ture where a th e T h u equation 9 ) a flows . B y m$_{\cdot}^{ta-t-i-n}$ $=_{a}^{ofe-w}$ $\omega$ n

$$A3 + (4k_{\omega +} \quad 4A \quad 2b + \quad (16\omega + 48)A \quad b + \quad 32\omega \quad + 80)^{3} \quad = 64. \qquad \text{have}$$

(

I t$_{ie}$ a s$^{y}$ t$_{s}^{o}$ e e$^{t}$ h at $4_{|}A$ a n

$$X3_{+}2(\omega + 1)X2^{Y} \quad + \quad 4\omega \quad + \quad 3_{)} \quad XY_{2} \quad + 2\text{parenleft} - \text{two} +_{\omega} \quad \text{parenright} - \text{five}Y3^{=}1. \tag{11}$$

i r R $i^{t}$ O 6 . $o\ n\ y$ $X$ $\in^{\mathcal{O}}$ $k\times$ $s\ i$ $n\ g$ $a$

$$({}^{2}-9\omega \quad 2 \quad -^{4^{\omega})} \, (-2 \quad -8\omega,^{-^{+}}8 \quad \omega^{)} \quad 2 \quad +^{1^{7}} \quad -$$
$$(9 + 2\omega, 1 - 2\omega) \quad (-3 - \quad \omega, -2$$
$$(-5 - \quad 2, 1 \quad +\omega, \quad (1 + \quad \omega - 1)(4 + omega - comma_{-} \quad \omega),$$
$$-2 - \quad \omega, ), (1, \quad ), (1 + \omega,_{-} 2)$$
$$( \quad +_{\omega}, 1 - \omega), \quad -\omega, ) \quad (_{-}3 - 2 + \quad \omega,$$
$$(7 - \quad 2\omega 11 - \quad 3\omega\text{parenright} - \text{comma}(1 \quad +, -9 + \quad \omega), (8 \quad +\omega, -_{2}$$

P r oo . L e $F$ $(X, Y)$ b e t e l $-$ e t had o ly o m$_{a}$ l $F$ $(X \quad , 1_{)}$ a n d l $-$ e$_{t}$ $L =$ $\mathcal{O}^{L} = \mathbb{Z}$ bracketleft $-$ xi] w h$_{4}$ r e 5 $=$ ( $2\sqrt{} + \quad 1\theta - 4\theta$ $\xi^{2}$ $4_{\xi}$ + + a n d $vline_{7} = 3 \quad - 12\xi \quad - 8\xi \quad 2 + 8\xi \quad 3_{+} \quad 2$ 4. Th ex ten $L$ /$\mathbb{Q}$ s Gais w thG alo s o p $\langle \sigma, \quad$ , whe e $\sigma$ n d $\tau$ a r$^{\xi}$ gi n b

i l o i g r 2 r$_{3}$ a 4 e$_{5}$ v y

$$\sigma\xi) = \quad -14 - 6 + \quad 49 \quad + \quad \xi \quad - \quad 8\xi \quad - \quad \xi,$$
$$tau - parenleft_{\xi}) = \quad -1 \quad -3\xi + 5\xi \quad + 43 - 4\xi 4 - \xi 5,$$

t sf y $\sigma^{3} \quad = 1, \tau = 1$ and $\sigma\tau = \tau\sigma$. Th s Gal ( m et r c$^{g}$ o u o fd e $-$ g ree period $-$ three T h c onj a te$_{s}$

$$\text{a}_{\text{f}_{\text{o}}}^{\text{s}} \text{lows :} \quad 1)$$
$$\xi_{(} \quad = \xi_{=} \quad - \quad \text{four} - \text{period}6017164..., \quad \xi) = 2(\xi) = \quad -04112$$
$$\xi) = \quad (\xi) = \quad -1^{period-two}776453$$

$$\mathrm{o}-\mathrm{f}\ \mathrm{fu}\ \mathrm{nd}\ \mathrm{am}\ \mathrm{e}\ \mathrm{n}$$

$$= -\xi,$$
$$= -5 - \quad 4\xi \quad + 1^{8}\xi 2 \quad + 5\xi 3 - \quad 9\xi$$
$$\varepsilon^{3} = - \quad 6\_8\xi + \quad 23\xi \quad +_{9} \xi - \quad 1_{3}^{\xi} \quad - \quad 3,$$
$$epsilon - four = 1 + \quad 3\xi - \quad 5\xi - 4\xi 3^{+} 4\xi 4^{+} \xi 5,$$
$$5 =$$

