1 . Int od u − c t o − i n . T he ob $j - e_{ctof} h - t$ is pap $e - r$ i st inv $t - s$ iga ethe

$$t_h$$

$F_o e_x$ ampel , ee $[bracketright - endash_{[}\ ], seven - bracketleftbracketright - comma[9 - -^{one-bracketleft}1]. F_o$ r $i - n$ s e $i_z e_{of} t_e$ c_{as} ν_m b e r o f t h e $co_m p^l$ x $q_u d_r a_i$ c n_m^u b_e r fi_e d a_{so}^s c i_{te}^a d w i $t_h \chi$. $A_{l\,o}^s$ i χ i s_t he n o $n - p_r$ inc i pa c_h a r a^{cer} m_d^o $l_o 4, t_{ent}^h$ $c_{at_{ion}}$ o f_t h $e^{\text{“}}$ “ 0 14 s t $''^{zr}_{e}$ o $L_{os}\chi)_{in} t^h$ ec r $t_{ica} s_t$ r i_p has a be ari ng o howt h e_p r e_s reds t^{r_i} ute d^i n t h r^e s d $u^e c^{la}$ s s 1 a n d 3 mo d , ers ec iv $l\cdot$ $a_n d_{in}$ p $a_{rticu_{la}}$, o n a p_h enom e n o n fi s t o $b_{ser} v_e$ d y $C^e b_y s_{he}$ v 4 $c_o n^c$ e n n g t_h e d_{isc} epa n_{ie} i n t h_e dis $i^{b_{ut}} o^{nof_p}$ r^m ein $d_{iff} r_e$ n rsid u cl a_s s . R o u $h - l_y$ s e a i n_g h rea r^e “^m_o r e ” $p_{rim} e_s$ ng $r_u e_n$ t o 3 m o d

$$\text{th } a^n \quad \text{c o n r } u_e \quad \text{t t o 1} \quad \text{mod 4}$$

I n $t^{h^s p}$ a per l e_t $K(_s)$ b ea c_o m p^e x f u n_{ction} s uc h h a | $K(_1/2 +$ it is a a i ly d^{ecr^e} a i_{ng} fu c i^n of $t > 0$. M o e $o_{ver,l} e - ta_{(}parenright - x$ b $e_{i^t} s_M$ e^{l-l} i ta nsf^o r m a_s d e_{fined} in (^{t} he $()x - t$ e $c_{ti_\sum}$ o $n_{.} D_{efi}$ n e $\sum$

$$FK(, D\ \ ) \quad = 1_2 K \quad 12 \quad D) - \quad 1 \quad e \quad - \quad d^{2/D2} \quad K(\varrho) Di\gamma_,$$

a s $ocate^{\varrho}$ ^{r} d wit s hh $evr_{eK} roneck^t$ er sy m o l $\chi f_{L=(d/\cdot).} t_{Un}$ e D r t h e Gen $raliz^t$ e

$$\begin{array}{c} \text{s i t b } d \\ D) \quad \text{w h o w t a} \\ = \quad -1 + \quad 2K \quad 2 \quad D \quad \alpha \quad a(D \quad -) \text{ l o g } \quad D \quad + o_1) \quad i_f \mid \alpha \mid < 1, \end{array}$$

$$\begin{array}{c} 0 + o(_) \quad f1 \quad <_\alpha \quad |< \quad 2 \\ as D \quad \to \infty. \end{array}$$

$$M \qquad a\ i \qquad S\ u\ j\ e \qquad C \qquad fi^c$$

n .

o u t

$f - o_r m_{@}$ o n_{fo} r $|\ \alpha_{|}\ < 2/3$ o n y . Or n e w r e ults h_{ow} s a $d_i co_{nt_{inut}}$ y i n F_K

at α = 1. W e h_a l s_e e t h t t_h s $d - i_{sc}$ o n_{in} ity i s r^e_{at} e d_t o t h_{ep} h $n - e_{om}$ eno fo C h e b y h evm e nt o_n e a bo $v^e_b u_t$ i n_a m $o - r_e g_{ene_{r^{alet}_t}}$ ng

W e theng ve t w_o c o^r_o l_a

p $l - i^e_s$ t h t fo r q ua^d ra $i_{cL}-$ u nto n st h e n o tr via z e_r o ne a t_h e r e

a_{xs} a $r^e_{\text{“}_s} p_{ar}$ e r ” h an he x p_{ct} d u m b e_r o f_{uch} z e^r_o s . T^h_s $i_{s_i} n_d i - r_c$ c n $t^r s_t t - o$ t h_e z ro - d $s - i_{tr}$ b ut_i on w h e^n la l D_i c $h_{le_t} L - f$ u $n - c_t o^{n_{saretake}}$ i $n - t$ o $a^{u-o}_{c^c}$ nt . T h^e_{oth} e r $c_{rol} a_{ry} im$ p esth t no t m $or^e_{th_a}$

g_e r s_d $h^{ave} t_e$ pr o pe_t y t $a_t 1/_2$ s a zero o $L_{(s}\chi d)._A$ sfa r a s_w

