Some statistically convergent difference sequence spaces defined over real 2 - normed linear spaces

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Abstract . In order to extend the notion of convergence of sequences , st a - t ist ical convergence of sequences was introduced by Fast [5] and Schoen - berg [26] . The main aim of this article is to study the concept of statistical convergence from difference sequence spaces point of view which are de - fined over real linear 2 - normed spaces .

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1 Introduction

Throughout the article $w(X), c(X), c_0(X), \bar{c}(X), \bar{c}_0(X), \ell_\infty(X), m(X)$ and $m_0(X)$ will represent the spaces of all , convergent , null , statistically convergent , statistically null , bounded , bounded statistically convergent and bounded statistically null X val - ued sequences spaces , where $(X, \| .,. \|)$ is a real linear 2 - normed space . The zero sequence is denoted by $\bar{\theta} = (\theta, \theta, \theta, ...)$, where θ is the zero element of X.

The notion of difference sequence space was introduced by Kizmaz [1 8] , who stud - ied the difference sequence spaces $\ell_\infty(\Delta), c(\Delta)$ and $c_0(\Delta)$. The notion was further generalized by Et and Colak [2] by introducing the spaces $\ell_\infty(\Delta^n), c(\Delta^n)$ and $c_0(\Delta^n)$. Another type of generalization of the difference sequence spaces is due to Tripathy and Esi [3 1] , who studied the spaces $\ell_\infty(\Delta_m), c(\Delta_m)$ and $c_0(\Delta_m)$.

Tripathy , Esi and Tripathy [32] generalized the above notions and unified these as

$\succ$:

Let m, n be non - negative integers , then for Z a given sequence space we have

$$Z(\Delta_m^n) = \{x = (x_k) \in w : (\Delta_m^n x_k) \in Z\},$$

where $\Delta_m^n x = (\Delta_m^n x_k) = (\Delta_m^{n-1} x_k - \Delta_m^{n-1} x_{k+m})$ and $\Delta_m^0 x_k = x_k$ for all $k \in N$, which is equivalent to the following binomial representation :

$$\Delta_m^n x_k = \sum \binom{v=0}{n} - 1)^v \begin{pmatrix} n \\ v \end{pmatrix} x_{k+mv}.$$

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Taking $m = 1$, we get the spaces $\ell_\infty(\Delta^n), c(\Delta^n)$ and $c_0(\Delta^n)$ studied by Et and Colak [2] . Taking $n = 1$, we get the spaces $\ell_\infty(\Delta_m), c(\Delta_m)$ and $c_0(\Delta_m)$ studied by Tripathy and Esi [3 1] . Taking $m = n = 1$, we get the spaces $\ell_\infty(\Delta), c(\Delta)$ and $c_0(\Delta)$ introduced and studied by Kizmaz [1 8] .

Recently H . Dutta introduced another type of difference operator $\Delta^n_{(v,m)}$, where m, n are non - negative integers and $v = (v_k)$ is a sequence of non - zero scalars . For details , one may refer to Dutta [4] .

The concept of 2 - normed spaces was introduced and studied by G ä hler , a German Mathematician who worked at German Academy of Science , Berlin , in a series of paper in German language published in Mathematische Nachrichten , see for example references [3 , 9 , 1 0 , 1 1 , 1 2] . This notion which seems to be a two dimensional analogue of a normed space got the attention of a wider audience after the publication of a paper by Albert George , White Jr . [34] of USA in 1 969 entitled 2 - Banach spaces . In the same year G ä hler [1 2] published another paper on this theme in the same j ournal . A . H . Siddiqi delivered a series of lectures on this theme in various conferences in India and Iran . His j oint paper with S . G ä hler and S . C . Gupta [1 7] of 1 975 also provide valuable results related to the theme of this paper . Results up to 1 977 were summarized in the survey paper by A . H . Siddiqi [28] . For other works in this direction one may refer to [7 , 8 , 1 3 , 1 6 , 24 , 29] . In the recent years , a number of articles devoted
to statistical convergence (see G ü rdal and Pehlivan [1 4]) and its generalization , ideal convergence (see G ü rdal and cedilla − S ahiner [1 5]) using 2 - norm , have been published .