$$\mathrm{s_o}\sigma_{\mathrm{f}} \quad \mathrm{a\ n\ d}\ \tau\ \mathrm{o\ n\ heu} \quad \mathrm{n-i\ sar\ e\ s-a\ o\ l\ ws}:$$

$$\varepsilon_{3}1 \quad \mathrm{i}i = \quad , \quad braceexbraceex4$$
$$4 \quad -1 \quad \mathrm{i^{f}}i = 2,$$
$$\varepsilon_{2}\varepsilon_{4} \quad \mathrm{i-f}i = 4, \quad i \quad braceleftbt \, , \varepsilon 1$$
$$braceleftbt 1\varepsilon 2\varepsilon 31^{4}5 \quad \mathrm{i}^{i} = \quad 5,$$

$$\mathrm{an_dS-a-two-N-i}L^{\mathrm{c-n}_{\varepsilon_3^{a}}\mathrm{e}_3}{}_{/k^{(\varepsilon}_{(\sigma_{(\theta)-}}} \quad \sigma \quad a_{\mathrm{val}}^{\mathrm{li})}\mathrm{m}_{\eta}^{\mathrm{e}_1}{}_{\mathrm{n}_{+}^{\mathrm{i}}}^{\mathrm{t^n}}(\dot{\sigma}^{\mathrm{t_i\,t_o}} \quad X, Y \quad \mathrm{w_e}eta - e - Y - o\mathrm{bt}_{+}^{\theta)\mathrm{th}}(\theta\_\sigma(^{\theta)}\sigma^{2\eta}$$
$$(i_{:=}^{=} =_0 1_{X}^{,}{}_{2}^{-}$$

$$\mathrm{h\ e\ nc\ e} \quad \theta - \sigma 2\theta \quad )$$

$$\sigma_{vline}\eta) \quad -1 = -\sigma \mathrm{parenleft} - \mathrm{line}^{\theta}) - \sigma^{2}(\theta) \quad \cdot \quad \eta$$
$$\mathrm{a} \quad \mathrm{t^y} \quad \varepsilon_{-1}\varepsilon 23^{3}\varepsilon\varepsilon_{4}^{-1=\varepsilon 1}{}_{d}{}_{1}24_{d}^{3^{3}\varepsilon^{4}},$$
$$\mathrm{wh\ ere}$$

$$b_2 = a^2 + 2a - 1 \quad b_3 = -2a_1$$

$$\mathrm{A\ s\ in}_{[\mathrm{K}} \quad ],[^{\mathrm{T}}\mathrm{d_W}][\mathrm{d_W}1]_{\mathrm{o}} \quad [\mathrm{d}\ \mathrm{W}\ ], \mathrm{w_e}\ \mathrm{s}\ \mathrm{t_i m - at_e} \quad \mathrm{n_a\ r\ o\ rm_s\ i\ n_h}$$
$$\mathrm{r\ i_t} \quad \mathrm{s} \quad braceleftttp$$
$$\sum 4 \quad braceex - braceex - braceleftmid\ \mathrm{l\ o} \quad \mathrm{g}\ \theta \quad - \quad \sigma(_{\theta} \quad \sigma(\eta i))vextendsinglevextendsingle$$
$$single_{\log vextendsingle_{\theta}\theta(}^{braceex-braceex-braceleftbt^{braceex}} \quad vextendsingle \quad i) \quad -\sigma_{\sigma}\mathrm{thinde_x-such\ t\ h} \quad (_{(i)}^{(_{\mathrm{t}|\eta}^{\theta_{\theta}i)}i} \quad \sigma_{\cdot\sigma line-two(}^{0)|}\eta_{\mathrm{equal-parenleft}} \quad )_{\mathrm{m}} vextendsingle_{\leq_{\mathrm{n}}}^{vextend}$$

in se h at $i0 =$

$$g \quad t \; t \quad d \quad t \quad s \quad e \quad \mathrm{fi}^{\mathrm{n_d}} \quad 18$$

$$o \quad 1 \left| > \right. \quad -4 \quad 0 \quad \cdot 10 \quad o \; g \; 21$$

o A p pl ing P $\mathrm{rp}^{\mathrm{b}}\mathrm{os}^{\mathrm{u}}$ it n 3 1 . o [ T $\mathrm{d}^{B}$ W $\leq_{]}$ t o 1 o u r c a 1 se b $\mathrm{y_t}$ a k − i n g th param et

