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CONVERGENCE OF ITERATES OF ASYMPTOTICALLY NONEXPANSIVE MAPPINGS IN BANACH SPACES WITH THE UNIFORM OPIAL PROPERTY

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Introduction . Throughout this paper X denotes a Banach space , C a subset of X(*not* necessarily convex) , and $T : C \to C$ a self - mapping of C. There appear in the lit erature two definitions of asymptotically nonexpan - sive mapping . The weaker definition (cf . Kirk [14]) requires that

$$\limsup_{n\to\infty} \sup_{y\in C} (\| T^n x - T^n y \| - \| x - y \|) \le 0$$

for every $x \in C$, and that T^N be continuous for some $N \ge 1$. The stronger definition (cf . Goebel and Kirk [8]) requires that each iterate T^n b e Lip - schitzian with Lipschitz constants $L_n \to 1$ as $n \to \infty$. For our it eration method we find it convenient t o introduce a definition somewhere b etween these two : T is *asymptotically nonexpansive in the intermediate s ense* pro - vided T i s uniformly continuous and

$$\limsup_{n\to\infty} \sup_{x,y\in C} (\| T^n x - T^n y \| - \| x - y \|) \le 0.$$

Many papers on the weak convergence of it erates of asymptotically non - expansive mappings have appeared recently ; their setting i s either a uni - formly convex space with a Fr é chet - differentiable norm or a uniformly con - vex space with the Opial property . In this paper we are primarily interested in a generalization of the second case . Our proofs are not only simpler , they are more general : when τ is a Hausdorff linear t opology and X satisfies the uniform $\tau-$ Opial property , we prove that $\{T^n x\}$ i s $\tau-$ convergent if and only if $\{T^n x\}$ i s $\tau-$ asymptotically regular , i . e .

$$T^{n+1}x - T^n x \xrightarrow{\tau} 0.$$

The $\tau-$ limit is a fixed point of T.

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In the second part of the paper we show how t o construct (in uniformly convex Banach spaces) a fixed point of a mapping which i s asymptotically nonexpansive in the intermediate sense as the $\tau-$ limit of a sequence $\{x_i\}$ defined by an it eration of the form

$$x_{i+1} = \alpha_i T^n i_{x_i} + (1-\alpha_i)x_i,$$

where $\{\alpha_i\}$ is a sequence in (0 , 1) bounded away from 0 and 1 and $\{n_i\}$ i s a sequence of nonnegative integers . Schu [25] has considered this it eration for $n_i \equiv i$, under the assumptions that X i s Hilbert , C i s compact , and T^n has Lipschitz constant $L_n \geq 1$ such that $\sum_n (L_n^2 - 1) < +\infty;$ our results considerably generalize this result .

Recall the classical definition of the Opial property : whenever $x_n \rightharpoonup x$, then

$$\limsup_n \| x_n - x \| < \limsup_n \| x_n - y \|$$

for all $y \neq x$, where $\rightharpoonup$ denotes weak convergence . Henceforth we shall de - note by τ a Hausdorff linear t opology on $X.$ The $\tau-$ Opial property is defined analogously t o the classical Opial property , replacing weak convergence by $\tau-$ sequential convergence . We say that X has the *uniform* $\tau-$ *Opial property* if for each $c > 0$ there exists $r > 0$ with the property that for each $x \in X$ and each sequence $\{x_n\}$ the conditions

$$x_n \tau\!\to 0, \quad 1 \leq \limsup_n \| x_n \| < +\infty, \quad \| x \| \geq c$$

imply that $\limsup_n \| x_n - x \| \geq 1 + r($ cf . Prus [2 1]) . Note that a uniformly convex space which has the $\tau-$ Opial property necessarily has the uniform $\tau-$ Opial property .

