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ON THE OSTROWSKI TYPE INTEGRAL INEQUALITY

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Abstract . In this note , we establish an inequality of Ostrowski - type involving functions of two independent variables newly by using certain integral inequalities .

1 . Introduction In [**3**] , Ujevi ć proved the following double integral inequality :

Theorem 1 . *Let* $f : [a, b] \to \mathbb{R}$ *be a twice differentiable mapping on* (a, b) *and suppose that* $\gamma \leq f''(t) \leq \Gamma$ *for al l* $t \in (a, b)$. *Then we have the double inequality*

$$(1.1)\frac{3S-\Gamma}{24}(b-a)^2 \leq \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(t)\mathrm{d}t \leq \frac{3S-\gamma}{24}(b-a)^2$$

$$where S = \frac{f'(b)-f'(a)}{b-a}.$$

In a recent paper [**2**] , Liu et al . have proved the following two sharp inequalities of perturbed Ostrowski - type

Theorem 2 . *Under the assumptations of Theorem 1 , we have*

$$\frac{\Gamma[(x-a)^3-(b-x)^3]}{12(b-a)} + \frac{1}{8}[\frac{b-a}{2} + \left| \begin{matrix} a+b \\ x-\text{-}2 \end{matrix} \right|^{]} 2(S-\Gamma)$$

$$(1.2) \quad \leq \frac{1}{2}[f(x) + \frac{(x-a)f(a)+(b-x)f(b)}{b-a}] - \frac{1}{b-a}\int_a^b f(t)\mathrm{d}t$$

$$\leq \frac{\gamma[(x-a)^3-(b-x)^3]}{12(b-a)} + \frac{1}{8}[\frac{b-a}{2} + \left| \begin{matrix} a+b \\ x-\text{-}2 \end{matrix} \right|^{]} 2(S-\gamma),$$

$$forallx \in [a, b],$$

where $S = \frac{f'(b)-f'(a)}{b-a}$. *If* γ, Γ *are given by*

$$\gamma = \min_{t\in[a,b]} f''(t), \quad \Gamma = t^{\max} \in [a, b]f''(t)$$

then the inequality given by (2) *is sharp in the usual s ense .*

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In [**1**] , Cheng has proved the following integral inequality

Theorem 3 . *Let* $I \subset \mathbb{R}$ *be an open interval* $, a, b \in I, \quad a < b. \quad f : I \to \mathbb{R}$ *is a differentiable function such that there exist constants* $\gamma, \Gamma \in \mathbb{R}$ *with* $\gamma \leq f'(x) \leq \Gamma$, $x \in [a, b]$. *Then we have*

$$\left| \frac{1}{2} f(x) - \frac{(x-b)f(b)-(x-a)f(a)}{2(b-a)} - \frac{1}{b-a} \int_a^b f(t)\mathrm{d}t \right| \leq \frac{(x-a)^2 + (b-x)^2}{8(b-a)} (\Gamma - \gamma), \tag{1.3}$$

for al l $x \in [a, b]$.

The main purpose of this paper is to establish new inequality similar to the inequalities (1 . 1) – (1 . 3) involving functions of two independent variables .

2 . Main Result

Theorem 4 . *Let* $f : [a, b] \times [c, d] \to \mathbb{R}$ *be an absolutely continuous fuction such that the partial derivative of order* 2 *exists and supposes that there exist constants* $\gamma, \Gamma \in \mathbb{R}$ *with* $\gamma \leq \frac{\partial^2 f(t,s)}{\partial t \partial s} \leq \Gamma$ *for al l* $(t, s) \in [a, b] \times [c, d]$. *Then , we have*

$$\begin{aligned} &\Big| \frac{1}{4} f(x, y) + \frac{1}{4} H(x, y) - \frac{1}{2(b-a)} \int_a^b f(t, y)\mathrm{d}t - \frac{1}{2(d-c)} \int_c^d f(x, s)\mathrm{d}s \\ &- \frac{1}{2(b-a)(d-c)} \int_a^b [(y-c)f(t, c) + (d-y)f(t, d)]\mathrm{d}t \\ &- \frac{1}{2(b-a)(d-c)} \int_c^d [(x-a)f(a, s) + (b-x)f(b, s)]\mathrm{d}s \\ &+ \frac{1}{2(b-a)(d-c)} \int_a^b \int_c^d f(t, s)\mathrm{d}s\mathrm{d}t \Big| \\ &\leq \frac{[(x-a)^2 + (b-x)^2][(y-c)^2 + (d-y)^2]}{32(b-a)(d-c)} (\Gamma - \gamma), \end{aligned} \tag{2.1}$$

