AN OVERVIEW OF THE APPLICATIONS OF

MULTISETS[1]

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Abstract . This paper presents a systemization of representation of multisets and basic operations under multisets , and an overview of the applications of multisets in mathematics , computer science and related areas .

1 . Introduction

A multiset (mset , for short) is an unordered collection of objects (called the elements) in which , unlike a standard (Cantorian) set , elements are allowed to repeat . In other words , an mset is a set to which elements may belong more than once , and hence it is a non - Cantorian set . In this paper , we endeavour to present an overview of basics of multiset and applications .

The term multiset , as Knuth ([46] , p . 36) notes , was first suggested by N . G . de Bruijn in a private communication to him . Owing to its aptness , it has replaced a variety of terms , viz . list , heap , bunch , bag , sample , weighted set , occurrence set , and fireset (finitely repeated element set) used in different contexts but conveying synonimity with mset .

As mentioned earlier , elements are allowed to repeat in an mset (finitely in most of the known application areas , albeit in a theoretical development infinite multiplicities of elements are also dealt with (see [1 1] , [37] , [5 1] , [72] and [30] , in particular) .

The number of copies (([1 7] , P . 5) prefers to call it ' multiples ') of an element appearing in an mset is called its multiplicity . Moreover , multiple o ccurrences of an element in an mset are treated without preference (perhaps to retain the force of classical concept of identity) . We mention [64] for an earliest extensive treatment of indistinguishability of repeated elements without any preference , and [85] for an alternative treatment .

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The number of distinct elements in an mset $M($ which need not be finite $)$ and their multiplicities j ointly determine its cardinality , denoted by $C(M)$.   In other words , the cardinality of an mset is the sum of multiplicities of all its elements .   An mset M is called finite if the number of distinct elements in M
and their multiplicities are both finite , it is infinite otherwise .   Thus , an mset M is infinite if either the number of elements in M is infinite or the multiplicity of one or more of its elements is infinite , i . e . $C(M) \geq \aleph_0$.

The root or support or carrier of an mset $M,$ denoted by M^*, is defined as

$$\succ:$$
$$M^* = \{x \in M \mid M(x) > 0\}.$$

The elements of the root set of an mset are called the generators of that mset .

A considerable amount of efforts have also gone into the study of msets with negative multiplicities (see [9] , [36] , [74] , [72] , [30] , [84] , [85] , in particular) .

2 .   Basics of Multiset

2 . 1 .   Representations of Multisets

2 . 1 . 1 .   Multiplicative form

Following Meyer and McRobbie [57] , the use of square brackets to represent an mset has become almost standard .   Thus , an mset containing one o ccurrence of $a,$ two occurrences of $b,$ and three o ccurrences of c is notationally written as
$[[a, b, b, c, c, c]]$ or $[a, b, b, c, c, c]$ or $[a, b, c]1, 2, 3$ or $[a^1, b^2, c^3]$ or $[a1, b2, c3]$, depend - ing on one ' s taste and convenience .

2 . 1 . 2 .   Linear form

Wildberger [85] puts forward a linear notation for multisets , which seems quite innovative , especially when negative multiplicities (integral as well as rational) are to be dealt with . For example , the mset $M = [a, b, c]1, 2, 3$ can be written as

$$M = [a] + 2[b] + 3[c].$$

Similarly , a rational mset can be represented , for example ,

$$N = 3^2[5] - 2^1[18].$$

In order to accommodate negative multiplicities round brackets are used : (a) in an mset stands for negative of $a;$ for example ,

$$[2, 4, (5), (5), 4] = [2] + 2[4] - 2[5].$$

In the same place , the distinction between the terms ' element ' and ' object ' o ccurring in an mset is made explicit as fallows :

Each individual o ccurrence of an object x in an mset A is called an element of $A.$ Thus , in the linear notation of M above ; $b,$ for example , is an object appearing twice , and every occurrence of b is an element of $M.$ It follows that the distinct elements of an mset are the objects .   An object is an element if its multiplicity is unity .

Further , the following notations used in ([85] , pp . 5 − 6) to represent data structures of set , ordered set , multiset and list , are quite instructive :

A collection containing $x, y, z, ...$ is denoted by $\{xyz...\}$ or $\{xyz \ \ ...\}$ if it is a set ; $\{x, y, z...\}$ if it is an ordered set ; $[xyz....] = [xyz...]$ if it is a multiset ; and $[x, y, z, ...]$ if it is a list .

Note that a list is an ordered sequence of elements with repetitions allowed , whereas an mset is a sequence with its ordering stripped off .

2 . 1 . 3 . Multiset as a Sequence

A multiset can also be represented as a sequence in which the multiplicity of an element equals the number of times the element o ccurs in the sequence , which is exactly Dedekind ' s ' frequency $- number$ '. *The* idea is to construct an mset as

a sequence (a function with domain $\aleph$, the set of natural numbers) and ignore the ordering of its elements , which can be done by taking all permutations of the domain of the sequence .