$k^{n^o_{w,}}$ n_{oq} $u_{ad} r^{aic_L}$ $-^u_{fn}$ $c_t i_{on} h^a_s$ b e n_f oun d w_i t h az er at $1/2_{\cdot}$ fa n xei stt h_e nm o^{re} t h_{an} l i_k ly t^{hed} e nit o_{fs} u c h d w ould e $0^{\cdot}$

2 P r l$_{\text{i}}$ m i − n a ie$_{\text{s}}$. Int $hi_{sse} cti^n_o$ w p_r o v a numb er_{of} e mma w h ic

$w_{ill^{lbe}}$ n e_d e in h^{ep} ro $o_f o f_o$ u r ma i_n t eo r m . W e a s u_m $e_t h_{@}$ Ks i ana ay i^{cfu_n} $c^t_{io} n_i$ n h e sr p $-1_{<} \Re s_{Kparenleft-s)} x - twos^c d - hs$ h t

$c{+}x5c{-}infinityminus{-}x54{-}x5d{-}x29_{x4a-i-infinity} i{-}x54{-}x20{-}x37{-}x39{-}x2e{-}x36{-}x20{-}x37{-}x46{-}x2fx32{-}$

K $1/2 -^u tl), an^o$ d $ax) is^f$ c nti $-^1_{nuu}$ ly differ n iale , o fb ouned ari^+ at r ea - vaued an dof com p at sup prto nt heine $v - a_l(0 \quad \infty$. More peci ca w eas su me $a(x) = 0$ ou sid e A, B fr s m e 0 r $<$ A

$infinity - x5cx54 - x5d - x29x4ax54 - x20 - x37 - x39 - x2e - x36 - x20 - x39 - x46 - x2fx32 - x2d - x2$

L EMM A 1. $d - \pi d/y = y + o() \quad a - s \quad y \to \infty$. $H \quad e - r \quad d$ den o_{t-e} th

s u m ove rall non

m d i a t re s u^l t o ft

L M M a_2 . d $e -^{\pi}_{d2}{}^{/}$ l o $bar - d = y$ lo $y +$ $O_{)}$ as $y \to \infty$

P ro of . Firs , w ew r $i - t$ e

$$\sum \quad -\pi d/y \quad \sum \quad -d2y2 \quad \sum \quad -\pi d/y^2$$

de o g

her h an d , by R iem $ea - minuspi - n_nu - S - slasht - y$ iel $t - j$ e

$$22 \quad = -^{2}2\infty_{x54-x5d-x29x4a} \quad infinity - x5c - x54 - x5d - x29e - one - x4a - x54 - x20 - x37 - x39 - x2e$$

I h e $s - e_c$ n $d - minus_{pi-x5c}n - x4aslash - tx54 - x20 - x37 - x39 - x2e - x36 - x20 - x39 - x46 - x2f - e - y_{x32-x2d-x20-x31-x33-x2e-x30-x2d-x20-x66-gx32-r}x32 - x2e - l_{w-x54-x20-x37x44}$ e i $n - t_{e-d}$ rateby $\leq$ p $a - y_{rts}$:

$$4 - minusx4a2x2f - x46 - x39 - x20 - x36 - x2e - x39 - x37 - x20 - x54x66 - x20 - x2d - x33 - x2e - x32$$

$$-4infinity - x5cminus - two - x54 - x5d - x29x4a - onex54 - x20 - x37 - x39 - x2e - x36 - x20 - x37 -$$

)

$$\infty_{x4a} \quad x54 - x20 - x37 - x39 - x2e - x36 - x20 - x39 - x46 - x2f x32 - x33 - x2e - x30 - x2d - x20 - x66$$

$$\text{n} \quad \text{e} \ (\tfrac{B^2}{t-h_e} fiu) \ \text{s} = \text{a} \ \{^{d}\}^{2ec^{o}n} \ \text{d p} \ e+$$

$$b-s1 \quad 1 \quad x54-x5d-x29-x5c-infinityx32-x2d-x20_{e\_t\_\pi}^{x4a^{x54-x20-x37-x39-x2e-x36-x20-x39-x46-x2f}} x32-x2ex30-x28$$

$$4 \ \text{N} \quad \text{xt} \quad no \ \begin{matrix} i \\ c \end{matrix} \ \eth \ \text{a} \quad [\quad]$$

$$-\pi u4y2 \quad \infty \quad 2\pi \quad 3 - \pi u$$
$$l_l \ \text{y} \quad \text{co d it} \quad \text{l} \quad /l \ \text{t} \quad y4/y2_o \ \text{b a} \quad \text{g} \quad 0$$