$\tau-$ **Convergence of it erates .** A common thread in each of our theorems is the convergence of a sequence of real numbers . We separate out the principle , but it is too trivial t o offer a proof :

LEMMA 1 . *Suppose* $\{r_k\}$ *is a bounded s equence of real numbers and* $\{a_{k,m}\}$ *is a doubly - indexed s equence of real numbers which satisfy*

$$\limsup_k \limsup_m a_{k,m} \leq 0, \quad r_{k+m} \leq r_k + a_{k,m} \quad \textit{for each } k, m \geq 1.$$

Then $\{r_k\}$ *converges to an* $r \in \mathbb{R};$ *if* $a_{k,m}$ *can be taken to be independent of* $k, a_{k,m} \equiv a_m,$ *then* $r \leq r_k$ *for each* $k.$

THEOREM 1 . *Suppose* X *has the uniform* $\tau-$ *Opial property* , C *is a norm - bounded* , *sequentially* $\tau-$ *compact subset of* X , *and* $T : C \to C$ *is asymp - totically nonexpansive in the weak s ense .* *If* $\{yn\}$ *is a s equence in* C *such that* $\lim_n \| yn - w \|$ *exists for each fixed point* w *of* T , *and if* $\{yn - T^k yn\}$ *is*

$\tau-$ convergent to 0 *for each* $k \geq 1$, *then* $\{yn\}$ *is* $\tau-$ *convergent to a fixed point of* T.

P r o o f . We shall begin by proving that if $\{yn_i\}$ is a subsequence such that $yn_i \tau\to z$, then $z = Tz$. Define

$$r_k = \limsup_i \| T^k yn_i - z \|, \quad a_m = \sup_{y \in C} (\| T^m y - T^m z \| - \| y - z \|).$$

By the Opial property (1) $r_{k+m} = \limsup_i \| T^{k+m} yn_i - z \| \leq \limsup_i \| T^{k+m} yn_i - T^m z \| \leq r_k + a_m$,

where $\limsup_m a_m \leq 0$ by the weak definition of asymptotically nonexpan - sive . By Lemma 1 , therefore , $\lim_k r_k = r$ exists and $r \leq r_k$ for each $k \geq 1$. Thus , given $\varepsilon > 0$, (1) implies that for sufficiently large k and m,

$$r \leq \limsup_i \| T^{k+m} yn_i - T^m z \| < r + \varepsilon.$$

By the uniform $\tau-$ Opial property , $\lim_m T^m z = z$. Since T^N is continuous , z is therefore a fixed point of T^N, and since

$$z = \lim_j T^{jN+1} z = \lim_j T T^{jN} z = Tz,$$

z is also a fixed point of T.

We have proved that $\tau-$ subsequential limits of $\{yn\}$ must be fixed points of T. Opial ' s classical argument [20] can now be followed to deduce that $\{yn\}$ is $\tau-$ convergent to a fixed point of T; for otherwise , by the sequential $\tau-$ compactness of C, there must exist $z_1 \neq z_2$ and subsequences $\{yn_i\}$ and $\{ym_i\}$ such that $yn_i \tau\to z_1$ and $ym_i \tau\to z_2$. By the Opial property ,

$$\limsup_i \| yn_i - z_1 \| < \limsup_i \| yn_i - z_2 \|$$

and

$$\limsup_i \| ym_i - z_2 \| < \limsup_i \| ym_i - z_1 \| .$$

But this is impossible ; the sequences $\{\| yn - z_1 \|\}$ and $\{\| yn - z_2 \|\}$ both converge , so the limsup ' s over subsequences are actually limits over the full sequence .

THEOREM 2 . *Suppose the Banach space* X *has the uniform* $\tau-$ *Opial prop - erty , and let* C *be a nonempty , norm - bounded , sequentially* $\tau-$ *compact subset of* X . *If* $T : C \to C$ *is asymptotically nonexpansive in the weak sense and* $x \in C$, *then* $\{T^n x\}$ *is* $\tau-$ *convergent if and only if it is* $\tau-$ *asymptotically regular . The* $\tau-$ *limit of* $\{T^n x\}$ *is a fixed point of* T .

P r o o f . It is obvious that if $\{T^n x\}$ is $\tau-$ convergent , then $T^{n+1}x - T^n x \tau\to 0$. Conversely , suppose that $T^{n+1}x - T^n x \tau\to 0$.

