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# A BOUNDARY VALUE PROBLEM OF FRACTIONAL ORDER AT RESONANCE

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Abstract . We establish solvability of a boundary value problem for a nonlin - ear differential equation of fractional order by means of the coincidence degree theory .

## 1 . Introduction

This article is a study of the boundary value problem of fractional order with non - lo cal conditions

$$\mathcal{D}^{\alpha}u(t) = f(t, u(t), u'(t)), \quad \text{a.e.} t \in (0,1),$$
$$\mathcal{D}^{\alpha-2}_{0+}u(0) = 0, \quad \eta u(\xi) = u(1),$$

where $1 < \alpha < 2, 0 < \xi < 1$ and $\eta\xi^{\alpha-1} = 1$. It will be shown that , with the present choice of boundary conditions , the boundary value problem is at resonance . We apply a well - known degree theory theorem for coincidences due to Mawhin [ 1 6 ] .

The monographs [ 1 0 , 20 , 2 1 , 22 ] are commonly cited for the theory of fractional derivatives and integrals and applications to differential equations of fractional or - der . Contributions to the theory of initial and boundary value problems for non - linear differential equations of fractional order have been made by several authors including a recent monograph [ 1 3 ] and the papers [ 1 , 2 , 9 , 1 5 , 24 ] . Although an ap - plication of the coincidence degree theory to a fractional order problem is not known to the author , we can account for several results that have been devoted to both the - oretical developments [ 5 , 1 7 , 1 9 ] and applications [ 23 ] to various types of boundary and initial value problems . A broad range of scenarios of resonant problems were studied in the framework of ordinary differential and difference equations [ 1 7 ] ( more generally , dynamic equations on time scales [ 3 , 1 1 ] ) on bounded and unbounded [ 1 2 ] domains with periodic [ 1 8 ] , non - lo cal boundary conditions [ 4 , 6 , 7 , 8 , 23 ] as well as boundary value problems with impulses [ 14 ] .

## 2 . Technical preliminaries

We start out by introducing the reader to the fundamental tools of fractional calculus and the coincidence degree theory .

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 The Riemann - Liouville fractional integral of order $\alpha > 0$ of a function $u \in L^p[0,1], 1 \leq p < \infty$, is the integral

$$\mathcal{I}^{\alpha} u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s) ds. \tag{2.1}$$

The Riemann - Liouville fractional derivative of order $\alpha > 0, n = [\alpha] + 1$, is defined by

$$\mathcal{D}^{\alpha} u(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dtn} \int_0^t (t-s)^{n-\alpha-1} u(s) ds. \tag{2.2}$$

Let $AC[0,1]$ denote the space of absolutely continuous functions on the interval [ 0 , 1 ] and $ACn[0,1] = \{u \in AC[0,1] : u^{(n)} \in AC[0,1]\}, n = 0, 1, 2, \ldots.$ We make use of several relationships between ( 2 . 1 ) and ( 2 . 2 ) that are stated in the next two theorems ( see [ 1 0 , 20 , 22 ] ) .

**Theorem 2 . 1 .** ( a ) *The equality* $\mathcal{D}^{\alpha} \mathcal{I}^{\alpha} g = g$ *holds for every* $g \in L^1[0,1]$;
( b ) *For* $u \in L^1[0,1], n = [\alpha] + 1, \beta > 0$, *if* $\mathcal{I}^{n-\alpha} u \in ACn - 1[0,1]$, *then*

$$\mathcal{I}^{\beta} \mathcal{D}^{\alpha} u(t) = \mathcal{D}^{\alpha-\beta} u(t) - \sum_{n-1}^{k=0} \frac{t^{\beta-k-1}}{\Gamma(\beta-k)} \Big(\frac{d^{n-k-1}}{dtn-k-1} \mathcal{I}^{n-\alpha} u\Big)(0).$$

For $\alpha < 0$, we introduce the notation $\mathcal{I}^{\alpha} = D^{-\alpha}$. **Theorem 2 . 2 .** *If* $\beta, \alpha + \beta > 0$ *and* $g \in L^1[0,1]$, *then the equality*

$$\mathcal{I}^{\alpha} \mathcal{I}^{\beta} g. = \mathcal{I}^{\alpha+\beta} g$$

**Definition 2 . 3 .** Let $X$ and $Z$ be real normed spaces . A linear mapping $L$ : dom $L \subset X \to Z$ is called a Fredholm mapping if the following two conditions hold :

( i ) ker $L$ has a finite dimension , and
( i i ) Im $L$ is closed and has a finite codimension .

If $L$ is a Fredholm mapping , its ( Fredholm ) *index* is the integer Ind $L =$ dim ker $L-$ codim Im $L$.