$c0$ appe a r n g he eto be $10^{\mathrm{one-zero0}}$, w eg e t am u hs mal rb o nd $B^{\mathrm{e}} \leq 719$. W

a − g ai ap pl t e s m e p r o p os ion

a n g $\mathrm{e_f}$ o s l $\mathrm{u_t}$ i o ns o f ( 1 2 ) a nd fi w h i hc $\mathrm{g_{iv}e^{i}_{nte}r_{al}}(1, a23, a)$ o o $\mathrm{w^{in_g}}$ t h $\mathrm{e^{ar}}$ $\mathrm{h^{c}_{se^{a}_{r}}}$ h t $\mathrm{a^{s}_{kel^{es}}}$ $\mathrm{t_{h^{a}}}$ n 1 m n $\mathrm{u_e}$ o $\mathrm{n_S pa^r}$ csta $\mathrm{i_o}$ S $\mathrm{S^{4}_{wi}1^{h}}a2^{a}3\mathrm{p}^{a-four}$ g $\mathrm{r_a}$ m

F o $\mathrm{re^{ac}h}(_{a}, \frac{a}{2}, a3, 4_{)}$ $\mathrm{w^e}$ s e w t h K S H t $\mathrm{a^{t}th_e}$ $\mathrm{n_{it}}\varepsilon 12\varepsilon 3^{\varepsilon}4$ $\mathrm{s_o}$

t $\mathrm{h_e}$ o r m $X^{-}$ $Y\theta \cdot$ W e l $\mathrm{i^{t}t_h}$ e s $\mathrm{l_{u}i^{n^{s}}_{o}}$ i n $\mathrm{T_a}$ b e $1 \cdot$

*vline*

T a b $l - e1$. T he so uions $f - o(11)$ a d $(1two - parenright$

$$l_t$$

$$a - one \quad two - a \quad a3 \quad a^4 \quad b_1 \quad b - two \quad 3 \quad four - bvline \quad X$$

$$44 \quad 0 \quad 8 \quad 3 \quad -3 \quad -8 \quad -two - three - 8\omega$$

$$-$$

$$4 \quad 1 \quad 4 \quad 4 \quad 0 \quad 3 \quad 2 \quad -$$

$$-$$

$$- \quad 0 \quad -9 - \quad 1$$

$$- \quad - \quad -_1 2 - \quad 7+$$

$$3 \quad - \quad - \quad - \quad 2 \quad 9+ \quad 1-$$

$$- \quad 2 \quad 3 \quad -3 \quad -2$$

$$- \quad -6 \quad 1$$

$$- \quad 1 \quad -5 - \quad 1$$

$$- \quad 0 \quad - \quad 1$$

$$2 \quad 4 \quad -$$

$$- \quad - \quad 1 \quad -2$$

$$0 \quad - \quad 1$$

$$1 \quad 1$$

$$1 \quad 2 \quad 3 \quad 1$$

$$0 \quad 1 \quad -$$

$$1 \quad 2$$

$$1 \quad 3 \quad 7 - \quad 1-$$

$$3 \quad 1 \quad 1 \quad -_9 +$$

$$0 \quad 1 \quad -8 \quad 2$$

$xy$ ) I $\mathrm{t^a}$ t r − n s t ou ttha o t nyt $\mathrm{h^o}$ efo owin gtwo$^{\mathrm{s}}$ s sa e ts yt e $\mathrm{n_c}$ on d $\mathrm{in}^4$ sof h

( , . u t l l if h i t

o S h i[m] ura ’ s ter

$$Y = X - 2_7x - X - 5_4y$$

t hnt h ea b o – v eg $t_i$ e s an[a]eipt[o] c cu[v]rve[e]tw ih go or edu tio no ∈ uts d e 2 .

algor th m ove quadra ticfi[l] e l d s i – s imple m nte dbyA . U m eg kio n S pa c – r

$$39 \overset{b}{,} \; ut \text{ t do e – s n t w r – o}$$

Th u – s wec o m l – p e t e th eproo fof Theor em .