$$x66 - x20 - x2d - x30 - x2e - x33 - x31 - x20 - x2d - x32_{x2e-x32x38-x20-x54}^{x32} 8\pi u$$
$$\infty_{x5cx29-x5d-x54x4a} x2f - x46 - x39 - x20 - x36 - x2e - x39 - x37 - x20 - x54$$
$$x54 - x5d - x29 \infty x4a^{B} \quad y^2_{x32-x2ex54-x20-x37} \quad (\quad 2 \quad / \quad \begin{matrix} 21 \\ 2 \end{matrix}$$
$$= \quad 2 \quad (y \quad v \quad) \quad y4 \quad - \quad y2$$
$$ty^{=} i_O^{(} \quad)_{/2.}^{e} m$$
$$m^m \quad a_2 four - period \quad Asx \to \quad \infty, \quad \begin{matrix} \{ \\ 1 \quad o \\ / \quad lo_p^g \end{matrix}$$

$$Her \ e \ \sum \quad p \ d \ eno \ est \ hesum$$

$$o_{hei.}$$

$$\text{I o} \quad \sum^{t} \quad 2 \quad =_{x5c}^{\infty x29-x5d-x54} a^{u}_{x66-x20-x2d-x30-x2e-x33-x32_{x2f-x46-x39-x20-x36-x2e-x39-x37-x20-x54_{x4a}}} \quad x20 - x2d - x32^{x32} x2e - x32x37 -$$

$$p$$

$$x1/2x5c-x54-x5d-x29x4a \quad x54-x20-x37-x39-x2e-x36-x20-x39-x46-x2fx32-x33-x2e-x30-x2d-x2$$

h r h nd , re cal t h at $ax)$ h $\infty x4a - E - o$ m $)p^d a$ s $u/x)$ ot e

$$x32 - x2d - x20 - x31 -$$

$$ua\prime(u2 \quad du = -^{x5c}x29 - x5d - x54x4a \quad x2f - x46 - x39 - x20 - x36 - x2e - x39 - x37 - x20 - x54^{x66-x20-}$$

$$x54 - x5d - x29x4a \quad x54 - x20 - x37 - x39 - x2e - x36 - x20 - x39 - x46 - x2f \quad x32 \quad x3$$

$$\sqrt{xv)^P} = o(\sqrt{Eradical} - v).$$ T r u , T h e o r^e m $(P^{\cdot} \quad a()v12_d$ v $v^{)} = (o$

$$m^a v)_h \; vd \quad \begin{matrix} x30 - x28 - x5b \\ R - ieoa \quad n - x_n \quad Hy \quad pot - h_e i^s thnE^{u-parenleft}) \end{matrix} \quad \ll \quad u2 \quad l$$

$$^{6-x39-x20-x36-x2e-x39-x37-x20-x54x66-x20-x2d-x30-x2e-x33-x31-x20-x2d-x32}x2e - x32_{x37-x20-x54x5b-x28-x30} \quad vone - slash_4 \text{ l o g } \quad 2x)$$

$$x54 - x20 - x37 - x39 - x2e - x36 - x20 - x39 - x46 - x2f$$

$$\text{l } xm^{a-i}a(v)^s dv \text{ b } i_s \text{ d . } {}^{4}_{\text{A}}$$

$$\sum ap/x) \text{ lo } p = K(1) \quad + \quad o - parenleftx) \quad u \;\; c - n \;\; ond \quad ti \;\; nal^{l-y},$$

$$(\rightarrow \quad \infty \quad a \quad v)dv = aK - parenleftu/xxE(u).$$

r On one han , $\sum$ $au/x)$

$$\infty E() \quad (_{u/}x_{)d}$$

s

$$x4a{-}lx54{-}x20{-}x37{-}x39{-}x2e{-}x36{-}x20{-}x39{-}x46{-}x2fx32{-}x2d{-}x20{-}x31{-}x33{-}x2e{-}x30{-}x2d{-}x20{-}x66x$$

A

$$\text{l o } radical - g_p = K(1)1/2 \quad + \quad \{_{ox1/2^{2})} \quad) \quad u \ nc \ \ o^{ndi}i_{on} \ a \ y$$

$$p \quad a \quad x \quad \infty x5c^{a} \quad x)x32 \quad \begin{matrix} u & \theta^{u}_{)} & u \\ 1x44 & \infty x54 - x5d - x29 & x54 - x20 - x37 - x39 - x2e - x36 - x20 - x39 - x46 \end{matrix}$$

$$h - a - ndx54 - x5d - x29 - x5c - infinity^{a}x4a - x54 - x20 - x37 - x39 - x2e - x36 - x2$$

$$\infty x54 - x5d - x29x4ax54 - x20 - x37 - x39 - x2e -$$

$$\rightarrow \infty \quad ,$$
$$p^{\Sigma} \quad ap$$
$$ogp_{=}K(zero - parenright +^{\{} \quad o(1$$