In this note we are concerned with a Fredholm mapping of index zero . From Definition 2 . 3 it follows that there exist continuous projectors $P : X \to X$ and

$$Q : Z \to Z \text{such that}$$

$$\text{Im } P = \ker L, \quad \ker Q = \text{Im } L, \quad X = \ker L \oplus \ker P, \quad Z = \text{Im } L \oplus \text{Im } Q$$

and that the mapping

$$L \mid_{\text{dom } L \cap \ker P} : \text{dom } L \cap \ker P \to \text{Im } L$$

is one - to - one and onto . The inverse of $L \mid_{\text{dom } L \cap \ker P}$ we denote by $K_P$ : Im $L \to$ dom $L \cap$ ker $P$. The generalized inverse of $L$ denoted by $K_{P,Q} : Z \to$ dom $L \cap$ ker $P$ is defined by $K_{P,Q} = K_P(I - Q)$.

If $L$ is a Fredholm mapping of index zero , then , for every isomorphism $J$ : Im $Q \to$ ker $L$, the mapping $JQ + K_{P,Q} : Z \to$ dom $L$ is an isomorphism and , for

$$\text{every} u \in \text{dom} L,$$
$$(JQ + K_{P,Q})^{-1} u = (L + J^{-1} P) u.$$

**Definition 2 . 4 .** Let $L$ : dom $L \subset X \to Z$ be a Fredholm mapping , $E$ be a metric space , and $N : E \to Z$ be a mapping . We say that $N$ is $L-$ compact on $E$ if $QN : E \to Z$ and $K_{P,Q}N : E \to X$ are continuous and compact on $E$. In addition ,
we say , that $N$ is $L-$ completely continuous if it is $L-$ compact on every bounded

$$E \subset X.$$

The existence of a solution of the equation $Lu = Nu$ will be shown using [ 1 6 , Theorem IV . 1 3 ] . **Theorem 2 . 5 .** *Let* $\Omega \subset X$ *be open and bounded* , $L$ *be a Fredholm mapping of index*

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*zero and* $N$ *be* $L-$ *compact on* $\Omega$. *Assume that the following conditions are satisfied :*

( i ) $Lu \neq \lambda Nu$ *for every* $(u, \lambda) \in ((\text{ dom } L \setminus \ker L) \cap \partial\Omega) \times (0,1)$;

( i i ) $N uelement - slash \operatorname{Im} L$ *for e very* $u \in \ker L \cap \partial\Omega$;

( ii i ) $\deg\,(JQN \mid_{\ker} L \cap \partial\Omega, \Omega \cap \ker L, 0) \neq 0$, *with* $Q : Z \to Z$ *a continuous projector such that* $\ker Q = \operatorname{Im} L$ *and* $J : \operatorname{Im} Q \to \ker L$ *is an is omorphism .*

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*Then the equation* $Lu = Nu$ *has at least one s o lution in* dom $L \cap \Omega$.

Suppose now that the function $f$ satisfies the Carath é odory conditions with respect to $L^p[0,1], p \geq 1$; that is , the following conditions hold :

( C 1 ) for each $z \in \mathbb{R}^n$, the mapping $t \mapsto f(t,z)$ is Lebesgue measurable ;

( C 2 ) for a . e .$t \in [0,1]$, the mapping $z \mapsto f(t,z)$ is continuous on $\mathbb{R}^n$;

( C 3 ) for each $r > 0$, there exists a nonnegative $\phi_r \in L^p[0,1]$ such that , for a . e .

$t \in [0,1]$ and every $z$ such that $\mid z \mid \leq r$, we have $\mid f(t,z) \mid \leq \phi_r(t)$.

## 3 . Main results

Consider the differential equation

$$\mathcal{D}^{\alpha} u(t) = f(t, u(t), u'(t)), \quad \text{a.e.} t \in (0,1), \tag{3.1}$$

of fractional order $1 < \alpha < 2$, subject to the boundary conditions

$$\mathcal{D}^{\alpha-2} u(0) = 0, \tag{3.2}$$

$$\eta u(\xi) = u(1), \tag{3.3}$$

where $0 < \xi < 1$ and

$$\eta \xi^{\alpha-1} = 1. \tag{3.4}$$

We let the following assumption stand throughout this article :

$$(\text{P}) p > \frac{1}{\alpha - 1} \text{and} q = line - p_{p-1}.$$

Let $AC_{\text{loc}}(0,1]$ be the space consisting of functions that are absolutely continuous on every interval $[a,1] \subset (0,1]$. We introduce the space

$$X_0 = \{u : u \in AC[0,1], u' \in AC_{\text{loc}}(0,1], \mathcal{D}^\alpha u \in L^p[0,1]\}.$$