Let X be a real vector space of dimension d, where $2 \leq d$. A real - valued function $\| .,. \|$ on X^2 satisfying the following four conditions :

(1) $\| x_1, x_2 \| = 0$ if and only if x_1, x_2 are linearly dependent ,

(2) $\| x_1, x_2 \|$ is invariant under permutation ,

(3) $\| \alpha x_1, x_2 \| = | \alpha | \| x_1, x_2 \|, \text{forany} \alpha \in R,$

(4) $\| x + x', x_2 \| \leq \| x, x_2 \| + \| x', x_2 \|$

is called a $2 - norm$ on X, and the pair $(X, \| .,. \|)$ is called a 2 - normed space . and^{The}notionofSchoenberg[26^{statistical}]convergenceindependently.Later^{was} on it^{studied} was @thefurtherinitialstageinv t $[^{[5]}_{[25]},$
Fridy [6] , Buck [1] , Sen and Tripathy [33] and many others . G ü rdal and Pehlivan [1 4] studied st atistical convergence in 2 - Banach space .

A subset E of N is said to have density $\delta(E)$ if

$$\delta(E) = \lim_{n \to \infty} 1_n \sum_{n}^{k=1} \chi E(k) \text{exists},$$

where χE is the characteristic function of E.

The following inequality will be used throughout the article :

Let $p = (pk)$ be a positive sequence of real numbers with $0 < pk \leq \sup pk = G$, $D = \max \{1, 2^{G-1}\}$. Then for all $a_k, b_k \in C$ for all $k \in N$, we have

$$| a_k + b_k |^{pk} \leq D\{| a_k |^{pk} + | b_k |^{pk}\}$$

$$\text{andfor} \lambda \in C,$$

$$| \lambda |^{pk} \leq \max\{1, | \lambda |^G\}.$$

The notion of paranormed sequence space was studied at the initial stage by Simons [27] and Nakano [22] . Later on it was further investigated by Maddox [2 1] , Lascarides [1 9] , Lascarides and Maddox [20] , Nanda [23] , B . c . Tripathy [30] , Tripathy and Sen [33] and a number of workers in the field of sequence spaces .

2 Definitions and background

A sequence space E is said to be *solid*(or *normal*) if $(x_k) \in E$ implies $(\alpha_k x_k) \in E$ for all sequences of scalars (α_k) with $\mid \alpha_k \mid \leq 1$ for all $k \in N$.

A sequence space E is said to be *symmetric* if $(x^{)}_{\pi(k)} \in E$ whenever $(x_k) \in E$, where π is a permutation on N.

A sequence (x_k) is said to be *statisticallyconvergent* to L if for every $\varepsilon > 0$,

$$\delta(\{k \in N :\mid x_k - L \mid \geq \varepsilon\}) = 0.$$

For $L = 0$, we say this is *statisticallynull*.

Throughout $\bar{c}, \bar{c}_0$ denote the classes of all statistically convergent and statistically null sequences respectively .

A sequence (x_k) in a 2 - normed space $(X, \| .,. \|)$ is said to *converge* to some $L \in X$ in the 2 - norm if

$$\lim_{k\to\infty} \| x_k - L, u_1 \| \quad = 0, \quad \text{forevery} u_1 \in X.$$

A sequence (x_k) in a 2 - normed space $(X, \| .,. \|)$ is said to be *Cauchy* with respect to the 2 - norm if

$$k, \lim l \to \infty \| x_k - x_l, u_1 \| \quad = 0, \quad \text{forevery} u_1 \in X.$$

If every Cauchy sequence in X converges to some $L \in X$, then X is said to be complete with respect to the 2 - norm . Any complete 2 - normed space is said to be 2 - Banach space .