Po o . $F_s()$ c

$$)$$
$$a_{(}^{\infty}$$

$$=^{K} zero - parenleftparenright - period$$
$$u \quad =_{x5c}^{-} x29 - x5d - x54(_{E}^{x2f-x46-x39-x20-x36-x2e-x39-x37-x20-x54} upare$$

$$=_{=} o(_{\infty}^{x})x4a \quad x54 - x20 - x37 - x39 - x2e - x36 - x20 - x39 - x46 - x2fx32 - x33 - x2e - x30 - x2d - x20$$

me Om a . g R

$$\infty,$$
$$= p \quad one - twoK(0) \quad + \quad \{o(\quad -/4 \quad 2$$
$$p$$

$$= 1_{2} - 1/2x29 - x5d - x54x4a \quad parenleft - x66 - x20 - x2d - x30 - x2e - x33 - x31 - x20 - x2d - x32 - r$$

$$\ll^{o} x^{-1/4} o_{l} 2 x^{u} \text{ a s u o m } i^{dti^{o}} . \text{ H}$$

$$p \quad o \quad p == \quad x5cx54 - x5d - x29^{x4a} infinity - x54 - x5d - x29(x4ax32 - x33 - x2e - x30 - x2d - x20 - x6$$

$$infinity - x5c - x29 - x5d - x54Ex4ax2f - x46 - x39 - x20 - x36 - x2e - x39 - x37 - x20 - x54x66 - x20$$

2

$$x1/infinity{-}two{-}x5cx29{-}x5d{-}x54{\infty}x4ax2f{-}x46{-}x39{-}x20{-}x36{-}x2e{-}x39{-}x37{-}x20{-}x54x66{-}x20{-}x2d{-}$$

3
o
t $o - f^{x}t^{and}Dn$ otrivalz t e^{e} re ϱ = n m β b $+\gamma$ e
i e t h $\sum$ xp re s i o n

$$e \quad \varrho d)$$

T h einner su m iso ver $a - l$ l t
o_{s} d w $i - c$ h $r - a$ e $i - n$ abs $l - o_{u}$ eva u $\sum \leq D.$ $F - o_{r}$ tc h n $c - i$ a rea onswe u $e - t_{h}$ w ie ghting $\sum de - \pi_{d}/D_{2}$ insteado f $|| D,$ which im pl_{i} e h_{t} e $c - a$ cu ati $n - o$
atone po int n th e p o o f $o - f$ t_{h} em a $n - i$ th $e - o_{rem}$ b eow where th Po^{s-i} so
s u m mtio nform ula $i - s$ ap led .

$$) \quad 1 \quad 2 \quad 1/2$$

$$+o \begin{pmatrix} D \\ (\end{pmatrix} x1^{slash-two}) \quad + \quad Ominx D$$

$$s-u \ m \ eth \ eG \qquad r-e-nl-i \ z \ edRi$$

$$\sum_{d)^K} \sum_{(\varrho x}$$

$$K(one-two^{)}Dx1/+IK(1)xD_12 \quad +aone-parenleft/x)D \ \mathrm{l \ o \ g} \ D$$

$$= \quad +O \begin{pmatrix} (\\ D3 \end{pmatrix} \begin{matrix} 3 \\ x2 \ \lg x \end{matrix} \quad) + O \quad (mi_{n(}x \quad D_{,D} \, l_o g D \qquad\qquad\qquad og_{x)})$$

$$\mathbf{braceex}^{\mathbf{braceleftbt}} Oparenleft-xg-o2 \quad x)+$$

$$n \quad a-o-lgf_{\cdot}^{\overset{T}{x^m}} s_{parenright-parenright}^{\overset{e \ n \ ts}{eW}} bya-t_{us_e}^{O(ements D)\cdot \underset{t}{h-teim} \underset{eexpi}{f_p} \underset{l^{ci-t}}{r_{e-i_{dco}}=} \underset{f_{or}m}{\overline{ul}}} ns-ta_{np}^a v_s d_{dep} n-e \ d \ o \ n \ y$$

$$\sum K(\quad \varrho)x \quad = K \quad (1)Eparenleft-chi)x \quad - \quad \sum \quad a(n/x \quad)\Lambda n) \quad d/n)$$

$$\varrho()$$

$$d \quad =$$

$$\sum \quad \pi d2/D2 \quad \sum \quad \varrho$$

$$\varrho^{)}x$$

$$\varrho(_{d)}$$

$$e \quad \sum \quad -d2D2$$

$$\infty \quad d \quad \sum \quad 2$$

$$- \quad \Lambda(n$$

$$C \quad = a(1/) \quad e \quad d \quad /D2log(|$$

$$O \quad O \quad i \quad d \quad e \quad \overset{l}{,_o} g^x \quad d \quad e$$

$$\mathrm{m} \quad 1 \quad \mathrm{a} \quad \mathrm{d} \, 2 \quad O \quad = \quad O(\quad \mathrm{n} \, (\quad xD \quad D$$