Let

$$X = \{u \in C[0,1] \cap C^1(0,1] : \lim_{\to_t 0^+} t^{2-\alpha} u'(t) \text{exists}\}$$

with the weighted norm $\| u \| = \max \{\| u \| 0, \| t^{2-\alpha} u' \| 0\}$, where $\| \cdot \| 0$ is the max - norm and $\| t^{2-\alpha} v \| 0 = \sup_{t \in (0,1]} | t^{2-\alpha} v(t) |$. Let $Z = L^p[0,1]$ with the usual norm $\| \cdot \|_p$, where $p$ satisfies $(P)$. Define the mapping $L : \text{dom } L \subset X \to Z$ with

dom $L = \{u \in X_0 : u$ satisfies ( 3 . 2 ) and ( 3 . 3 ) $\}$

$$\text{and} Lu(t) = \mathcal{D}^\alpha u(t).$$

 Define the mapping $N : X \to Z$ by

$$Nu(t) = f(t, u(t), u'(t)).$$

**Lemma 3 . 1 .** *The mapping* $L : \text{dom } L \subset X \to Z$ *is a Fredholm mapping of index zero .* *Proof .* It is easy to see that $\ker L = \{ct^{\alpha-1} : c \in \mathbb{R}\}$. We claim that

$$\text{Im}L = \{g \in Z : \eta\mathcal{I}^\alpha g(\xi) = \mathcal{I}^\alpha g(1)\}.$$

Let $g \in Z$ and

$$u(t) = \mathcal{I}^\alpha g(t) + ct^{\alpha-1}, \quad c \in \mathbb{R}.$$

Then $\mathcal{D}^\alpha u(t) = g(t)$, a . e . in ( 0 , 1 ) . By Theorem 2 . 2 ,

$$\mathcal{D}^{\alpha-2}u(t) = \mathcal{I}^{2-\alpha}u(t)$$
$$= \mathcal{I}^{2-\alpha}\mathcal{I}^\alpha g(t) + c\mathcal{I}^{2-\alpha}(t^{\alpha-1})$$
$$= \mathcal{I}^2 g(t) + c\Gamma(\alpha)t,$$

so that $\mathcal{D}^{\alpha-2}u(0) = 0$. One can readily verify that , in view of (3.4), $u$ satisfies ( 3 . 3 ) provided $\eta\mathcal{I}^\alpha g(\xi) = \mathcal{I}^\alpha g(1)$. It is obvious that $u \in AC[0,1]$. Then $u'$ exists , for a . e .$t \in (0,1]$, and , by Theorem 2 . 2 ,

$$u'(t) = \mathcal{I}^{\alpha-1}g(t) + c(\alpha-1)t^{\alpha-2}.$$

Moreover ,

$$\lim_{\to_t 0^+} t^{2-\alpha}u'(t) = c(\alpha-1)$$

since

$$\lim_{t\to 0^+} t^{2-\alpha} \mid \mathcal{I}^{\alpha-1}g(t) \mid \leq \lim_{\to_t 0^+} \frac{t^{1/q} \parallel g \parallel_p}{\Gamma(\alpha-1)((\alpha-2)q+1)^{1/q}} = 0.$$

Let $t_1, t_2 \in (0,1)$ and $t_1 < t_2$. Then

$$\mid \mathcal{I}^{\alpha-1}g(t_2) - \mathcal{I}^{\alpha-1}g(t_1) \mid$$
$$= \frac{1}{\Gamma(\alpha)} \left| \int_0^{t_2} (t_2-s)^{\alpha-2}g(s)ds - \int_0^{t_1} (t_1-s)^{\alpha-2}g(s)ds \right|$$
$$= \frac{1}{\Gamma(\alpha)} \left| \int_{t_1}^{t_2} (t_2-s)^{\alpha-2}g(s)ds + \int_0^{t_1} ((t_2-s)^{\alpha-2} - (t_1-s)^{\alpha-2})g(s)ds \right|$$
$$\leq \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2-s)^{\alpha-2} \mid g(s) \mid ds + \frac{1}{\Gamma(\alpha)} \int_0^{t_1} ((t_1-s)^{\alpha-2} - (t_2-s)^{\alpha-2}) \mid g(s) \mid ds$$
$$\leq C_1(t_2-t_1)^{\alpha-2+\frac{1}{q}} \parallel g \parallel_p + C_1[\int_0^{t_1} ((t_1-s)^{\alpha-2} - (t_2-s)^{\alpha-2})^q ds]1/q_{\parallel} g \parallel_p$$
$$\leq C_1(t_2-t_1)^{\alpha-2+\frac{1}{q}} \quad \parallel g \parallel p + C_1[\int_0^{t_1} ((t_1-s)^{(\alpha-2)q} - (t_2-s)^{(\alpha-2)q})ds]1/q_{\parallel} g \parallel_p$$
$$\leq C_1(t_2-t_1)^{\alpha-2+\frac{1}{q}} \parallel g \parallel_p$$
$$+C_1(1_t^{(\alpha-2)q+1} - 2_t^{(\alpha-2)q+1} + (t_2-t_1)^{(\alpha-2)q+1})1/q \parallel g \parallel_p,$$