If w is a fixed point of T, define

$$r_k = \| T^k x - w \|, \quad a_m = \sup_{y \in C}(\| T^m y - w \| - \| y - w \|),$$

so that $r_{k+m} \leq r_k + a_m$. By the asymptotic nonexpansiveness of T, $\limsup_m a_m \leq 0$, hence by Lemma 1, $\{r_n\}$ converges. We have proved that $\{\| T^n x - w \|\}$ converges for each fixed point w of T. By the $\tau-$ asymptotic regularity of T,

$$T^n x - T^k T^n x \xrightarrow{\tau} 0 \quad \text{as } n \to \infty$$

for each integer $k \geq 1$. Theorem 1 now shows that $\{T^n x\}$ is $\tau-$ convergent to a fixed point of T. (In particular, this proves that T *has* a fixed point.)

R e m a r k 1. In the case X is a Hilbert space and τ is its weak topology, Theorem 1 was proved by Bruck in [4]. In this case the result also follows from the nonlinear mean ergodic theorem [1, 22, 23]. See [2, 5, 10, 11, 19, 24, 26, 28] for more recent results and a comprehensive and updated bibliography.

R e m a r k 2. There is still another definition of "asymptotically nonexpansive" mapping which appears in the literature:

$$\limsup_n \| T^n x - T^n y \| \leq \| x - y \| \quad \text{for each } x, y \in C.$$

However, this is unsatisfactory from the point of view of fixed point theory: Tingley [27] has constructed an example of a bounded closed convex C in Hilbert space and a continuous but fixed-point-free $T: C \to C$ which actually satisfies

$$\lim_n \| T^n x - T^n y \| = 0 \quad \text{for each } x, y \in C.$$

In his example it is even true that $\{T^n e_1\}$ is weakly convergent to 0, but of course 0 is not a fixed point.

The proof of Theorem 1 can also be applied to asymptotically nonexpansive commutative semigroups. Let C be a nonempty subset of a Banach space X. Let $\mathcal{T} = \{T(t) : t \geq 0\}$ be a family of mappings from C into itself. $\mathcal{T}$ is called an *asymptotically nonexpansive semigroup on* C if $T(t+s) = T(t)T(s)$ for all $s, t \geq 0, T(t_0)$ is continuous for some $t_0 > 0$, and

for each $x \in C$,

$$\limsup_{t \to +\infty} \sup_{y \in C} (\| T(t)x - T(t)y \| - \| x - y \|) \leq 0.$$

THEOREM 3. *In the setting of Theorem 1, a trajectory $\{T(t)x\}$ of an asymptotically nonexpansive semigroup $\mathcal{T}$ on C is $\tau-$ convergent as $t \to +\infty$ iff $T(t+s)x - T(t)x \xrightarrow{\tau} 0$ as $t \to +\infty$ for each $s \geq 0$. The limit is a common fixed point of $\mathcal{T}$.*

R e m a r k 3 . Theorems 2 and 3 can be easily generalized t o metric spaces

$$(X, d).$$

R e m a r k 4 . Theorems 2 and 3 can be proved in the nonexpansive case under the weaker assumption that X has the Opial property and τ i s " locally metrizable " (see Dye , Kuczumow , Lin and Reich [6] and Kuczumow [1 5]) .

An averaging iteration of Schu . J . Schu [25] considered the averaging it eration

$$x_{i+1} = \alpha_i T^i x_i + (1 - \alpha_i) x_i$$

when $T : C \to C$ i s asymptotically nonexpansive in the stronger , Lip - schitzian sense . Here $\{\alpha_i\}$ is a sequence in (0 , 1) which i s bounded away from 0 and 1 . We shall consider , instead , the more general it eration

$$x_{i+1} = \alpha_i T^{n_i} x_i + (1 - \alpha_i) x_i, \tag{2}$$

where $\{n_i\}$ is a sequence of nonnegative integers (which need *not* b e increas - ing) . A strictly increasing sequence $\{m_i\}$ of positive integers will be called *quasi - periodic* if the sequence $\{m_{i+1} - m_i\}$ i s bounded (equivalently , if there exists $b > 0$ so that any block of b consecutive positive integers must contain a t erm of the sequence) .