ma 2 ,

$$D \to \infty.$$

$$e^{r} B$$

$$B1 = - \quad \sum$$

$$\textbf{summationdisplay} - \textbf{p} \quad \sum d \quad 2 \quad 2$$

$$\sum p \quad \sum d \quad 2$$

$$B \quad 3 \quad p \quad p \quad / \quad xl \quad d - comma - pd^{e} \quad \pi_{d} \quad D \quad ,$$

$$B \quad 4 \quad pmm \geq 3a(p/_{x} l - og^{p} \quad e \quad -$$

$$/ \quad zero - u \text{ n l } \sum^{s} \text{ s } pm/x_{\in} comma - AB] \quad a^{n} \text{ d } \sum^{b}$$

$$\ll \quad /x) \mid$$

$$, \quad \sum \quad m \quad d \quad /3$$

$$\ll D \quad o$$

$$pm{\cdot}m \geq$$

$$i_{2}th \quad \sum \quad - \quad p \quad 2/D2 \quad \sum$$

$$)^{l-o} g^{p}$$

$$1 \quad m \quad p$$

$$2_{)}$$

yLem mas 1 , 4 , an
m m s a (1() /42)
2 o

$$= -1 \quad K$$

a $\sum_{i-n}$ g $R.period - H_{,}$ w $h - i$ e

$$B \quad = ($$

$$= -two - one2K(^{12)} x - Done - slash2 + \quad O(one - x^{/}2) \qquad +oDx1/2)$$

y . 2 Fina $l - y$ w e $o - c$ nsi

$$(\quad / \quad \mathbf{vextendsingle} x5c \mathbf{vextendsingle} \infty x4a \leq x2f - x46 - x39 - x20 - x36 - x2e - x39 - x37 - x20 - x54$$
$$\ll D_1 2p1 \quad ox4a^{g}_{px54-x20-x37-x39-x2e-x36-x20-x39-x46-x2fx32-x33-x2e-x30-x2d-x20-x66^{x32-x2d-x}}$$

$$\text{Bu} \quad \text{t e } nb_y \quad \text{L}$$
$$B^1 \quad \ll \quad pa(p \quad /)^p \quad \text{l o} \quad \text{g} \quad plessmuch - x \quad \text{l o g } x$$
$$\overset{\text{a}}{s^{(} \text{ e} \quad \text{e } period - g - period^{[5parenright-bracketright'}} \quad \text{B}$$

$$x \quad \log p \quad \sqrt{}_p \quad \sum \quad p^m e_{-\pi m}$$
$$e_{n_{ow}} \text{ w} \quad i - te^B 1 \quad = \quad B_1 \quad + B12+)B13, \text{ wher e}$$
$$\sum \quad (p \quad ogp$$
$$\sum p(px_{)} \quad pp\mathbf{summationdisplay} - \mathbf{m} - \mathbf{equal}\square - m$$
$$B \quad D \quad pa \quad x\sqrt{p} \quad m = \square$$
$$B1 \quad D \quad \sum a^{(}pgl^{)}o_{\surd} \quad \sum \quad \begin{matrix}(m) - \pi \\ m\end{matrix}$$
$$\text{s } 3, 5, , n - d7(\text{ a } n - parenrightbiggua - s \text{ s} \quad n - i_{\surd} \quad \text{R} \quad \text{H}) ,$$

$$p$$
$$pa_{\sum} \quad x_{(}p - radical \quad p) \quad D_1 -^2 \sum \qquad /_p)$$
$$= \quad - \quad p \quad ax \quad g \quad + 2$$
$$+OD/2summationdisplay - p \ \begin{matrix}(p \\ a\end{matrix} \ x^{o-lgp}_{p}$$

$$B12 \quad = \quad D \quad pa \quad x \quad \sqrt{p_m} \quad e - m \quad p2^{D} \binom{}{}$$

$$\mathbf{vextendsingle}^{\mathbf{vextendsingle}} \quad \sum \quad (p) \text{ l o g } i_{de} r^{B} 1 period - S \text{ n}$$

$$c \binom{e}{2} parenleft - two - au) = 2_{x} \mathbf{vextendsingle} - \mathbf{i} - \mathbf{f}_{2\mathbf{vextendsingle} \sum} u \quad > B_{,)} \text{ w e}$$

$$p \quad = \quad x5c\infty \quad x4a - summationdisplay - x - lessequal \quad \sqrt{(} \\ x54 - x20 - x37 - x39 - x2e - x3$$

$$= \quad infinity - x5c - one - x29 - x5d - x54^{p - x4a} lessequal - x2f - x46 - x39 - x20 - x36 - x2e - x39 - x37$$

