ON THE OSTROWSKI TYPE INTEGRAL INEQUALITY

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Abstract. In this note, we establish an inequality of Ostrowski-type involving functions of two independent variables newly by using certain integral inequalities.

1. Introduction In [3], Ujević proved the following double integral inequality:

Theorem 1. Let \( f : [a, b] \to \mathbb{R} \) be a twice differentiable mapping on \((a, b)\) and suppose that \( \gamma \leq f''(t) \leq \Gamma \) for all \( t \in (a, b) \). Then we have the double inequality

\[
\frac{3S - \Gamma}{24}(b - a)^2 \leq \frac{f(a) + f(b)}{2} - \frac{1}{b - a} \int_a^b f(t)dt \leq \frac{3S - \gamma}{24}(b - a)^2
\]

where \( S = \frac{f'(b) - f'(a)}{b - a} \).

In a recent paper [2], Liu et al. have proved the following two sharp inequalities of perturbed Ostrowski-type

Theorem 2. Under the assumptions of Theorem 1, we have

\[
\frac{\Gamma[(x-a)^3 - (b-x)^3]}{12(b-a)} + \frac{1}{8} \left( \frac{b-a}{2} \right) + \left| \frac{a+b}{x-2} \right| 2(S - \Gamma)
\]

\[
\leq \frac{1}{2} \left[ f(x) + \frac{(x-a)f(a) + (b-x)f(b)}{b-a} \right] - \frac{1}{b - a} \int_a^b f(t)dt
\]

\[
\leq \gamma \frac{(x-a)^3 - (b-x)^3}{12(b-a)} + \frac{1}{8} \left( \frac{b-a}{2} \right) + \left| \frac{a+b}{x-2} \right| 2(S - \gamma),
\]

for all \( x \in [a, b] \),

where \( S = \frac{f'(b) - f'(a)}{b - a} \). If \( \gamma, \Gamma \) are given by

\[
\gamma = \min_{t \in [a,b]} f''(t), \quad \Gamma = \max_{t \in [a,b]} f''(t)
\]

then the inequality given by (2) is sharp in the usual sense.

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In [1], Cheng has proved the following integral inequality

**Theorem 3.** Let $I \subset \mathbb{R}$ be an open interval, $a, b \in I$, $a < b$. $f : I \to \mathbb{R}$ is a differentiable function such that there exist constants $\gamma, \Gamma \in \mathbb{R}$ with $\gamma \leq f'(x) \leq \Gamma$, $x \in [a, b]$. Then we have

$$\left| \int \frac{b}{2(b-a)} f(x) - \frac{(x-b)f(b)-(x-a)f(a)}{2(b-a)} - \frac{1}{b-a} \int f(t) dt \right| \leq \frac{(x-a)^2 + (b-x)^2}{8(b-a)}(\Gamma - \gamma),$$

for all $x \in [a, b]$.

The main purpose of this paper is to establish new inequality similar to the inequalities (1.1) – (1.3) involving functions of two independent variables.

**2. Main Result**

**Theorem 4.** Let $f : [a, b] \times [c, d] \to \mathbb{R}$ be an absolutely continuous function such that the partial derivative of order 2 exists and suppose that there exist constants $\gamma, \Gamma \in \mathbb{R}$ with $\gamma \leq \frac{\partial^2 f(t,s)}{\partial t \partial s} \leq \Gamma$ for all $(t, s) \in [a, b] \times [c, d]$. Then, we have

$$\left| \int \frac{1}{4} f(x, y) + \frac{1}{4} H(x, y) - \frac{1}{2(b-a)} \int f(t, y) dt - \frac{1}{2(d-c)} \int f(x, s) ds \right|$$

$$\leq \frac{1}{2(b-a)(d-c)} \left[ (y-c)^2 + (d-y)^2 \right] \left( \frac{(x-a)^2 + (b-x)^2}{32(b-a)(d-c)}(\Gamma - \gamma) \right),$$

for all $(x, y) \in [a, b] \times [c, d]$ where
\[(x - a)[(y - c)f(a, c) + (d - y)f(a, d)] + (b - x)[(y - c)f(b, c) + (d - y)f(b, d)]
\]

\[
\begin{aligned}
&\quad \frac{H(x, y)}{H(x, y)}
&\quad + \frac{(x - a)f(a, y) + (b - x)f(b, y)}{b - a} + \frac{(y - c)f(x, c) + (d - y)f(x, d)}{d - c}.
\end{aligned}
\]
Proof. We define the functions $p : [a, b] \times [a, b] \to \mathbb{R}$, $q : [c, d] \times [c, d] \to \mathbb{R}$ given by

$$p(x, t) = \begin{cases} \frac{ab + 2x}{2}, & t \in ([a, x]_b) \\ \end{cases}$$

and

$$q(y, s) = \begin{cases} \frac{c + y}{2}, & s \in ([c, y]_d) \\ \end{cases}$$

By definitions of $p(x, t)$ and $q(y, s)$, we have

$$\int \int p(x, t)q(y, s) \frac{\partial^2 f(t, s)}{\partial t \partial s} \, ds \, dt$$

$$= \int \int (t - \frac{a + x}{2})(s - \frac{c + y}{2}) \frac{\partial^2 f(t, s)}{\partial t \partial s} \, ds \, dt$$

$$+ \int \int (t - \frac{a + x}{2})(s - \frac{d + y}{2}) \frac{\partial^2 f(t, s)}{\partial t \partial s} \, ds \, dt$$