THEOREM 4 . *Suppose X is a uniformly convex Banach space , C is a bounded convex subset of X , and $T : C \to C$ is asymptotically nonexpansive in the intermediate sense . Put*

$$c_n = \max(0, \sup_{x, y \in C}(\| T^n x - T^n y \| - \| x - y \|)),$$

so that $\lim_n c_n = 0$. Suppose $\{n_i\}$ is a s equence of nonnegative integers such that

$$\sum_i c_{n_i} < +\infty$$

and such that

$$\mathcal{O} = \{i : n_{i+1} = 1 + n_i\}$$

is quasi - periodic . Then for any $x_1 \in C$ and $\{x_i\}$ generated by (2) for $i \geq 1$, we have $\lim_i \| x_i - T x_i \| = 0$. If , in addition , τ is a Hausdorff linear topology such that C is sequentially $\tau-$ compact and X has the $\tau-$ Opial property , then $\{x_i\}$ is $\tau-$ convergent to a fixed point of T .

P r o o f . We have not assumed C i s closed , but since T i s uniformly continuous it (and it s it erates) can b e extended t o the (norm) closure C with the same modulus of uniform continuity and the same constants c_n, so it does no harm t o assume C it self is closed . By a theorem of Kirk [14], T has at least one fixed point w in C.

We begin by showing that for a fixed point w, the limits $\lim_i \| x_i - w \|$ and $\lim_i \| T^n i_{x_i} - w \|$ exist and are equal . From (2) we have

$$\begin{aligned} \| x_{k+1} - w \| &\leq \alpha_k \| T^n k_{x_k} - w \| + (1-\alpha_k) \| x_k - w \| \\ &= \alpha_k \| T^n k_{x_k} - T^n k_w \| + (1-\alpha_k) \| x_k - w \| \\ &\leq \alpha_k (\| x_k - w \| + c_{n_k}) + (1-\alpha_k) \| x_k - w \| \leq \| x_k - w \| + c_{n_k}, \end{aligned}$$

and hence that

$$\| x_{k+m} - w \| \leq \| x_k - w \| + \sum_{i=k}^{k+m-1} c_{n_i}. \tag{3}$$

Applying Lemma 1 with $r_k = \| x_k - w \|$ and $a_{k,m} = \sum_{i=k}^{k+m-1} c_{n_i}$, we see that $\lim_i \| x_i - w \| = r$ exists for each fixed point w of T.

If $r = 0$ then we immediately obtain

$$\| Tx_i - x_i \| \leq \| Tx_i - w \| + \| w - x_i \| = \| Tx_i - Tw \| + \| w - x_i \|,$$

and hence by the uniform continuity of T, that $\lim_i \| x_i - Tx_i \| = 0$. There - fore we must also have

$$\| T^n i_{x_i} - w \| = \| T^n i_{x_i} - T^n i_w \| \leq c_{n_i} + \| x_i - w \| \to 0 \quad \text{as} i \to \infty.$$

If $r > 0$, we shall prove that $\lim_i \| T^n i_{x_i} - w \| = r$ by showing that for any increasing sequence $\{i_j\}$ of positive integers for which $\lim_j \| T^{n_i} j x_{i_j} - w \|$ exists , it follows that the limit is r. Without loss of generality we may assume that the corresponding subsequence $\{\alpha_{i_j}\}$ converges to some α; we shall have $\alpha > 0$ b ecause $\{\alpha_i\}$ i s assumed t o be bounded away from 0 .

Thus we have

$$\begin{aligned} r &= \lim_i \| x_i - w \| = \lim_j \| x_{i_j+1} - w \| \\ &= \lim_j \| \alpha_{i_j} T^{n_i} j x_{i_j} + (1-\alpha_{i_j}) x_{i_j} - w \| \\ &\leq \alpha \liminf_j \| T^{n_i} j x_{i_j} - w \| + (1-\alpha) r \\ &\leq \alpha \limsup_j \| T^{n_i} j x_{i_j} - w \| + (1-\alpha) r \\ &\leq \alpha \limsup_j (\| x_{i_j} - w \| + c_{n_{i_j}}) + (1-\alpha) r \\ &\leq \alpha \limsup_j \| x_{i_j} - w \| + (1-\alpha) r = r. \end{aligned}$$