2 . 1 . 4 . Multiset as a Family of Sets

A multiset can also be represented as a family of sets , which is altogether a generalization of the idea of a sequence described above . Thus , the family of sets $F = \{F_i\}, i \in I$, where $F_i = F_j$, if $i = j$, which identifies a repeated element , represents an mset . Clearly , such a family F is a function : $I \rightarrow \{F_i \mid i \in I\}$, which in turn , is a sequence if $I = \aleph$.

2 . 1 . 5 . Multiset as a Numeric - valued function

Representation of an mset as a numeric - valued or cardinal - valued function abo - unds , especially in the application areas . Formally , an mset is just a mapping from some ground or generic or universal set into some set of numbers . For example , an mset $A = [x, y, z]1, 2, 3$ is a mapping from a ground set S to $\aleph$, the set of natural numbers with zero , defined by

$$\alpha(t) = braceex - braceex - braceleftmid - braceex - braceex - braceleftbt \quad \begin{array}{lll} 1, & \text{if} & t = x \\ 2, & \text{if} & t = y \\ 3, & \text{if} & t = z \\ 0, & \text{for} & all the remaining t \in S. \end{array}$$

In general terms , for a given ground set S and a numeric set T, we call a mapping

$$braceex - braceex - braceleftmid - braceex - braceex - braceleftbt \quad \begin{array}{l} a set, if T = \{0, 1\}; \\ a multiset, if T = \aleph, the set of natural numbers; \\ a signed multiset (or, hybrid/shadow set) if T = \\ a fuzzy (or hazy) set if T = [0, 1] \subseteq R, a 2 - valu \end{array}$$

2 . 1 . 6 .   Multiset as a generalized characteristic function

Similar to the representation of a set by its characteristic function (function
whose range is { 0 , 1 }) , a multiset or hybrid set is determined by its generalized
characteristic function (whose range is the set of integers , positive , negative or
zero) , see [84] for details .

2 . 2 .   Operations under Mset

The monograph [46] can be considered as the earliest reference describing intuitively prop-
erties of msets in a sufficient detail .   During the recent years , a good number of papers ([5 1
] ,   [36]   ,   [37] ,   [8] ,   [85] , and others) have appeared . We endeavour to present
an overview of various approaches in this regard . We will adhere to function - approach and use
Dom (f), Ran (f) to denote the domain and range respectively of a given function f .

Definition 1 .   *Multiset*

Let $D = \{x_1, x_2, ..., x_j, ...\}$ be a set . An mset A over D is a cardinal - valued
function , i . e . $A : D \to \aleph = \{0, 1, 2, ...\}$ such that for $x \in$ Dom (A) implies $A(x)$
is a cardinal and $A(x) = m_A(x) > 0$, where $m_A(x)$ denotes the number of t imes an object x o
ccurs in A, i . e .   a counting function of A.   The set D is called the ground or generic set of
the class of all msets containing objects from D.

An mset A can also be represented by the set of pairs as follows :

$$A = \{\langle m_A(x_1), \quad x_1\rangle, ..., \langle m_A(x_j), \quad x_j\rangle, ...\}$$
$$\text{or}, A = \{m_A(x_1).x_1, ..., m_A(x_j).x_j, ...\}.$$

Relatedly , an mset is called ' regular ' or ' constant ' if all it s objects occur with the same
multiplicity .   Also , an mset is called ' simple ' if all its objects are the same , for example
, $[x]3$ is a simple mset containing x as its only object (see [8]) . Clearly , the root set of every
simple mset contains a single object .

Definition 2 .   *Dressed epsilon symbol* $, \in +$

The symbol $\in_+$ was first introduced by Singh and Singh [80] . For any object x o ccurring
as an element of an mset A, i . e . $m_A(x) \quad > 0$, we write $x \in_+ A$,
where $\in_+$ (dressed epsilon is a binary predicate intended to mean ' belongs to at least once ' ,
as $\in$ is ' belongs to only once ' in the case of sets .   Thus , $m_A(x) = 0$

implies$_{x \in^k A}$means$^{xelement-slashA,}x$:$^{\text{and}}x \in_+^k$ belongs$A_{k\text{times}}^{\text{implies}}toA'$:$_x$belongs$_{\text{Themset}}toA$for@$_{\text{any}}$,$_{\text{least}k\text{timesground}'}^{\text{set}D}$ however$_{\text{isc}}$

empty , denoted by $\varnothing$ or [ ] , if $m_\varnothing(x) = 0$ for all $x \in D$.   Further , in order to make our
presentation concise , we shall follow some terminologies introduced in ([37] , pp . 2 1 2 $-$ 2 1
3) : (' A (x) denotes the number of copies of x , including x itself ,
belonging to Dom (A) ') , which is exactly the Dedekind ' s frequency number .