We introduce the following definitions in this article . Let m and n be two non - negative integers and $p = (pk)$ be a sequence of strictly positive real numbers . Then

$$\bar{c}(\| .,. \|, \Delta^n_{(m)}, p) =$$

$$\{(x_k) \in w(X) : (\| \Delta^n_{(m)} x_k - L, z \|)^{pk} st \overset{@}{\to} 0, \quad \text{for every } z \in X \text{ and some } L \in X\},$$

$$\bar{c}_0(\| .,. \|, \Delta^n_{(m)}, p) = \{(x_k) \in w(X) : (\| \Delta^n_{(m)} x_k, z \|)^{pk} st@-0, \quad \text{forevery} z \in X\},$$

We procure the following definition for the sake of completeness :

$$\ell_\infty(\| .,. \|, \Delta^n_{(m)}, p) = \{(x_k) \in w(X) : \sup_{k\geq 1}(\| \Delta^n_{(m)} x_k, z \|)^{pk} < \infty, \quad \text{forevery} z \in X\},$$

The following definition is introduced :

$$W(\| .,. \|, \Delta^n_{(m)}, p) =$$

$$\left\{ (x_k) \in w(X) : \lim_{j\to\infty} \sum_{k=1}^{j} (\| \Delta^n_{(m)} x_k - L, z \|)^{pk} = 0, \quad \text{forevery} z \in X \text{andsome} L \in X \right\}.$$

$m(\| .,. \|, \Delta^n_{(m)}, p) = \bar{c}(\| .,. \|, \Delta^n_{(m)}, p) \cap \Rightarrow notdefperiod - notdef, .notdefbardbl - notdef, notdef\Delta notdef - n_{)not}$

$m_0(\| .,. \|, \Delta^n_{(m)}, p) = \bar{c}_0(\| .,. \|, \Delta^n_{(m)}, p) \cap \Rightarrow notdefperiod - notdef, notdef.bardbl - notdef, notdef\Delta not$

and

where $(\Delta^n_{(m)} x_k) = (\Delta^{n-1}_{(m)} x_k - \Delta^{n-1}_{(m)} x_{k-m})$ and $\Delta^0_{(m)} x_k = x_k$ for all $k \in N$, which is equivalent to the following binomial representation :

$$\Delta^n_{(m)} x_k = \sum_{n}^{v=0} (-1)^v \begin{pmatrix} n \\ v \end{pmatrix} x_{k-mv}.$$

In the above expansion we take $x_k = 0$ for non - positive values of k.

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values of m and n. In particular for $n = 0$, the above spaces reduce to the spaces

$\bar{c}(\| .,. \|, p), \bar{c}_0(\| .,. \|, p), \ell_\infty(\| .,. \|, p), W(\| .,. \|, p), m(\| .,. \|, p)\text{and}m_0(\| .,. \|, p)\text{respec}-$

tively .

Again if we replace the base space X, which is a real linear 2 - normed space by C, complete normed linear space , we get the spaces $\bar{c}(\Delta^n_{(m)}, p)$, $\bar{c}_0(\Delta^n_{(m)}, p)$,

$$\ell_\infty(\Delta^n_{(m)}, p), W(\Delta^n_{(m)}, p), m(\Delta^n_{(m)}, p)\text{and}m_0(\Delta^n_{(m)}, p)\text{respectively}.$$

Further if we take $X = C, pk = l$, a constant for all $k \in N$ and $n = 0$, we get the spaces $\bar{c}, \bar{c}_0, \ell_\infty, W, m$ and m_0 respectively .

First we procure some known results ; those will help in establishing the results of this article .

Lemma 2 . 1 . ([*33*]) *For two s equences pk and* (t_k) *we have* $m_0(p) \supseteq m_0(t)$ *if and*
only if $\lim\inf_{k \in K} p^k_{t_k} > 0$, *where* $K \subseteq N$ *such that* $\delta(K) = 1$.