This completes the proof that

$$\lim_i \| x_i - w \| = r = \lim_i \| T^n i_{x_i} - w \| .$$

Let $\delta : [0,2] \to [0,1]$ b e the modulus of uniform convexity of X, so that whenever $0 < \alpha < 1$ and at least one of u, v is not $0_{line-comma}$ then

$$2\min(\alpha, 1-\alpha)\delta\left(\begin{array}{c}\| u - v \| \\ \max(\| u \|, \| v \|)\end{array}\right) \quad \leq 1- \quad \| \alpha u + \max_{line-parenleft}(^{1}_{\|u\|} -,_{\|} \alpha_{v})^{v\|}_{\|}) \quad .$$

Take $u = T^n i_{x_i} - w$ and $v = x_i - w$; then

$$2\min(\alpha_i, 1-\alpha_i)\delta\left(\begin{array}{c}\| T^n i_{x_i} - x_i \| \\ \max(\| u \|, \| v \|)\end{array}\right) \quad \leq 1- \quad \| \alpha_i u + \max_{(\|}(_u 1 - \|, \quad \|^{\alpha \; i)}_{\quad v\|)}{}^{v} \; \|$$

$$= 1 - \max(\|\|\|^{x_{i+} \; 1}_{n_T i \; x_i -} - ww\|^{\|}_{, \; \|x_i} - w \|).$$

Since $\| T^n i_{x_i} - w \|$, $\| x_i - w \|$ and $\| x_{i+1} - w \|$ all converge to $r > 0$ as $i \to \infty$, and since $\{\alpha_i\}$ remains bounded away from 0 and 1 , we conclude that

$$\lim_i \delta(\| T^n i_{x_i} - x_i \| /r) = 0.$$

Therefore

$$\lim_i \| T^n i_{x_i} - x_i \| = 0. \tag{4}$$

This i s equivalent t o

$$\lim_i \| x_i - x_{i+1} \| \quad = 0. \tag{5}$$

We claim that $x_j - Tx_j \to 0$ as $j \to \infty$ through $\mathcal{O}$. Indeed , since $n_{j+1} = 1 + n_j$ for such j, we have

$$\begin{aligned} \| x_j - Tx_j \| \quad &\leq \| x_j - x_{j+1} \| \quad + \quad \| x_{j+1} - T^n j + 1_{x_{j+1}} \| \\ + \| T^n j + 1_{x_{j+1}} - T^n j + 1_{x_j} \| \quad &+ \quad \| TT^n j_{x_j} - Tx_j \| \\ &\leq \| x_{j+1} - x_j \| \quad + \quad \| x_{j+1} - T^n j + 1_{x_{j+1}} \| \\ + \| x_{j+1} - x_j \| \quad &+ c_{n_{j+1}} + \quad \| TT^n j_{x_j} - Tx_j \| . \end{aligned} \tag{6}$$

By (4) – (6) and the uniform continuity of T, we conclude that $\| x_j - Tx_j \| \quad \to 0$

$$\text{as} j \to \infty \text{through} \mathcal{O}.$$

But since $\mathcal{O}$ is quasi - periodic , there exists a constant $b > 0$ such that for each positive integer i we can find $ji \in \mathcal{O}$ with $| ji - i | \leq b$. Thus (5) and the uniform continuity of $I - T$ imply $x_i - Tx_i \to 0$ as $i \to \infty$ through all of $\mathbb{N}$.

If X has the $\tau-$ Opial property and C is $\tau-$ sequentially compact , the strong convergence of $\| x_i - Tx_i \|$ t o 0 implies $x_i - Tx_i \tau \to 0$. Applying Theorem 1 , we conclude that $\{x_i\}$ i s $\tau-$ convergent to a fixed point of T.

R e m a r k 5 . Schu [25] assumed that X i s Hilbert and that the iterates T^n have Lipschitz constants $L_n \geq 1$ such that $\sum_n (L_n^2 - 1)$ converges . Even for Schu ' s original it eration ($n_i \equiv i$), Theorem 4 i s more general , since the convergence of $\sum_n (L_n^2 - 1)$ implies that of $\sum_n (L_n - 1)$, which in turn assures the convergence of our $\sum_n c_n$.