Integrating by parts twice, we can state:
\[
x \int y \int \left( t - \frac{a + x}{2} \right) \left( s - \frac{c + y}{2} \right) \frac{\partial^2 f(t, s)}{\partial t \partial s} \, ds dt \quad a \leq c
\]
\[
= \frac{(x - a)(y - c)}{4} \left[ f(x, y) + f(a, y) + f(x, c) + f(a, c) \right]
\]
\[
- \frac{y - c}{2} x \int [f(t, y) + f(t, c)] \, dt - \frac{x - a}{2} y \int [f(x, s) + f(a, s)] \, ds
\quad a \leq c
\]
\[
+ \int \int f(t, s) \, ds dt.
\quad a \leq c
\]
\[\int \int (t - \frac{a + x}{2}) (s - \frac{d + y}{2}) \frac{\partial^2 f(t, s)}{\partial t \partial s} \, ds \, dt\]

\[= \frac{(x - a)(d - y)}{4} [f(x, y) + f(x, d) + f(a, y) + f(a, d)]\]

(2.4) \[= \frac{d - y}{2} \int [f(t, d) + f(t, y)] \, dt - \frac{x - a}{2} d \int [f(x, s) + f(a, s)] \, ds\]

\[= \frac{(b - x)(y - c)}{4} [f(x, y) + f(b, y) + f(x, c) + f(b, c)]\]

(2.5) \[- \frac{y - c}{2} b \int [f(t, c) + f(t, y)] \, dt - \frac{b - x}{2} y \int [f(x, s) + f(b, s)] \, ds\]

\[= \frac{(b - x)(d - y)}{4} [f(x, y) + f(x, d) + f(b, y) + f(b, d)]\]

(2.6) \[- \frac{d - y}{2} b \int [f(t, d) + f(t, y)] \, dt - \frac{b - x}{2} d \int [f(x, s) + f(b, s)] \, ds\]
Adding (2.3) – (2.6) and rewriting, we easily deduce

\[
\int \int p(x,t) q(y,s) \frac{\partial^2 f(t,s)}{\partial t \partial s} \, ds \, dt = \frac{1}{4} ((b-a)(d-c)f(x,y) \nonumber \\
+ [(x-a)f(a,y) + (b-x)f(b,y)](d-c) \nonumber \\
+ [(y-c)f(x,c) + (d-y)f(x,d)](b-a) \nonumber \\
+ [(y-c)f(b,c) + (d-y)f(b,d)](b-x) \nonumber \\
+ [(y-c)f(a,c) + (d-y)f(a,d)](x-a) \nonumber \\
+ [(y-c)f(a,y) + (b-x)f(b,y)](d-c) \nonumber \\
+ [(y-c)f(b,c) + (d-y)f(b,d)](b-x) \nonumber \\
+ [(y-c)f(a,c) + (d-y)f(a,d)](x-a) \nonumber \\
+ [(y-c)f(b,c) + (d-y)f(b,d)](b-x) \nonumber \\
+ \int \int p(x,t) q(y,s) \, ds \, dt = 0. \tag{2.7}
\]

We also have

\[
\int \int p(x,t) q(y,s) \, ds \, dt = 0. \tag{2.8}
\]

Let \( M = \text{line} - \text{gamma} \Gamma + 2 \). From (2.7) and (2.8), it follows that

\[
\int \int p(x,t) q(y,s) \frac{\partial^2 f(t,s)}{\partial t \partial s} \, ds \, dt - M \, ds \, dt = \frac{1}{4} ((b-a)(d-c)f(x,y) \nonumber \\
+ [(x-a)f(a,y) + (b-x)f(b,y)](d-c) \nonumber \\
+ [(y-c)f(x,c) + (d-y)f(x,d)](b-a) \nonumber \\
+ [(y-c)f(b,c) + (d-y)f(b,d)](b-x) \nonumber \\
+ [(y-c)f(a,c) + (d-y)f(a,d)](x-a) \nonumber \\
+ [(y-c)f(a,y) + (b-x)f(b,y)](d-c) \nonumber \\
+ [(y-c)f(b,c) + (d-y)f(b,d)](b-x) \nonumber \\
+ [(y-c)f(a,c) + (d-y)f(a,d)](x-a) \nonumber \\
+ [(y-c)f(b,c) + (d-y)f(b,d)](b-x) \nonumber \\
+ \int \int p(x,t) q(y,s) \, ds \, dt = 0. \tag{2.9}
\]
On the other hand, we get

$$\left| \int \int p(x, t)q(y, s)\frac{\partial^2 f(t, s)}{\partial t \partial s} - M \right| \, ds \, dt$$

$$\leq (t, s) \max \in [a, b] \times [c, d] \left| \frac{\partial^2 f(t, s)}{\partial t \partial s} \right| - M \int \int |p(x, t)q(y, s)| \, ds \, dt.$$  \hspace{1cm} (2.10)
We also have

\[(t, s) \max \in [a, b] \times [c, d] \left| \frac{\partial^2 f(t, s)}{\partial t \partial s} - M \right| \leq \frac{\Gamma - \gamma}{2} \quad (2.11)\]

and

\[
\int \int |p(x, t)q(y, s)| \, dsdt = \frac{[(x - a)^2 + (b - x)^2][(y - c)^2 + (d - y)^2]}{16}. \quad (2.12)
\]

From (2.10) to (2.12), we easily get

\[
\left| \int_a^b \int_c^d p(x, t)q(y, s) | \frac{\partial^2 f(t, s)}{\partial t \partial s} - M | dsdt \right| \leq \frac{[(x - a)^2 + (b - x)^2][(y - c)^2 + (d - y)^2]}{32} (\Gamma - \gamma). \quad (2.13)
\]

From (2.9) and (2.13), we see that (2.1) holds. \( \Box \)

References


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