$$x5c^{p} ux66 - x20 -$$

$$x_{2} minus - one_{-1}^{2x29 - x5d - x54} \infty x29 - x5d - x54 px4a_{\leq} x66 - x20 - x2d -$$

fo o sw m me dat $e_{\sum}$ y r om th ei $d_{n}t^{i}$ t
$\varrho(K - parenleft\varrho^{)} =^{a} 1)$ l og | $d \mid +$ $O()$ a s $|_{\overrightarrow{d|}}^{\rightarrow}$ $\infty_{.}$
Th e^{T} heo em i onw $e_{s}t_{ablsh_{e}}$ d
W enows et $x = D\alpha$ and $d - e_{fi}$ n e

$alpha - comma) =$ $D^{-}1$ $\sum$ $e - \pi d2D2one - two$ K $))^{-1}\sum$ $K - parenleft_{\varrho}$

$h - e$ n san imm ed $a - i$ te coro $l - l$ a ryto C ORO LAR Y $._{1}$ As ume G . R .

$$alpha - comma_{\{}D)$$
$$-1$$
$$= \quad 0 + o(\quad 21) \quad 2 \quad a \quad D \quad ($$

$eimpiedco$ n s nt sep nd oly onh ek r l $K.$

N oi c et
of $greater - n - f1Ge_{Th}^{R}{}^{H}$.œ b e_{m} c a u e $ofour_{v}$ eryr o u_{gx} e $i - t$ m a t e $nt_{FK}h_{(\dot{\alpha}D)}^{e,\ u}{}^{m} =$ o
T $h - e$ Th eorem rem ain sv id i K (s) srep l $e - c_{d}$
$parenleft - s$ $)^{=}$ $1/_{s,}$ nw h i_{h} ca s
$a(_{x})$ $=^{\xi}$ $[comma - zero]parenleft - x_{),t}e - h_{c}h_{ar}$ a t e s i fu n_{c} o no
th e^{u} n t $i_{n}t - ev - r_{al \cdot Th}$ eon l y cha nge s n t^{h-e} p $r_{of}^{o}o_{f}$ th e T_{h} eoe m
coc u r_{i}
th e r r o $r_{e}s - tm - iates_{i}n_{Lm}$ $ma_{s}7$ nd 8 , wh i^{c} h d $o_{no}t_{a}ff$ c tt h^{e} e
s u $2t - s$ i

th e s t e_{m}
$he_{o}em_{the}$ n $aket^{h}$ e fo l o i n g o r m :

$$\sum \quad - \quad d2/D2\sum \quad x^{i} \quad \gamma$$
$$e\pi \quad \varrho$$
$$d \quad \textbf{bracelefttp} \quad \varrho(d_{)}$$

braceleftmid $-$ $D + Ix1/2D12 +$ $x -$ $2D$ lo g $D +$ Ox_{l} o $g_{x}) =$ $+O(Dx -_{1}/_{6}$
l og $x) +$ $O($ m i $(x12^{D}$

$$\sum^{O}(^{x} \quad p - parenleft_{x} \quad _{-}2_{x)}^{\pi} \qquad Sn$$

n e e o $v_{r}^{ed})_{,}$

p r o o f o f t h e The o $r - e$ m w e s ee

$$p2 \leq x \quad og \quad + D \quad x \quad \leq xp$$
$$d \quad \square e \quad \{ \quad p$$
$$= \quad 1 \quad + \quad (1^{)} \quad if x \quad o(D,$$
$$0 + o(1 \quad i_{fD}$$
$$\rightarrow \quad \infty \quad - \quad 1 \quad slash - two \quad \sum \quad -\pi 2 \quad 2 \quad \sum \quad p - g = \quad) \qquad a$$
$$\text{o} \quad 1$$

$$m, \quad p \leq x$$
$$D -_{x} -_{/2}$$
$$d^{\neq \square} \quad p \leq^{x} \quad \{$$
$$+^{o_{(1}}$$

$$e - r \quad \rightarrow_{s} \quad \infty. \text{ p } p - e \text{ n} \quad er^{w} \text{ s } n^{h} o - f \text{ p rime s} \quad \leq$$
$$\text{bet} \quad \text{d} \quad d$$
$$d^{\alpha} parenright - ao^{\downarrow}_{emo} \alpha \mid^{>}_{e-r} \text{o r} \quad wesher^{a-so-f_{r}}_{v^{n}_{e} \quad ly \quad d} \text{ i} \quad t - r \begin{matrix} i \\ u \end{matrix} t \text{ ed .}$$

.4 **A p$^{\mathbf{pl}}$ i a i $-$ o$_{\mathbf{n}}$ s** W e n o w_{inv} t g e th e dst bt $o - i$ n o f then $o^{n} t - r_{ivi}$

z r $o - so_{f}$ $Lcomma - s\chi_{d)wh} i - c$ ha r c o $s^{eto_{t} he}$ r ea a i^{x}.