**A ckn o wle dg m e nts .** W ew u ld lket oexpress ourt ha n k s toPr o

K . Hash moto fo r into d u cig uto th ss u b e – c t to K . A oki f r hs m ation a b – o utvar usn um b e rt h o iysof tw r esuch’ a K ASH and S IM

whch wer ein d i s p esa b efor thi w or k . W ea so t h ank A . U meg a s – e v r – e a usful d scu s – i ons .

*Re*$_{fe}$ r e ces

[*BW*] *A*· *Bke*$_r$ *a*

*a* – *b* eian v *r* – *a* ie *C*$_W$] J . Coa *e*$_s$*a* – *n* d A . W i es , On *h* – *t* e con *j* – *e* c ure of B *r* – *i* cha n d *S*$_w$*i*$_n$*n* – *e* r *o* – *t* n

$$ii - d.3_9(197), 22_3 - -2_5period - one$$

*bracketleft* – *C* ] S . C o mal d , E *i* – *lp* – *tc* – *i* c rve with ri ial co n uctoro *e* – *r* qu *r* – *d* atc *e*$_l$ s , .*M*$_a$ th .

*D*$_R$] P . D e *ig*$_n$ ee M . Ra po p or *t* – *comma* Le s ch *é* m asdem *odulesd*$_e$ sco r es *eli*[p] n : Modu *la*$_r$ *Funct*$_i$ nsof O n V ara b *eII*, L *ec*$_t$

$$1_{97}, 14endash - three_1period - six$$

[*Fr*] R . *Fri*$_c$ k e , Di e

L *i*$_p$*z* – *i*$_{g,}$1922. *H*$_a$*one* – *bracketright* Y Ha s e g aw *comma* – *a*$\mathbb{Q}$$_-$*cur*$_v$ *H*$_a$*two* – *bracketright* — , T ab *e*$_o$ c u p f *r* – *o*[ms] o n Γ( N ) wit *r* – *el* – *a* qua dr tc h *a* – *r* acte rs , u *npubl*$_i$ H M ] Y . H aseg a wa K . Ha shim ot o a nd F . M o m o se *Mod*$_u$ ar c *o*$_{n-j}$ e *t* – *c* ure

*cu*$_r$*ves*$_a$

*bracketleft* – *H* ] E .*H*$_e$ k e Le ctures on heT heoryo A *g* – *l* brai Num *b* – *e* rs , G *a* – *r*$_d$ .*Tex*$_{t-s}$*i*$_n$ 7 7 , S *p*$_r$ [*I*] H . Ih *i*$_i$ T he non - exs ence of e *l* – *l*[p]*i* – *t* c *c* – *u* rves withe *e* – *v* ryw h *e*$_{r-e}$ g *o* – *o*[d]*r*$_e$

o v r *c* – *e* rtain *u* – *q* a d *r* – *ai*$_c$*e* – *fi* l *s* – *comma* Ja pan . J . M ath .12(1986),$_4$ 5 – 52 .

$$@ \quad o-n \quad f-o \quad S-hi$$

A r th 77 $1^{1}_{96)}$ 5 – 171 [ K $i^{2}_{]}$ — $,pivat^{e}$ $c_{o}$ m u $n_{ic}a_{t}$ o n $[K-K_{]}$ M K $i_{d}a_{andT}$ $K_{a}$ a w $a_{,}N^{o}_{n}$ $i-x_{s}t-en^{c}$ e o f $el^{l}_{i^{p}}{}^{t}$ i $c_{u}$ rve $w^{h}_{it}$ g $o_{od}$

$$ev_{e}rywh_{e}$$

$K^{r}_{]}$ A $._{K}$ r $u_{s,Q}u^{q}_{el}$ u s r m a $qu^{e}_{s}\grave{a}p^{r}$ o p o s $d^{e}_{s}$ invar ants $4,^{six-e}$ t $\Delta_{d'_{u}}$
$l^{lipi}q_{u}$ e , A c aA $[MS^{Z]}$ H . H . M $\ddot{u}^{l}$ e $r_{,}$ H
0 – 6 1 . a $l^{jiv}$ $r_{an}o^{v}e_{rqua}$ d r t $ic^{fi}e^{l}_{d}$ s J . Re $i^{e}$ A $n^{gew}.M_{@h}3$ 1 1
$P_{1]}$ R G $.E_{.}$ P n ch Elp $ic$ c u $r^{ve}o_{ve}$ r nu $m_{ber}$ $fi_{e}d_{s}$ P $..^{t}$ hs s , O $x_{f}$
$[^{P}i^{2}_{]}$ $emdash-commaE_{l}p-i^{i}_{t}c_{c}$ rv $e^{s}$ wi $t_{h}go_{o}$ d r