Lemma 2 . 2 . ([*33*]) *Let* $h = \inf pk$ *and* $G = \sup pk$, *then th e following are equivalent :*

(i) $G < \infty$ *and* $h > 0$,

$$(ii) m(p) = m.$$

We now cite the following two known 2 - normed spaces . **Example 2 . 1 .** *Consider the spaces* Z *where* $Z = \ell_\infty, c$ *and* c_0 *of real s equences . Let us define :*

$\| x, y \| = \sup \sup \mid x_i y_j - x_j yi \mid,$ *where* $x = (x_1, x_2, ...)$ *and* $y = (y1, y2, ...) \in Z$.

$$i \in Nj \in N$$

Then $\| .,. \|$ *E is a 2 - norm on* Z. **Example 2 . 2 .** *Let us take* $X = R^2$ *and consider the function on* X *defined as :*

$$\| x_1, x_2 \| E = abs(\begin{vmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{vmatrix}^{)}, \quad where x_i = (x_{i1}, x_{i2}) \in R^2 foreachi = 1, 2.$$

Then $\| .,. \|$ *E is a 2 - norm on* X *known as Euclidean 2 - norm .* **Remark 2 . 1 .** *Every clos ed linear subspace of an arbitrary linear normed space* E, *different from* E, *is a nowhere dense s e t in* E.

3 Main results

In this section we mainly investigate several linear topological and algebraic properties relevant to the spaces $\bar{c}_0(\| .,. \|, \Delta^n_{(m)}, p)$, $\bar{c}(\| .,. \|, \Delta^n_{(m)}, p), m(\| .,. \|, \Delta^n_{(m)}, p)$ and

$$m_0(\| .,. \|, \Delta^n_{(m)}, p)\text{respectively.}$$

Theorem 3 . 1 . *Let $p = (pk)$ be a bounded s equence of strictly positive real numbers . Then th e class es of s equences $\bar{c}_0(\| .,. \|, \Delta^n_{(m)}, p), \bar{c}(\| .,. \|, \Delta^n_{(m)}, p), m_0(\| .,. \|, \Delta^n_{(m)}, p)$ and $m(\| .,. \|, \Delta^n_{(m)}, p)$ are linear spaces .*

Proof . We prove the theorem only for the space $\bar{c}(\| .,. \|, \Delta^n_{(m)}, p)$ and for the other spaces it will follow on applying similar arguments .

Let $(x_k), (yk) \in \bar{c}(\| .,. \|, \Delta^n_{(m)}, p)$. Then there exist $L, J \in X$ such that for every

$$z \in X$$
$$(\| \Delta^n_{(m)} x_k - L, z \|)^{pk} st@\to 0$$

and

$$(\| \Delta^n_{(m)} yk - J, z \|)^{pk} stat\to 0.$$

Let α, β be scalars . Then we have for every $z \in X$

$$(\| \Delta^n_{(m)}(\alpha x_k + \beta yk) - (\alpha L + \beta J), z \|)^{pk} = (\| \Delta^n_{(m)} \alpha(x_k - L) + \Delta^n_{(m)} \beta(yk - J), z \|)^{pk}$$
$$\leq (| \alpha | \| \Delta^n_{(m)} x_k - L, z \| \quad + | \beta | \| \Delta^n_{(m)} yk - J, z \|)^{pk}$$
$$\leq D \mid \alpha \mid^G (\| \Delta^n_{(m)} x_k - L, z \|)^{pk} + D \mid \beta \mid^G (\| \Delta^n_{(m)} yk - J, z \|)^{pk}, \quad \text{where} G = \sup pk.$$
$$stat\to 0, \quad \text{as} k \to \infty.$$

Hence $\bar{c}(\| .,. \|, \Delta^n_{(m)}, p)$ is a linear space . □ **Theorem 3 . 2 .** *The spaces $m_0(\| .,. \|, \Delta^n_{(m)}, p)$ and $m(\| .,. \|, \Delta^n_{(m)}, p)$ are paranormed spaces , paranormed b y*

$$g(x) = \sup_{N_{\in_k}, z \in X} (\| \Delta^n_{(m)} x_k, z \|) p_k H, \quad where H = \max\{1, \sup_k pk\}.$$

Proof . Clearly $g(x) = g(-x); x = \theta$ implies $g(\theta) = 0$. Now

$$g(x + y) = k \in \sup_{N, z \in X} (\| \Delta^n_{(m)}(x_k + yk), z \|) p_k H.$$
$$\leq k \in \sup_{N, z \in} X(\| \Delta^n_{(m)} x_k, z \|) p_k H + k \in \sup_{N, z \in} X(\| \Delta^n_{(m)} yk, z \|) p_k H.$$