We can *always* choose a sequence $\{n_i\}$ satisfying the conditions of The - orem 4 : since $\lim_n c_n = 0$, we can choose a subsequence $\{c_{m_i}\}$ such that $\sum_i c_{m_i} < +\infty$ and $\sum_i c_{1+m_i} < +\infty$, then put $n_{2i} = m_i$ and $n_{2i+1} = 1 + m_i$. If T is nonexpansive we can take $n_{2i} = 1, n_{2i+1} = 0$, recovering a well - known result on the iteration of averaged mappings (although it i s not as general as the theorems of Ishikawa [1 3] and Edelstein and O ' Brien [7] on asymptotic regularity) .

Theorem 4 would be more satisfying if we had no condition of quasi - periodicity on $\{n_i\}$, but we do not know whether such a result i s true .

The uniform Opial property . We conclude by recalling a few exam - ples of spaces with the uniform Opial property .

EXAMPLE 1 . If X i s a Banach space with a weakly sequentially continu - ous duality map J_Φ associated with a gauge function Φ which i s continuous , strictly increasing , with $\Phi(0) = 0$ and $\lim_{t\to+\infty} \Phi(t) = +\infty$, then X has the uniform Opial property with resp ect t o the weak t opology (cf . Gossez and Lami - Dozo [1 2]) . In particular , ℓ^p has the uniform Opial property with resp ect t o the weak topology for $1 < p < +\infty$.

EXAMPLE 2. $\ell^1 = c^*0$ has the uniform Opial property with respect to the weak $-*$ t opology (cf . Goebel and Kuczumow [9] , Lim [1 8]) .

EXAMPLE 3 . The James Tree $JT = B^*(B$ i s generated by the biorthog - onal functionals $\{f_n, i\}$ corresponding t o the basis $\{e_{n,i}\})$ has the uniform Opial property with resp ect to its weak $-*$ t opology . This is also true for the James space $J = I^*$ $(I$ i s generated by the biorthogonal functionals $\{fi\}$ corresp onding t o the basis $\{e_1 + ... + e_n\})$. See Kuczumow and Reich [1 6] for details .

EXAMPLE 4 . It i s known that $L^p[0, 1]$ does not have the Opial property for $1 \leq p \leq +\infty$ and $p \neq 2$ (Opial [20]) . Nevertheless , if (Ω, Σ, μ) is a positive $\sigma-$ finite measure space , then for $1 \leq p < +\infty$ the space $L^p(\mu)$ does have the uniform Opial property with resp ect t o the t opology of convergence lo cally in measure (cf . Brezis and Lieb [3] , Lennard [1 7]) .

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[28] H . - K . Xu , *Existence and convergence for fixed points of mappings of asymptotically nonexpansive type* , Nonlinear Anal . 1 6 (1 99 1) , 1 1 39 – 1 146 .

Added in proof . It seems worthwhile to point out that Schu ' s iteration is valid in the class of spaces in which the nonlinear mean ergodic theorem is usually set :

T HEOREM 5 . *If , in Theorem* $4, \tau$ *is the weak topology , then the conclusion remains valid if the hypothesis that* X *has the* $\tau-$ *Opial* $prope^{r-t}$ *y is replaced by the hypothesis that* X *has Fr é chet differentiable norm , and the assumption that* T *is asymptotically nonex - pansive in the intermediate sense is strengthened to the strong (Lipschitzian) asymptotic nonexpansiveness of* T.

We sketch the proof : first , as in Theorem 4 we have $\lim_i \| x_i - Tx_i \| = 0.$ Xu [2 8] has proved that $I - T$ is demiclosed , which in our context means :

(7) All weak subsequential limits of $\{x_i\}$ are fixed points of T.

To prove the uniqueness of the weak subsequential limit we use an " orthogonality " relationship between fixed points , as in the proof of the nonlinear mean ergodic theorem . The idea is adapted from S . Reich [*Weak convergence theorems for nonexpansive mappings in Banach spaces* , J . Math . Anal . Appl . 67 (1 9 79) , 274 – 276] .