where $C_1$ is a generic constant that depends only on $\alpha$ and $p$. Thus , $u' \in AC_{\text{loc}}(0,1]$. Combining the preceding observations , we obtain that $u \in \text{dom } L$. So , $\{g \in Z :$

$$\eta\mathcal{I}^\alpha g(\xi) = \mathcal{I}^\alpha g(1)\} \subseteq \text{Im}L.$$

 Let $u \in$ dom $L$. Then , for $\mathcal{D}^{\alpha}u \in$ Im $L$, we have , by Theorem 2 . 1 ( b ) and ( 3 . 2 ) ,

$$\mathcal{I}^{\alpha}\mathcal{D}^{\alpha}u(t) = u(t) - \frac{\mathcal{D}^{\alpha-1}u(0)}{\Gamma(\alpha)}t^{\alpha-1} - \frac{\mathcal{D}^{\alpha-2}u(0)}{\Gamma(\alpha-1)}t^{\alpha-2} = u(t) - \frac{\mathcal{D}^{\alpha-1}u(0)}{\Gamma(\alpha)}t^{\alpha-1},$$

which , due to the boundary conditions ( 3 . 2 ) , ( 3 . 3 ) together with ( 3 . 4 ) , implies that $\mathcal{D}^{\alpha}u$ satisfies $\eta\mathcal{I}^{\alpha}\mathcal{D}^{\alpha}u(\xi) = \mathcal{I}^{\alpha}\mathcal{D}^{\alpha}u(1)$. Hence , Im $L \subseteq \{g \in Z : \eta\mathcal{I}^{\alpha}g(\xi) =$

$$\mathcal{I}^{\alpha}g(1)\}. \text{Therefore}, \text{Im}L = \{g \in Z : \eta\mathcal{I}^{\alpha}g(\xi) = \mathcal{I}^{\alpha}g(1)\}.$$

$$\text{Define} Q : Z \to Z \text{by}$$

$$Qg(t) = \kappa(\eta\mathcal{I}^{\alpha}g(\xi) - \mathcal{I}^{\alpha}g(1))t^{\alpha-1},$$

where

$$\kappa = \frac{\Gamma(2\alpha)}{\Gamma(\alpha)(\xi^{\alpha}-1)}.$$

Then

$$\begin{aligned} Q^2g(t) &= \kappa(\eta\mathcal{I}^{\alpha}Qg(\xi) - \mathcal{I}^{\alpha}Qg(1))t^{\alpha-1} \\ &= \kappa(\frac{\eta}{\Gamma(\alpha)}\int_0^{\xi}(\xi-s)^{\alpha-1}Qg(s)ds - \frac{1}{\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}Qg(s)ds)t^{\alpha-1} \\ &= \kappa(\frac{\eta}{\Gamma(\alpha)}\int_0^{\xi}(\xi-s)^{\alpha-1}s^{\alpha-1}ds - \frac{1}{\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}s^{\alpha-1}ds)Qg(t) \\ &= \kappa\frac{\Gamma(\alpha)}{\Gamma(2\alpha)}(\eta\xi^{2\alpha-1}-1)Qg(t) \\ &= Qg(t) \end{aligned}$$

in view of ( 3 . 4 ) . Therefore , $Q : Z \to Z$ is a continuous linear projector with

$$\text{Ker}Q = \text{Im}L.$$

Let $g \in Z$ be written as $g = (g - Qg) + Qg$ with $g - Qg \in$ Ker $Q =$ Im $L$ and $Qg \in$ Im $Q$. Hence , $Z =$ Im $L+$ Im $Q$. Let $g \in$ Im $L\cap$ Im $Q$ and set $g(t) = ct^{\alpha-1}$ to obtain that

$$0 = \gamma\mathcal{I}^{\alpha}g(\xi) - \mathcal{I}^{\alpha}g(1) = \frac{c\Gamma(\alpha)}{\Gamma(2\alpha)}(\eta\xi^{2\alpha-1}-1) = \frac{c}{\kappa},$$