This implies that

$$g(x + y) \leq g(x) + g(y).$$

 The continuity of the scalar multiplication follows from the following equality :

$$g(\lambda x) = k \in \sup_{N,z\in X}(\| \Delta^n_{(m)}(\lambda x_k), z \|)p_k H.$$
$$= k \sup_{N_\in, z\in} X(| \lambda |\| \Delta^n_{(m)} x_k, z \|)p_k H.$$
$$\leq \max(1, | \lambda |)k \sup_{N_\in, z\in} X(| \lambda |\| \Delta^n_{(m)} x_k, z \|)p_k H.$$
$$= \max(1, | \lambda |)g(x).$$

Hence the spaces $m_0(\| .,. \|, \Delta^n_{(m)}, p)$ and $m(\| .,. \|, \Delta^n_{(m)}, p)$ are paranormed by g.□ **Remark 3 . 1 .** *For any two s equences* $p = (pk)$ *and* $t = (t_k)$ *of positive real num - bers and for any two 2 - norms* $\| .,. \| 1$ *and* $\| .,. \| 2$ *on* X *we have* $Z(\| .,. \| 1, \Delta^n_{(m)}, p) \cap$

Proof . The proof follows from the fact that the zero element belongs to each of the classes of sequences involved in the intersection . □ **Theorem 3 . 3 .** *The spaces* $Z(\| .,. \|, \Delta^n_{(m)}, p)$ *are not s o lid in general ,* *where* $Z =$

$$\bar{c}, m, \bar{c}_0, m_0.$$

Proof . To show that the spaces are not solid in general , consider the following exam - ples .

Example 3 . 1 . *Let* $m = 3, n = 1$ *and consider the 2 - normed space as defined in Example 2 . 1 .* *Let* $pk = 5$ *for all* $k \in N$. *Consider the s equence* (x_k), *where* $x_k = (i_{x_k})$ *is defined by* $(i_{x_k}) = (k, k, k, ...)$ *for each fixed* $k \in N$. *Then* $(x_k) \in Z(\| .,. \|, \Delta^1_{(3)}, p)$ *for* $Z = \bar{c}, m$. *Let* $\alpha_k = (-1)^k$, *th en* $(\alpha_k x_k) slash - element Z(\| .,. \|, \Delta^1_{(3)}, p)$ *for* $Z = \bar{c}, m$. *Thus* $Z(\| .,. \|, \Delta^n_{(m)}, p)$ *for* $Z = \bar{c}, m$ *are not so lid in general* . □

Example 3 . 2 . *Let* $m = 3, n = 1$ *and consider the 2 - normed space as defined in Example 2 . 1 .* *Let* $pk = 1$ *for all* k *odd and* $pk = 2$ *for al l* k *even . Consider the s equence* (x_k), *where* $x_k = (i_{x_k})$ *is defined by* $(i_{x_k}) = (3, 3, 3, ...)$ *for each fixed* $k \in N$. *Then* $(x_k) \in Z(\| .,. \|, \Delta^1_{(3)}, p)$ *for* $Z = \bar{c}_0, m_0$. *Let* $\alpha_k = (-1)^k$, *th en* $(\alpha_k x_k)$ $slash - element$ $Z(\| .,. \|, \Delta^1_{(3)}, p)$ *for* $Z = \bar{c}_0, m_0$. *Thus* $Z(\| .,. \|, \Delta^n_{(m)}, p)$ *for* $Z = \bar{c}_0, m_0$ *are not s o lid in general* . □

Theorem 3 . 4 . *The spaces* $Z(\| .,. \|, \Delta^n_{(m)}, p)$ *are not symmetric in general ,* *where*

$$Z = \bar{c}, m, \bar{c}_0, m_0.$$

Proof . To show that the spaces are not symmetric in general , consider the following example .