Co RO LL R Y 2 . *Su pp o* $- s_{e} parenleft - r\alpha)$ *i a n e v e n functio nde fi ned on* $(-\infty, \quad \infty$ *s uch t* $- h$ *ati sFou i* $- re - r$ *tr s* $- n$ *fo m* , $r(ue - 2\pi i\alpha udu,$

$$r - hatwide - parenleft\alpha) = \quad x54 - x20 - x31 - x2e \quad x54\infty$$
$$exi\ t\ s\ a\ n\ d\ h \quad a\ s\ u\ p\ o\ rt \quad n_{[-,2\,]} a_{n} \quad dt\ h\ t \quad -$$

$$= 2 \quad x5c_{x54-x5d-x29x4a} \quad x54 - x20 - x37 - x39 - x2e - x36 - x20 - x30 - x31 - x46 - x2f x34 - x2e - x35 -$$

$$h - e^{re} e - hi^{mpl-iedc} \quad o \quad n_{s} a - t\ n\ t \quad d\ e\ p \quad n - d\ on\ l\ y\ o \quad n \begin{matrix} theke \\ r \end{matrix} \quad n$$

$$But \quad \infty x5cx29 - x5d - x54x4ax2f - x46 - x31 - x30 - x20 - x36 - x2e - x39 - x37 - x20 - x54x66 - x20 -$$

$$\mathrm{B} \quad y_a \quad \mathrm{c} \quad \mathrm{h\ a\ n\ g\ eo}\ (D \quad v - a_r b^a \quad - e \quad) = \quad D \quad - \beta, \mathrm{t\ h\ i\ s\ l}\ a_s n^i\ \mathrm{e} \quad \mathrm{r} \quad e_q^{ual}$$

$$dt$$

$$a \quad (tt1 \quad t \quad @) t^{12} (2t \quad pi - i\alpha /$$

$$x2e - x32x37 - x20 - x54$$

$$= \quad \mathrm{lo\ g} \quad D \quad t + \quad \log \quad D\ c_o\ \mathrm{h\ t}$$

$$e_2 \quad \mathrm{go\ l} \quad tge \quad D^{(} \quad \mathrm{lo} \quad)$$

$$2\alpha / ogD \ \overset{e}{og_{\ll}} \ {}_{l} \ t$$

$$t) /_{two-parenleftttt} i_{2^\pi} \alpha / olD \quad - 1) \ \overset{t}{d}$$

$$1 \quad x4a \quad x32x2e - x32$$

$$\mathrm{a} \quad \mathrm{i} \quad \mathrm{n} \quad \mathrm{p\ n\ de}\ \alpha n_{to}\ \mathrm{f} \quad \mathrm{a}$$

$$infinity - x5c \quad x4ax2f$$

$$n \quad e \quad infinity - x5c_{x54-x5d-x29} \quad x4ax54 - x20 - x37 - x39 - x2e - x36 - x20 - x30 - x31 - x46 - x2f^{K} x3$$

$$= \quad () \quad) \sum \quad e\pi \sum_{\sum}^{2} \varrho \quad \infty x29 - x5d - x54x4ax66 -$$

$$=_{r} (-\alpha) =^{r-parenleft} \alpha \quad .h - T \text{ sesta bls } e - s \text{ heco} \quad \text{o} \quad \text{a ry}$$

$$\text{z e ro} \quad \text{lD}$$

$$\text{da r } a^{\pi} \text{ c } / \begin{pmatrix} \\ L \end{pmatrix} \text{ un) n } o - n_{o}, \quad \text{c } o_{l}^{o} l^{r} v - i - r_{mes} \text{ l l ze o } s - v \begin{matrix} e \\ ha_{r} \end{matrix} \quad ben \quad r - e \text{ r}$$

$$\text{r}$$

$$f - r \text{ om} \quad \text{the } e_{r} \text{ a l a } i - x \text{ s}$$

$$\text{T he } n_{e} \text{ x} \quad \text{ap p } l_{cat} \text{ i n } g_{i} \text{ vs} \quad \text{a n} \quad u_{p} \text{ p er bo} \quad \text{u n} \quad \text{o } t^{h} \quad \text{e} \quad n_{um} \quad b_{e} \text{ r of } d_{'sf}$$

$$/12,_{\chi d}) =^{0} . \qquad w_{hi} c^{h}($$

Cor ollar y 3 *Suppo se* $1_{/}2$ *is a "zero" o f* $L(s, \chi d)$ *ofm uti l i − c i t y* $m \geq 0$. *A* $s - s_{um}$ *in gG . R . H .* ,

$$e^{-} \quad d \quad m \quad d \leq \quad 8D + \quad O \log D \quad a_{s} D \quad \to \infty.$$

$$\sum_{d}$$

$$\infty x2$$

$$= -1 \quad x5c - x29 - x5d - x541x$$

$$-[\quad \alpha 2]1 \quad (1 \quad (1 \quad - \quad K(\quad 2)infinity - x5cx29 - x5d - x54x4ax66 - x20 - x2d - x35 - x2e - x34 - x2f -$$