which implies that $c = 0$. Hence $\{0\} =$ Im $L\cap$ Im $Q$ and so $Z =$ Im $L\oplus$ Im $Q$. Note that Ind $L =$ dim ker $L-$ codim Im $L = 0$; that is , $L$ is a Fredholm mapping of index zero . □

$$\text{Define} P : X \to X \text{by}$$

$$Pu(t) = \frac{1}{\Gamma(\alpha)}\mathcal{D}^{\alpha-1}u(0)t^{\alpha-1}.$$

Since $0 < \alpha - 1 < 1$,

$$\mathcal{D}^{\alpha-1}u(t) = \frac{1}{\Gamma(2-\alpha)}\frac{d}{dt}\int_0^t(t-s)^{1-\alpha}u(s)ds.$$

Then

$$P^2u(t) = \frac{1}{\Gamma(\alpha)}\mathcal{D}^{\alpha-1}(Pu)(0)t^{\alpha-1}$$

$$= \frac{1}{\Gamma(2-\alpha)}\frac{1}{\Gamma(\alpha)}(\frac{d}{dt}\int_0^t (t-s)^{1-\alpha}s^{\alpha-1}ds)|_{t=0}Pu(t)$$

$$= Pu(t).$$

 We have that $P : X \to X$ is a continuous linear projector . Note that ker $P = \{u \in$

$$X : \mathcal{D}^{\alpha-1}u(0) = 0\}.\text{For} u \in X,$$
$$\| Pu \| 0 = \frac{1}{\Gamma(\alpha)} \mid \mathcal{D}^{\alpha-1}u(0) \mid$$

and

$$\| t^{2-\alpha}(Pu)' \| 0 = \frac{1}{\Gamma(\alpha-1)} \mid \mathcal{D}^{\alpha-1}u(0) \mid .$$

Hence ,

$$\| Pu \| = \frac{1}{\Gamma(\alpha)} \mid \mathcal{D}^{\alpha-1}u(0) \mid . \tag{3.5}$$

Define $K_P : \text{Im } L \to \text{dom } L\cap \ker P$ by

$$K_P g(t) = \mathcal{I}^\alpha g(t), \quad t \in (0,1).$$
$$\text{For} g \in \text{Im} L,$$
$$LK_P g(t) = \mathcal{D}^\alpha \mathcal{I}^\alpha g(t) = g(t)$$

by Theorem 2 . 1 ( a ) . For $u \in \text{dom } L\cap \ker P$, we have $\mathcal{D}^{\alpha-2}u(0) = 0$ and $\mathcal{D}^{\alpha-1}u(0) = 0$ . Hence , by Theorem 2 . 1 ( b ) ,

$$K_P Lu(t) = \mathcal{I}^\alpha \mathcal{D}^\alpha u(t)$$
$$= u(t) - \frac{\mathcal{D}^{\alpha-1}u(0)}{\Gamma(\alpha)} t^{\alpha-1} - \frac{\mathcal{D}^{\alpha-2}u(0)}{\Gamma(\alpha-1)} t^{\alpha-2}$$
$$= u(t).$$

Thus ,

$$K_P = (L \mid \text{dom} L \cap \ker P) - 1.$$

Furthermore , using ( P ) , we have

$$\| t^{2-\alpha}(K_P g)' \| 0 = t^{\max} \in (0,1] |t^{2-\alpha}(K_P g)'(t)|$$
$$\le t^{\max} \in (0,1] \frac{t^{2-\alpha}}{\Gamma(\alpha-1)} \int_0^t (t-s)^{\alpha-2} \mid g(s) \mid ds$$
$$\le t^{\max} \in (0,1] \frac{t^{2-\alpha}}{\Gamma(\alpha-1)} (\int_0^t (t-s)^{(\alpha-2)q} ds) 1/q_{\|} g \|_p$$
$$= \frac{\alpha-1}{\Gamma(\alpha)} \frac{1}{((\alpha-2)q+1)^{1/q}} \| g \|_p .$$

Similarly ,

$$\| K_P g \| 0 = t^{\max} \in [0,1] \mid K_P g(t) \mid$$
$$\le t^{\max} \in [0,1] \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \mid g(s) \mid ds$$
$$\le t^{\max} \in [0,1] \frac{1}{\Gamma(\alpha)} (\int_0^t (t-s)^{(\alpha-1)q} ds) 1/q_{\|} g \|_p$$
$$= \frac{1}{\Gamma(\alpha)} \frac{1}{((\alpha-1)q+1)^{1/q}} \| g \|_p .$$