Example 3 . 3 . *Let* $m = 2, n = 2$ *and consider the 2 - normed space as defined in Example 2 . 2 .* *Let* $pk = 2$ *for all* k *odd and* $pk = 1$ *for al l* k *even . Consider the s equence* (x_k) *defined by* $x_k = (k, k)$ *for ea ch fixed* $k \in N$. *Then* $\Delta^2_{(2)} x_k = x_k -$

 $2x_{k-2}+x_{k-4}, k \in N$. *Hence* $(x_k) \in Z(\| .,. \|, \Delta^2_{(2),}p)$ *for* $Z = \bar{c}_{,}m, \bar{c}_{0,}m_0$. *Let* (yk) *be a rearrangement of* (x_k), *which is defined as follows :*

$$(yk) = \{x_1, x_2, x_4, x_3, x_9, x_5, x_{16}, x_6, x_{25}, x_7, x_{36}, x_8, x_{49}, x_{10}, ...\}.$$

Then $(yk) element - slash Z(\| .,. \|, \Delta^2_{(2),}p)$ *for* $Z = \bar{c}_{,}m, \bar{c}_{0,}m_0$. *Hence for* $Z = \bar{c}_{,}m, \bar{c}_{0,}m_0$, *the spaces* $Z(\| .,. \|, \Delta^n_{(m),}p)$ *are not symmetric in general* . □ **Remark 3 . 2 .** *For two s equences* (pk) *and* (t_k) *we have*

$$m_0(\| .,. \|, \Delta^n_{(m),}p) \supseteq m_0(\| .,. \|, \Delta^n_{(m),}t)$$

if and only if $\lim\inf_{k\in K} p^k_{t_k} > 0$, *where* $K \subseteq N$ *such that* $\delta(K) = 1$. *Proof* . If we take $(yk) = (\| (m)^{\rho}_{\Delta^n} x_k, z \|)$ for all $k \in N$, then the result follows from the Lemma 2.1. □

Remark 3 . 3 . *For two s equences* (pk) *and* (t_k) *we have*

$$m_0(\| .,. \|, \Delta^n_{(m),}p) = m_0(\| .,. \|, \Delta^n_{(m),}t)$$

if and only if $\lim\inf_{k\in K} p^k_{t_k} > 0$ *and* $\lim\inf_{k\in K} t_k p_k > 0$, *where* $K \subseteq N$ *such that* $\delta(K) = 1$.

Proof . This result is a consequence of the above result . □ **Remark 3 . 4 .** *Let* $h = \inf pk$ *and* $G = \sup pk$, *then the fo l lowing are equivalent :*

$$(i) G = \sup pk and h > 0,$$
$$(ii) m(\| .,. \|, \Delta^n_{(m),}p) = m(\| .,. \|, \Delta^n_{(m)})$$

Proof . Taking $(yk) = (\| (m)^{\rho}_{\Delta^n} x_k, z \|)$ for all $k \in N$ and using the Lemma 2 . 2 , we get the result . □

Theorem 3 . 5 . *Let* $p = (pk)$ *be a s equence of non - negative bounded real numbers such that* $\inf pk > 0$. *Then* $m(\| .,. \|, \Delta^n_{(m),}p) = W(\| .,. \|, \Delta^n_{(m),}p) \cap \Rightarrow notdef notdef - period - commanotdef. \| notdef$ Delta − notdef$negationslash - notdef - notdef n - notdef,)_{notdef negationslash - notdef - notdef - p}\{.notdef - infinitynotdef$

Proof . Let $(x_k) \in W(\| .,. \|, \Delta^n_{(m),}p) \cap \Rightarrow notdef notdef - period, .notdef notdef - bardbl, notdef \Delta n - notdef notdef)_{,notdef - notdef - negationslash - p}\{notdef - notdef - period$ T − notdef $infinity - h - e$ n $notdef$o − notdef − notdef − f − notdef − notdef r $notdef notdef - a$g − notdef$notdef - v$ en $\varepsilon > 0$ eh ve

$$(\| \Delta n_{)^{x}k} - Lz \|)pk \text{ c}_{\text{a}}\text{rd}k \leq j(: \| \Delta^n_{)} xk - Lz \|)pk \geq \varepsilon \quad epsilon - period$$

1 From the above inequality , it follows that $(x_k) \in m(\| .,. \|, \Delta^n_{(m),}p)$.

Conversely let $(x_k) \in m(\| .,. \|, \Delta^n_{(m),}p)$ and $\rho > 0$ be such that

$(\| \Delta^n_{(m)} x_k - L, z \|)^{pk} stat \to 0$, for every $z \in X$ and some $L \in X$.