1

$$(\quad d) \quad rho1 - parenleft - d$$

d

R EMA R $k_{\cdot}S_in-c$ e md i

$/\mathbf{e-mn-e\ s}$ l d_s'. ld il e o h a nk P o e

g o m_{ey}^{r} f o b_r i g^i g t_h i p r_{ob} e m too ur a t

Added in pr o − o$_{\mathbf{f}}$. S ince t e ac_c epta nce for p ubicati onof t h i sp ap e

t h eB AM Sart c $l-e$ o N . K a t z

$c_{\cdot}3_{619}9_9)_{,^{126.}}T_h$ $i-r_a$ tic le c o n_ta_n s t h e t_te_{me} nt o_{fo} u C $r-o_{lla_r}$ 2 (A $p_{ro}o-fd_i$ ffe_r n tf_ro^m ou r m_a y b e fo u n^d i_n

S e_e r m o^{thsl-a} $e_{,we}^{r}$ h_a c om $m_{unic}t_{io}$ w ih P e^t e S a n_a^k i^w h h_e o ff_{ee} s o m_e i n p c_{on_c} en i g o_u ar t c e . We w oud_l ik t_{oth} n_k P

fe or S ana k f_o rr ai_{no}^{g} u r m_a nus c_{rt}^{p} a n_d o ff e_{rngi} m p r ov $m_{nts}w_h$ i

eh v^e t r d t ore a i_{zen} his rt c e_sfi na l or m .

$Al^so-commat-i$ ha s^b ee n b $r-o$ $ugt-h$ too ur a^t t_n t

$._S$ oun d aar a j_a n h a pr o_e n $unond^t$ io n_{ll}^{a} ht a $L(_{1/}2_{,\chi d)}=$ 0 f_r^o a m $o_{t1}/$ o fa d'^s

R f er$_{\mathbf{en}}$c$_{\mathbf{s}}^{\mathbf{e}}$

] H . J . B ent z , *scr pncies thedstribuin o pime n u m be r* , J . m b rTh or

[1 15 19 82) 2 5 2

$l-s$ *che e* $p-x$ l $i-z$ i

[P L_{Che_b} y_s^h e v $,rLtt$ *e de* $M_{\cdot}l^e$ *p r* $o_{fs^{eu}}^{s}$ *r T c heby che* $\grave{}_{aM}$. *F* u_{ss} *r u n* n_{ouve}

t $h^{\acute{o}r\grave{}}m_e$ re_a *t iv*

$B_ul_{Cla_s}$ e P $y.A^c$ a $d_{._I}$ m p $.S_{ci.t.}^{S}$. P e_{trs} u g $1_{(}1^85_3)2_{8.}^{0}$

H $D_{aven}p_or_{,}M_u$ *li* $l^{icai}v_e$ *N u* m b_r

o

t

[H L M o n t_g o m e_ry^T h $e_ai_rc_r$ *el* a i *n o* $f_{thez}e^{ro}$ *so f h e* $z^{ta^fu}n_ct_o^n$, i n P

: ro

$S_ym_po_{\cdot}^{P}$ u r M a_h2_4, A mr . Ma $h^{\cdot S}$ o $c_{\cdot}$ P r_o v d_n

[H L M $o_{n^t}g_{om}e_{ry}$ an $d_{P\cdot J}$ W e i n be rg $r_{,}$ N $o_{t^e}s_{on}$ s m_al_{cla} *ssn um bes* , A c

A $rt^{h}\ddot{\cdot}4(19_{4}^{7}),52-52$.

A . EO zl uk O_nt_{heq} $a_{na_o}g_u^e$ o_{f_t} h $p_{air}c_o$ *r e a* o^n *c n ect*$_u$

$oy_{59}{}_{19^{6)},}$ $9\_\_351^{\cdot}$

[A E O $l_{u_C^{kand}}$ $S_ny^{d^e}$ r *Sm* a_{ll} *ze o s o quadr a* $t^{cL}-fu$ n_c *t ion* $s_{,}B_ul.A_{usr_a}$

Ma^{th} $S_{o.}4_{7(1}9_93)$ 3_{731_9}.

D $s_a n - d \quad t - S_a$ t s $t - i$ s

O $o_{,} M_E \quad 04_4 9_{,}$ U $S - period$ A

a n

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s n y d $r_{@g} a^m_{us.u^e}$ ma t $m^e_{a_i n} \cdot^e_{du}$

$a \quad n^d \; i \quad r_e \; vs$