for al l $(x, y) \in [a, b] \times [c, d]$ *where*

$$H(x,y)$$

$$\frac{(x-a)[(y-c)f(a,c)+(d-y)f(a,d)]+(b-x)[(y-c)f(b,c)+(d-y)f(b,d)]}{(b-a)(d-c)}$$

$$+\frac{(x-a)f(a,y)+(b-x)f(b,y)}{b-a}+\frac{(y-c)f(x,c)+(d-y)f(x,d)}{d-c}.$$

 Proof . We define the functions : $p : [a, b] \times [a, b] \to \mathbb{R}, q : [c, d] \times [c, d] \to \mathbb{R}$ given by

$$p(x,t) \quad = \quad braceex-braceex-braceleftmid-braceex-braceex-braceleftbt \quad t^t -_- \frac{ab_+ + 2_x}{2}^{x_-,} \quad t^t \in^\in ({}^{[a,\ x]}_{x,\ t}$$

and

$$q(y,s) \quad = \quad braceex-braceex-braceleftmid-braceex-braceex-braceleftbt \quad s^s -_- \frac{\frac{c+y}{d2_+ y}}{2}^{,} \quad s^s \in^\in ({}^{[c,\ y]}_{y,\ d]}.$$

By definitions of $p(x,t)$ and $q(y,s)$, we have

$$\begin{aligned}
& \int_a^b \int_c^d p(x,t)q(y,s)\frac{\partial^2 f(t,s)}{\partial t \partial s}\mathrm{d}s\mathrm{d}t \\
= \quad & x\int_a^y \int_c^c (t - \frac{a+x}{2})(s - \frac{c+y}{2})\frac{\partial^2 f(t,s)}{\partial t \partial s}\mathrm{d}s\mathrm{d}t \\
& + \int_a^x \int_c^d (t - \frac{a+x}{2}) \quad (s - \frac{d+y}{2})\frac{\partial^2 f(t,s)}{\partial t \partial s}\mathrm{d}s\mathrm{d}t \\
& + b\int_a^y \int_x^c (t - \frac{b+x}{2}) \quad (s - \frac{c+y}{2})\frac{\partial^2 f(t,s)}{\partial t \partial s}\mathrm{d}s\mathrm{d}t \\
& + b\int_x^d \int_x^y (t - \frac{b+x}{2}) \quad (s - \frac{d+y}{2})\frac{\partial^2 f(t,s)}{\partial t \partial s}\mathrm{d}s\mathrm{d}t.
\end{aligned} \tag{2.2}$$

Integrating by parts twice , we can state :

$$
\begin{aligned}
& x\int_a^x y\int_c^y (t-\frac{a+x}{2})(s-\frac{c+y}{2})\frac{\partial^2 f(t,s)}{\partial t\partial s}\mathrm{d}s\mathrm{d}t \\
&= \frac{(x-a)(y-c)}{4}[f(x,y)+f(a,y)+f(x,c)+f(a,c)] \\
&-\frac{y-c}{2}x\int_a^x [f(t,y)+f(t,c)]\mathrm{d}t-\frac{x-a}{2}y\int_c^y [f(x,s)+f(a,s)]\mathrm{d}s \\
&+\int_a^x\int_c^y f(t,s)\mathrm{d}s\mathrm{d}t.
\end{aligned} \tag{2.3}
$$

$$\int_a^x\int_y^d (t-\frac{a+x}{2})\quad (s-\frac{d+y}{2})\frac{\partial^2 f(t,s)}{\partial t\partial s}\mathrm{d}s\mathrm{d}t$$

$$=\frac{(x-a)(d-y)}{4}[f(x,y)+f(x,d)+f(a,y)+f(a,d)]$$

$$-\frac{d-y}{2}x\int_a^x[f(t,d)+f(t,y)]\mathrm{d}t-\frac{x-a}{2}d\int_y^d[f(x,s)+f(a,s)]\mathrm{d}s \tag{2.4}$$

$$+\int_a^x\int_y^d f(t,s)\mathrm{d}s\mathrm{d}t.$$

$$b\int_y^d y\int_x^b (t-\frac{b+x}{2})(s-\frac{c+y}{2})\frac{\partial^2 f(t,s)}{\partial t\partial s}\mathrm{d}s\mathrm{d}t$$

$$=\frac{(b-x)(y-c)}{4}[f(x,y)+f(b,y)+f(x,c)+f(b,c)]$$

$$-\frac{y-c}{2}b\int_x^b[f(t,c)+f(t,y)]\mathrm{d}t-\frac{b-x}{2}y\int_c^y[f(x,s)+f(b,s)]\mathrm{d}s \tag{2.5}$$