Hence

$$\| K_P g \| \leq \Lambda \| g \|_p, \tag{3.6}$$

 where

$$\Lambda = \frac{1}{\Gamma(\alpha)} \max\{\frac{1}{((\alpha-1)q+1)^{1/q}}, \frac{\alpha-1}{((\alpha-2)q+1)^{1/q}}\}. \tag{3.7}$$

We introduce

$$\begin{aligned} QNu(t) &= \kappa(\eta \mathcal{I}^{\alpha} Nu(\xi) - \mathcal{I}^{\alpha} Nu(1)) t^{\alpha-1} \\ &= \frac{\kappa}{\Gamma(\alpha)}(\eta \int_0^{\xi} (\xi - s)^{\alpha-1} f(s, u(s), u'(s)) ds \\ &\quad - \int_0^1 (1-s)^{\alpha-1} f(s, u(s), u'(s)) ds) t^{\alpha-1} \end{aligned}$$

and

$$K_{P,Q} Nu(t) = K_P(I - Q) Nu(t) = \frac{\kappa}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} (Nu(s) - QNu(s)) ds.$$

Now we are in position to prove the existence results . We impose the condiitions ( H 1 ) there exists a positive constant $K$ such that $u \in$ dom $L\backslash$ Ker $L$ with

$$\min_{t \in [0,1]} \mid \mathcal{D}^{\alpha-1}(t) \mid > K \text{implies} QNu(t) \neq 0 \text{on} (0,1];$$

( H 2 ) there exist $\delta, \beta, t^{\alpha-2}\gamma, \rho \in L^p[0,1]$ and a continuous nondecreasing function $\phi : [0,\infty) \to [0,\infty)$ and $x_0 > 0$ with the properties :

(a)

$$\| \beta \| p + \quad \| t^{\alpha-2}\gamma \|_p < \frac{\Gamma(\alpha)}{1 + \Gamma(\alpha)\Lambda};$$

( b ) for all $x \geq x_0$

$$x \geq + \frac{K + (1 + \Gamma(\alpha)\Lambda) \| \delta \|_p}{\Gamma(^{\alpha)-}_{\Gamma\ (\alpha)}(^1_- + (1(1 + \Gamma(\alpha)\Lambda) \| \rho \|^{+\Gamma(\alpha)\Lambda)(\|\beta\|p \ p^{+} \ \|t\alpha-2}_{\Gamma(\alpha)\Lambda)(\|\beta\|p+ \ p_{\|t\alpha-2_\gamma\|} \ p\gamma)} \|} _{-p)\phi(x);} \tag{3.8}$$

(c) $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$satisfies

$$\mid f(t,x,y) \mid \leq \delta(t) + \beta(t) \mid x \mid + \gamma(t) \mid y \mid + \rho(t)\phi(\mid x \mid);$$

( H 3 ) there exists a constant $B > 0$ such that , for every $c \in \mathbb{R}$ satisfying $\mid c \mid > B$ we have

$$\text{sgn}[c(\eta \mathcal{I} u_c(\xi) - \mathcal{I} u_c(1))] \neq 0,$$
$$\text{where} u_c(t) = ct^{\alpha-1}.$$

**Theorem 3 . 2 .** *If the hypotheses* ( P ) , ( H 1 ) - ( H 3 ) *are satisfied , th en th e boundary value problem* ( 3 . 1 ) - ( 3 . 4 ) *has a s o lution .*

*Proof .* Let $\Omega_1 = \{u \in \text{dom } L\backslash \text{ Ker } L : Lu = \lambda Nu \text{ for some } \lambda \in (0,1)\}$. Applying ( H 1), $QNu(t) = 0$ for all $t \in [0,1]$. Hence there exists $t_0 \in (0,1]$ such that
$\mid \mathcal{D}^{\alpha-1}(t_0) \mid \leq K$. By Theorem 2 . 1 with $\beta = 1$,

$$\begin{aligned} \mathcal{I}\mathcal{D}^{\alpha} u(t_0) &= \mathcal{D}^{\alpha-1} u(t_0) - \mathcal{D}^{\alpha-1} u(0) - \mathcal{D}^{\alpha-2} u(0) t_0^{-1} \\ &= \mathcal{D}^{\alpha-1} u(t_0) - \mathcal{D}^{\alpha-1} u(0) \end{aligned}$$

since $u \in \operatorname{dom} L$. That is ,

$$\mathcal{D}^{\alpha-1}u(0) = \mathcal{D}^{\alpha-1}u(t_0) - \int_0^{t_0} \mathcal{D}^{\alpha}u(s)ds,$$

 which implies

$$\begin{aligned}| \mathcal{D}^{\alpha-1}u(0) | &\leq | \mathcal{D}^{\alpha-1}u(t_0) | + \int_0^{t_0} | \mathcal{D}^{\alpha}u(s) | \, ds \\ &\leq K + \| Lu \| \\ &< K + \| Nu \|_p .\end{aligned}$$