For a given $\varepsilon > 0$, let $B = \sup_k(\| \Delta^n_{(m)} x_k - L, z \|)p_k H \quad < \infty$, where $H = \max\{ 1 , \sup pk\}$. Let $L_j \quad = \quad \{k \leq j : (\| \Delta^n_{(m)} x_k - L, z \|)^{pk} \geq \varepsilon 2\}$. Since $(x_k) \quad \in \quad m(\| ., . \|, \Delta^n_{(m)}, p)$, so
$Card\{jL_j\} \quad \to 0$ as $j \to \infty$. Let $n_0 > 0$ be such that $Card\{jL_j\} \quad < \quad 2\varepsilon BH$ for all $j > n_0$.
Then for all $j > n_0$, we have

$$1j \sum_{j}^{k=1} (\| \Delta^n_{(m)} x_k - L, z \|)^{p_k} \quad = 1j \sum_{kelement-slashL_j} ($$

$$\leq$$

$$\text{Hence}(x_k) \in W(\| ., . \|, \Delta^n_{(m)}, p) \cap \Rightarrow notdefnotdef - periodnotdef.notdef - bardbl, notdef\Delta n - notdefnotdef)_{,ne}$$

The following result is a consequence of the above theorem . **Corollary 3 . 1 .** *Let* (pk) *and* (t_k) *be two bounded s equences of real numbers such that* $\inf pk > 0$ *and* $\inf t_k > 0$. *Then*

$$W(\| ., . \|, \Delta^n_{(m)}, p) \cap \Rightarrow notdefnotdef - period, .notdefnotdef - bardbl, notdef\Delta n - notdef_{)comma-notdefp-negationsla}$$

orem 3period − six *Let* $(X, \| ., . \|)$ *be a 2 - Banach space , then the spaces* $m(\| ., . \| , \Delta^n_{(m)}, p)$ *and* $m_0(\| ., . \|, \Delta^n_{(m)}, p)$ *are complete .*
Proof . We prove the result for the space $m_0(\| ., . \|, \Delta^n_{(m)}, p)$ and for the other space it will follow on applying similar arguments . Let (x^i) be a Cauchy sequence in $m_0(\| ., . \|, \Delta^n_{(m)}, p)$. Then for a given $\varepsilon (0 < \varepsilon < 1)$, there exists a positive integer n_0 such that $g(x^i - x^j) < \varepsilon$ for all $i, j \geq n_0$. This implies that

$$\sup_{N_{\in_k}, z \in X} (\| \Delta^n_{(m)} i_{x_k} - \Delta^n_{(m)} j_{x_k}, z \|)p_k H \quad < \varepsilon$$

for all $i, j \geq n_0$. It follows that for every $z \in X$,
$(\| \Delta^n_{(m)}(i_{x_k} - j_{x_k}), z \|) < \varepsilon$ for each $k \geq 1$ and $i, j \geq n_0$. Hence $(\Delta^n_{(m)} i_{x_k})$ is a Cauchy sequence in the 2 - Banach space X for all $k \in N$. Thus
$(\Delta^n_{(m)} i_{x_k})$ is convergent in X for all $k \in N$. For simplicity , let $\lim_{i\to\infty} \Delta^n_{(m)} i_{x_k} = yk$ for each $k \in N$. Let $k = 1$, then we have

$$\lim_{i\to\infty} \Delta^n_{(m)} i_{x_1} = \lim_{i\to\infty} \sum_{n}^{v=0} (-1)^v \begin{pmatrix} n \\ v \end{pmatrix} i_{x_1 - mv} = \lim_{i\to\infty} i_{x_1} = y1. \tag{3.1}$$

Similarly , we have ,

$$\lim_{i\to\infty} \Delta^n_{(m)} i_{x_k} = \lim_{i\to\infty} i_{x_k} = yk, \quad \text{for} k = 1, ..., nm. \tag{3.2}$$