Put $S_i = \alpha_i T^{n_i} + (1-\alpha_i)I$ and , for $k \geq j, S(k,j) = S_{k-1}S_{k-2}...S_j$, so that $x_k 4 = Sthat^{(k,j)x_j}_{\sum c_{n_i}} <^{Let} L^{kjdenote}_{\infty_+ implies} that^{the}$ Lipschitz constant of $S(k,j)$. The condition of Theorem

$$\lim_{j\to\infty} \sup_{k\geq j} L_{kj} = 1. \tag{8}$$

The proof of Theorem 4 that $\{\| x_i - w \|\}$ converges for each fixed point w of T is still valid , but we need a stronger result :

(9) $\{\| tx_i + (1-t)w_1 - w_2 \|\}$ converges for all fixed points w_1, w_2 and all $0 < t < 1$.

It follows from R . E . Bruck [*A simple proof of the mean ergodic theorem for nonlinear contractions in Banach spaces* , Israel J . Math . 32 (1 979) , 1 7 – 1 1 6] that th $line-e$ re exists a strictly increasing , continuous convex function $\gamma : [0,+\infty) \to [0,+\infty)$ with $\gamma(0) = 0$ such that for each $S : C \to C$ with Lipschitz constant L,

$$\| S(tu_1 + (1-t)u_2) - tSu_1 - (1-t)Su_2 \| \leq L\gamma^{-1}(\| u_1 - u_2 \| -1L \| Su_1 - Su_2 \|)$$

for all $u_1, u_2 \in C$ and $0 < t < 1$. Applying this to $u_1 = x_j, u_2 = w_1$, a fixed point of T, and $S = S(k,j)$ for $k \geq j$, we see by virtue of (8) and the convergence of $\{\| x_i - w_1 \|\}$ that

$$\lim_{j\to\infty} \sup_{k\geq j} \| S(k,j)(tx_j + (1-t)w_1) - tx_k - (1-t)w_1 \| = 0. \tag{10}$$

Since

$$\begin{aligned}\| tx_k + (1-t)w_1 - w_2 \| &\leq \| tx_k + (1-t)w_1 - S(k,j)(tx_j + (1-t)w_1) \| \\ &\quad + \| S(k,j)(tx_j + (1-t)w_1) - w_2 \| \\ &\leq \| tx_k + (1-t)w_1 - S(k,j)(tx_j + (1-t)w_1) \| \\ &\quad + L_{kj} \| tx_j + (1-t)w_1 - w_2 \|,\end{aligned}$$

(9) follows from (8) and (1 0) by first taking the lim sup as $k \to \infty$ and then taking the lim inf as $j \to \infty$.

 Put $gi(t) = (1/2) \| tx_i + (1-t)w_1 - w_2 \|^2$. We have proved that $\lim_i gi(t)$ exists . By
the hypothesis of Fréchet differentiability of the norm ,

$\lim_{\to_t 0+}$

exists$(_{1/2)\|}$*uniformly.* $\|^2)$*.It is in an i, where J elementary exercise is the normalized in analysis duality that if a seque*

pointwise convergent and equidifferentiable from the right at a point , then the sequence
of derivatives converges at the point ; thus (11) $\lim \langle x_i - w_1, J(w_1 - w_2)\rangle$ exists for any fixed points w_1, w_2 of T.

$$i \to \infty$$

In particular , if w_1 and w_2 are weak subsequential limits of $\{x_i\}$, then when we first let $i \to \infty$ through a subsequence so $x_i \rightharpoonup w_1$, then through a subsequence such that $x_i \rightharpoonup w_2$, the resulting subsequential limits in (11) must be equal , i . e .

$$0 = \langle w_1 - w_1, J(w_1 - w_2)\rangle = \langle w_2 - w_1, J(w_1 - w_2)\rangle = - \| w_1 - w_2 \|^2 .$$

This proves the uniqueness of weak subsequential limits of $\{x_i\}$ and completes the proof that $\{x_i\}$ converges weakly .

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