$$+\int_x^b\int_c^y f(t,s)\mathrm{d}s\mathrm{d}t.$$

$$\int_x^b\int_y^d (t-\frac{b+x}{2})(s-\frac{d+y}{2})\frac{\partial^2 f(t,s)}{\partial t\partial s}\mathrm{d}s\mathrm{d}t$$

$$=\frac{(b-x)(d-y)}{4}[f(x,y)+f(x,d)+f(b,y)+f(b,d)]$$

$$-\frac{d-y}{2}b\int_x^b[f(t,d)+f(t,y)]\mathrm{d}t-\frac{b-x}{2}d\int_y^d[f(x,s)+f(b,s)]\mathrm{d}s \tag{2.6}$$

$$+\int_x^b\int_y^d f(t,s)\mathrm{d}s\mathrm{d}t.$$

 Adding (2 . 3) – (2 . 6) and rewriting , we easily deduce

$$\int_a^b\int_c^d p(x,t)q(y,s)\frac{\partial^2 f(t,s)}{\partial t\partial s}\mathrm{d}s\mathrm{d}t = \frac{1}{4}\{(b-a)(d-c)f(x,y)$$
$$+[(x-a)f(a,y)+(b-x)f(b,y)](d-c)$$
$$+[(y-c)f(x,c)+(d-y)f(x,d)](b-a)$$
$$+[(y-c)f(a,c)+(d-y)f(a,d)](x-a)$$
$$+[(y-c)f(b,c)+(d-y)f(b,d)](b-x)\}$$
$$-\frac{d-c}{2}\int_a^b f(t,y)\mathrm{d}t-\frac{b-a}{2}\int_c^d f(x,s)\mathrm{d}s \tag{2.7}$$
$$-\frac{1}{2}\int_a^b[(y-c)f(t,c)+(d-y)f(t,d)]\mathrm{d}t$$
$$-\frac{1}{2}\int_c^d[(x-a)f(a,s)+(b-x)f(b,s)]\mathrm{d}s$$
$$+\int_a^b\int_c^d f(t,s)\mathrm{d}s\mathrm{d}t.$$

We also have

$$\int_a^b\int_c^d p(x,t)q(y,s)\mathrm{d}s\mathrm{d}t = 0. \tag{2.8}$$

Let $M = line - gamma\Gamma +_2$. From (2 . 7) and (2 . 8) , it follows that

$$\int_a^b\int_c^d p(x,t)q(y,s)[\frac{\partial^2 f(t,s)}{\partial t\partial s}-M]\mathrm{d}s\mathrm{d}t$$
$$= b\int_a^b d\int_c^d{}_p(x,t)q(y,s)\frac{\partial^2 f(t,s)}{\partial t\partial s}\mathrm{d}s\mathrm{d}t. \tag{2.9}$$

On the other hand , we get

$$
\begin{aligned}
& \left| \int_a^b \int_c^d p(x,t)q(y,s)\left[\frac{\partial^2 f(t,s)}{\partial t \partial s} - M\right] \mathrm{d}s\mathrm{d}t \right| \\
& \leq (t,s) \max \in [a,b] \times [c,d] \left| \frac{\partial^2 f(t,s)}{\partial t \partial s} - M \right| \int_a^b \int_c^d \mid p(x,t)q(y,s) \mid \mathrm{d}s\mathrm{d}t.
\end{aligned} \tag{2.10}
$$

 We also have

$$(t,s)\max\in[a,b]\times[c,d]|\frac{\partial^2 f(t,s)}{\partial t\partial s}-M|\leq\frac{\Gamma-\gamma}{2} \tag{2.11}$$

and

$$\int\int|\,p(x,t)q(y,s)\,|\,\mathrm{d}s\mathrm{d}t=\frac{[(x-a)^2+(b-x)^2][(y-c)^2+(d-y)^2]}{16}\Bigg|_a^b\Bigg|_c^d. \tag{2.12}$$

From (2 . 1 0) to (2 . 1 2) , we easily get

$$\begin{aligned}&|\int_a^b\int_c^d p(x,t)q(y,s)[\frac{\partial^2 f(t,s)}{\partial t\partial s}-M]\mathrm{d}s\mathrm{d}t| \\ &\leq\frac{[(x-a)^2+(b-x)^2][(y-c)^2+(d-y)^2]}{32}(\Gamma-\gamma).\end{aligned} \tag{2.13}$$

From (2 . 9) and (2 . 1 3) , we see that (2 . 1) holds . □

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