Thus from (3 . 1) and (3 . 2) , we have $\lim_{i\to\infty} i_{x_1+nm}$ exists . Let $\lim_{i\to\infty} i_{x_1+nm} = x_{1+nm}$. Proceeding in this way inductively , we have $\lim_{i\to\infty} xi_k = x_k$, say exists for each $k \in N$.
Now we have for all $i, j \geq n_0$,

$$\sup_{N_{\in_k}, z\in X} (\| \Delta^n_{(m)}(i_{x_k} - j_{x_k}), z \|)p_k H \quad < \varepsilon$$

$$\Rightarrow \lim_{j\to\infty} \{_{k\in N} \sup_{,z\in X} (\| \Delta^n_{(m)}(i_{x_k} - j_{x_k}), z \|)p_k H\} < \varepsilon \quad \text{forall} i \geq n_0$$

$$\Rightarrow \sup_{N_{\in_k}, z\in X} (\| \Delta^n_{(m)}(i_{x_k} - x_k), z \|)p_k H \quad < \varepsilon \quad \text{forall} i \geq n_0.$$

It follows that $(x^i - x) \in m_0(\| .,. \|, \Delta^n_{(m)}, p)$. Since $(x^i) \in m_0(\| .,.(\|, \Delta^n_{(m)}, p)$ and $m_0(\| .,. \|, \Delta^n_{(m)}, p)$ is a linear space , so we have $x = x^i - (x^i - x) \in m_0 \quad \| .,. \|, \Delta^n_{(m)}, p)$. This completes the proof . □
As consequence , it follows that $m_0(\| .,. \|, \Delta^n_{(m)}, p)$ and $m(\| .,. \|, \Delta^n_{(m)}, p)$ are closed subspaces of $\ell_\infty(\| .,. \|, \Delta^n_{(m)}, p)$. Since the inclusion relations

$$m_0(\| .,. \|, \Delta^n_{(m)}, p) \subset \ell_\infty(\| .,. \|, \Delta^n_{(m)}, p), \quad m(\| .,. \|, \Delta^n_{(m)}, p) \subset \ell_\infty(\| .,. \|, \Delta^n_{(m)}, p)$$

are strict , we have the following result .
Corollary 3 . 2 . *The spaces* $m_0() \| .,. \|, \Delta^n_{(m)}, p)$ *and* $m(\| .,. \|, \Delta^n_{(m)}, p)$ *are nowhere dense subsets of* $\ell_\infty(\| .,. \|, \Delta^n_{(m)}, p$.

4 Conclusions

In this paper we introduce the difference sequence spaces $\bar{c}_0(\| .,. \|, \Delta^n_{(m)}, p)$,

$$\bar{c}(\| .,. \|, \Delta^n_{(m)}, p), m(\| .,. \|, \Delta^n_{(m)}, p), m_0(\| .,. \|, \Delta^n_{(m)}, p), \ell_\infty(\| .,. \|, \Delta^n_{(m)}, p) \text{and} W(\| .,. \|,$$

$\Delta^n_{(m)}, p)$ with base space a real linear 2 - normed space . We study the spaces $\bar{c}_0(\| .,. \|, \Delta^n_{(m)}, p), \bar{c}(\| .,. \|, \Delta^n_{(m)}, p), m(\| .,. \|, \Delta^n_{(m)}, p)$ and $m_0(\| .,. \|, \Delta^n_{(m)}, p)$ with the help of the

$$\text{spacesearity, existence}^{\ell_\infty(\|.,.\|,}\Delta^n_{\text{of}}(_{\text{paranorm}} m)^{,} p)\text{and}_{\text{and}} W(_{\text{investigate}} \| .,. \|, \Delta^n_{(m)}{}^{;}_{\text{the}} p)\text{for}_{\text{spaces}}\text{different}_{\text{forsolidity}}\text{propert}$$

$$\text{Furtherweproveparanormedthat}^{\text{the}}_{\text{spaceswhen}}\text{spacesthe} m\text{base}^{(}_{\text{space}}{}^{\|.,.\|,}\Delta^n_{\text{is}}(_{\text{a}} m)^{,} 2^{p)\text{and} m}_{-\text{Banac}}$$

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