$$P_{hl}sS_{c.}10_{1(}198)4-{-^{5}_{49}}$$

[ R $o]$ M $.\dot{I}$ R o $s^{e}$ n , S o me $c^{onfi_{r}}m^{i}_{n}$ g $in_{s_{t}}a-ne_{sof}$ $h_{eB}$ c $endash-h$ S $w-i_{n}$ ner $o^{n-}$ D $y_{e}$
G $r_{uy}$ t $,19^{9}0^{.}_{4}$ 93 – 49 .
u ] $Kfie$ d , u Inv $Tnt.$ h Mth $.10c^{j}_{3(19^{91},25}$ – 8 . w a $a_{th}$ e y or $m_{ag}$ n a $r_{y}$
s e a

S ] 2 9 4 – 17 . , Gro u es d e $S_{elm}$ er e c o r $p_{s}c\ {}^{u}_{bi}\ q_{u}e_{s,J.Nu_{m}}$ $e_{r}$ T eor y 2 3 [
S $e^{r]}$ J . - P $Ser_{r}$ e , P

t u $e_{,}$ I $n^{n}_{vet}$ M $ta$ h $._{1}5(1972_{,2}593$ 1
$[Se_{t}$ B S $t_{ze}r_{,}$ E ll $p^{tc}$ c $u_{r}ve_{s}wi_{t}$
a n ha $v^{in}g_{r}$ t $o_{nlj}-$ i var $i^{a}$ t $,I^{l}_{li^{n}}$ o J $M_{@}$ h $2(1_{981)}233^{--245}.$

$$[Sh_{i}m]$$

$$P \quad {}^{.J}\ {}^{p}_{a}$$

P u $l_{M}$ at $h_{.}Soc$ a n $11I_{wa}$ n $m_{iS_{h}}o^{t}_{e}n_{an}$ d $rPi_{nc}$ o n $U_{ni}$. P $re$
$[Sho_{]}$ K . S $h_{ot}a_{,}$ O n t he $e_{x}pic_{i}$ m $o_{d}$
J p a n $3^{8}_{(}1986),964-6^{59}$
$[i-l_{]}$ J . H $.^{S}$ i v $re$ m an , T $h_{e}$ Ar $ih^{m}$ $t-e_{i}o_{f}E_{l}$ i p c Cu $v_{e}$ s , G $rad_{.}$

S p r $n_{ge_{r}}1_{9}86$ $[St_{r}]$ R $J_{S}$ t $r_{oe}$ k r , $R^{n}_{eucto}$ of $e_{li}$ tc c $u^{e}_{vs}$ $o_{ve}i^{a}_{mg_{i}}n_{aryq}$ u $adr_{a}$

$$fi_{e} \quad d_{,}Pa_{c}ficJ\ {}^{\cdot}_{ah}\ {}^{M}\ {}_{.1}0_{8} \quad (98^{3}_{)4}5^{1}--463^{\cdot}$$
$$[^{Ta} \quad J^{\cdot} \quad Tat_{e},A_{l}gor_{t}$$

S p rn er $,975,a33-52.u^{n}_{c}$ o $n^{s}o\ {}^{f}_{n}\ {}^{O}\ e_{Varia}$ e I V $L_{ect_{u}}reo_{te}$ s i nM

i

T W ] e q u a i o n , J . Nu *amb* rT h M .*eoMry*31(198) e ,9*O*−−132$^{ht}$. ep a $c^{ica_l}s^o_u$ ion $o_f$

t e

$U_{m]}$ r $eprin^m$ t . g a ki A c $o^st^r$ u c i $o_n$ o $f^e$ v e y $h^e$ r e g o $d^{\mathbb{Q}} - ucr$ ve w $i^t$ h p p

[$_d$ W 1 B . M . M . de

$3a_{--1}56$; C o ec on : b . 4 4 1 ( 1 93 , 21 7 – $21J_{\dot{8}.R}$ e n e $A^n$ e w . $M_a$ h .42

1 7 r r ti i i d 9 )

ng

$$1 \quad O_k u_o S_h \quad n_{k u_u^{-k}} \qquad 3$$
$$y 16_9 J_{apa_n}$$