By ( 3 . 5 ) ,

$$\| Pu \| = \frac{1}{\Gamma(\alpha)} | \mathcal{D}^{\alpha-1}u(0) | < \frac{1}{\Gamma(\alpha)}(K + \| Nu \|_p).$$

Since $(I-P)u \in$ dom $L \cap$ Ker $P =$ Im $K_P$, for $u \in \Omega_1$, $\| (I-P)u \| < \Lambda \| Nu \|_p$ by ( 3 . 6 ) and ( 3 . 7 ) . Also $Pu \in$ Im $P =$ Ker $L \subset$ dom $L$ and , therefore ,

$$\| u \| \leq \| Pu \| + \| (I-P)u \| < \frac{K}{\Gamma(\alpha)} + (\frac{1}{\Gamma(\alpha)} + \Lambda) \| Nu \|_p .$$

From ( H 2 ) and the previous inequality , it follows that

$$\begin{aligned}\| t^{2-\alpha}u' \| 0 &< \frac{K}{\Gamma(\alpha)} + (\frac{1}{\Gamma(\alpha)} + \Lambda)(\| \delta \| p + \| \beta \| p \| \| u \| 0 \\ &+ \| t^{\alpha-2}\gamma \| p \| t^{2-\alpha}u' \| 0 + \| \rho \|_p \phi(\| u \| 0))\end{aligned}$$

or

$$\begin{aligned}\| t^{2-\alpha}u' \| 0 < &\frac{K + (1+\Gamma(\alpha)\Lambda) \| \delta \| p}{\Gamma(\alpha) - (1+\Gamma(\alpha)\Lambda) \| t\alpha - 2_\gamma \|_p} + \frac{(1+\Gamma(\alpha)\Lambda) \| \beta \|_p}{\Gamma(\alpha) - (1+\Gamma(\alpha)\Lambda) \| t\alpha - 2_\gamma \|_p} \| u \| 0 \\ &+ \frac{(1+\Gamma(\alpha)\Lambda) \| \rho \|_p}{\Gamma(\alpha) - (1+\Gamma(\alpha)\Lambda) \| t\alpha - 2_\gamma \|_p} \phi(\| u \| 0).\end{aligned}$$

( 3 . 9 ) Combining the above inequality with

$$\begin{aligned}\| u \| 0 &< \frac{K}{\Gamma(\alpha)} + (\frac{1}{\Gamma(\alpha)} + \Lambda)(\| \delta \| p + \| \beta \| p \| \| u \| 0 \\ &+ \| t^{\alpha-2}\gamma \| p \| t^{2-\alpha}u' \| 0 + \| \rho \|_p \phi(\| u \| 0))\end{aligned}$$

we obtain

$$\begin{aligned}\| u \| 0 &< \frac{K + (1+\Gamma(\alpha)\Lambda) \| \delta \| p}{\Gamma(\alpha) - (1+\Gamma(\alpha)\Lambda)(\| \beta \| p + \| t\alpha - 2_\gamma \|_p)} \\ &+ \frac{(1+\Gamma(\alpha)\Lambda) \| \rho \| p}{\Gamma(\alpha) - (1+\Gamma(\alpha)\Lambda)(\| \beta \|_p + \| t\alpha - 2_\gamma \|_p)} \phi(\| u \| 0),\end{aligned}$$

for all $u \in \Omega_1$. Suppose that $\Omega_1$ is unbounded . If $\{\| t^{2-\alpha}u' \| 0 \ : u \in \Omega_1\}$ is un - bounded , then , by ( 3 . 9 ) , so is $\{\| u \| 0 \ : u \in \Omega_1\}$. So , it suffices to consider the case that $\{\| u \| 0 \ : u \in \Omega_1\}$ is unbounded . Then , in view of ( 3 . 8 ) , we arrive at a
contradiction . Therefore , $\Omega_1$ is bounded .

Set $\Omega_2 = \{u \in \ker L : Nu \in \text{Im } L\}$. Hence $u_c \in \ker L$ is given by $u_c(t) = ct^{\alpha-1}$, $c \in \mathbb{R}$. Then $(QN)(ct^{\alpha-1}) = 0$, since $Nu \in \text{Im } L = \ker Q$. It follows from ( H 3 ) that $\| u_c \| = \max \{\| u_c \| 0, \| t^{2-\alpha}u_c' \| 0\} = \max \{| c |, (\alpha-1) | c |\} =| c | \leq B$; that is , $\Omega_2$ is bounded .

Define the isomorphism $J : \text{Im } Q \to \ker L$ by $Ju_c = u_c, u_c(t) = ct^{\alpha-1}$ for $c \in \mathbb{R}$. Let $\Omega_3 = \{u \in \ker L : -\lambda J^{-1}u + (1-\lambda)QNu = 0, \quad \lambda \in [0,1]\}$, if sgn $[c(\eta\mathcal{I}u_c(\xi) - \mathcal{I}u_c(1))] = -1$. Then $u \in \Omega_3$ implies $\lambda c = (1-\lambda)(\eta\mathcal{I}u_c(\xi) - \mathcal{I}u_c(1))$. If $\lambda = 1$, then

$c = 0$ and , if $\lambda \in [0, 1)$ and $| c |> B$, then $0 < \lambda c^2 = (1 - \lambda)c(\eta \mathcal{I}u_c(\xi) - \mathcal{I}u_c(1)) < 0$, which is a contradiction . Let $\Omega_3 = \{u \in \ker L : \lambda J^{-1}u + (1 - \lambda)QNu = 0, \lambda \in [0, 1]\}$ if sgn $[c(\eta \mathcal{I}u_c(\xi) - \mathcal{I}u_c(1))] = 1$, and we arrive at a contradiction , again . Thus ,

$$\| u_c \| \leq B, \text{forall} u_c \in \Omega_3.$$

Let $\Omega$ be open and bounded such that $\cup_{i=1}^3 \overline{\Omega_i} \subset \Omega$. Then the assumptions ( i ) and ( i i ) of Theorem 2 . 5 are fulfilled . It is a straightforward exercise to show that

the mapping $N$ is $L-$ compact on $\overline{\Omega}$. Lemma 3 . 1 establishes that $L$ is a Fredholm mapping of index zero .

Define

$$H(u, \lambda) = \pm \lambda \text{Id} u + (1 - \lambda)JQNu.$$

By the degree property of invariance under a homotopy , if $u \in \ker L \cap \partial\Omega$, then deg $(JQN \mid_{\ker} L \cap \partial\Omega, \Omega \cap \ker L, 0) =$ deg $(H(\cdot, 0), \Omega \cap \ker L, 0)$

$$= \deg(H(\cdot, 1), \Omega \cap \ker L, 0)$$

$=$ deg $(\pm$ Id $, \Omega \cap \ker L, 0) \neq 0$. Therefore , the assumption ( i i i ) of Theorem 2 . 5 is fulfilled and the proof is completed .

□

Suppose that the hypothesis ( H 2 ) is replaced by ( H 2 ' ) there exist $\delta, \beta, t^{\alpha-2}\gamma, t^{\alpha-2}\rho \in L^p[0, 1]$ and a continuous nondecreasing function $\phi : [0, \infty) \to [0, \infty)$ and $y0 > 0$ with the properties :

(a)

$$\| \beta \| p + \| t^{\alpha-2}\gamma \|_p < \frac{\Gamma(\alpha)}{1 + \Gamma(\alpha)\Lambda};$$

( b ) for all $y \in [0, \infty)$ and $t \in [0, 1]$,

$$t^{2-\alpha}\phi(y) \leq \phi(t^{2-\alpha}y);$$

( c ) for all $y \geq y0$,

$$y \geq \frac{K + (1 + \Gamma(\alpha)\Lambda) \| \delta \| p}{\Gamma(\alpha) - (1 + \Gamma(\alpha)\Lambda)(\| \beta \| p + \| t\alpha - 2_\gamma \|_p)} + \frac{(1 + \Gamma(\alpha)\Lambda) \| t^{\alpha-2}\rho \| p}{\Gamma(\alpha) - (1 + \Gamma(\alpha)\Lambda)(\| \beta \| p + \| t\alpha - 2_\gamma \|_p)}\phi(y);$$

(d) $f : [0, 1] \times \mathbb{R}^2 \to \mathbb{R}$satisfies

$$| f(t, x, y) | \leq \delta(t) + \beta(t) | x | + \gamma(t) | y | + \rho(t)\phi(| y |).$$

Then we have the following existence criterion whose proof is analogous to that of Theorem 3 . 2 .

**Theorem 3 . 3 .** *If the hypotheses* ( P ) , ( H 1 ) , ( H 2 ' ) , ( H 3 ) *are satisfied , then the boundary valu e problem* ( 3 . 1 ) - ( 3 . 4 ) *has a s o lution .*

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