Definition 3 .   *Multisubsets   (or msubsets , for short)*

Let A and B be two msets , A being an msubset or a submultiset of B, written as $A \subseteq B$
or $B \supseteq A$, if $m_A(x) \leq m_B(x)$ for all $x \in D$.   Also , if $A \subseteq B$

and $A \neq B$, then A is called a proper submset of B. An mset is called the parent in relation to its msubsets .

It follows from the definition of multisets that $A = B$ if and only if for $x \in D$, $m_A(x) = m_B(x)$. Also , $A = B \implies A^* = B^*$, but the converse need not hold .

It is easy to see that $\subseteq$ is antisymmetric , i . e . $A \subseteq B$ and $B \subseteq A \implies A = B$, and it is a partial ordering on the class of msets defined on a given generic domain . Clearly , $\varnothing$ is a submset of every mset . Note that the terms ' element ' and ' object ' are being distinguished throughout , and coincide if a generic set is in consideration . We wish to emphasize that introduction of $\in_+$ greatly enhance the language of msets . For example , $A \subseteq B$ stands for

$$\forall z \forall k (z \in^k A \rightarrow z \in_+^k B).$$

Relatedly , a ' whole ' msubset of a given mset contains all multiplicities of common elements ; while a ' full ' msubset contains all objects of the parent mset , and accordingly , every mset contains a unique full msubset , its root set . Clearly , for any two msets A and B, if $A \subseteq B$ and $Dom(A) = Dom(B)$, then A is a full msubset of B.

Definition 4 . *Similar msets*

Two msets A and B are said to be ' cognate ' or ' similar ' if $\forall x (x \in A \iff x \in B)$, where x is an object . Thus , similar msets have equal root sets but need not be equal themselves .

Definition 5 . *Ordered pair of two mset te r − m s*

Ordered pair of two mset terms u and v, denoted by $[u, v]$, can be defined as follows :

$[u, v] = \{u, v\}$ if $u \neq v$, and $[u, v] = \{[u]2\}$ if $u = v$.

Here , $[u]$ is written as $\{u\}$, and $\langle u, v \rangle$ is actually the ordered pair set , where Set (u) stands for $u = \varnothing \vee \forall x \forall n (x \in^n u \implies n = 1)$, though x itself may be an mset term , (see [8] pp . 42 – 44 , for details) .

Definition 6 . *Power multiset*

In Cantorian spirit , the power multiset of a given mset A, denoted by $\wp(A)$ to distinguish it from the symbol $\wp(A)$ used for power set of A, is the multiset of all submultisets of A. For example , let $A = [x, y]2, 1 = [x, x, y]$. Then ,

$$\wp(A) = [\phi, \{x\}, \{x\}, [x]2, \{y\}, \{x, y\}, \{x, y\}, [x, y]2, 1].$$

In this sense , $C(\wp(A)) = 2^{C(A)}$, for any mset A. However , as has been voiced by many researchers in the area of msets and their applications (see [37] , p . 2 1 3 and [8] , p . 45 , in particular) , there is no ' good ' reason for admitting repeated elements into a power multiset . Hence , a power multiset needs to be

called $\wp(A)$ a = power $[_{\phi, \{x\}}$ set$^{\text{only}}_{,[x]}$ 2, $\{y\}, \{x,$ denoted$^{\text{by}}_{y\}, [x,y]2,1}$ $\wp_{\text{and}}(A)$. Accordingly$^{\text{for}}_{\text{hence} C(\wp(A))}$ $A^=_< $ $2^C \binom{[x,}{(A)} y]2, 1;$ which

implies that Cantor ' s power set theorem : $C(A) < C(\wp(A))$ fails . However , for

D . Singh , A . M . Ibrahim , T . Yohanna , J . N . Singh finite msets , Cantor ' s theorem holds for power mset (see [8] , p . 45 , for related

inherent difficulties if the mset in consideration is infinite) .

Definition 7 . *Union* $(\cup),$ *Intersection* $(\cap \Rightarrow$ $notdefnotdefnotdef \quad \propto \quad notdefnotdefnotdef - notdef - existentialnotdef \quad \exists \quad \{notdef - negationslash - notdefnegationslash - notdefnotdef - notdefnotdef - notdef - notdef \quad \cup \Leftarrow$

$$\biguplus o \quad r(+) \qquad\qquad\qquad notdef - notdef\}notdef$$

Let A and B be two msets over a given domain set D.

1. $A \cup B$ is the mset defined by $m_{A\cup B}(x) = m_A(x) \cup m_B(x) = \text{maximum } (m_A(x), m_B(x))$, being the union of two numbers . That is , an object

z o ccurring a times in A and b times in B, o ccurs maximum (a, b) t imes in $A \cup B$, if such a maximum exists ; otherwise the minimum of (a, b) is taken , which always exists .

It follows that for any given mset x there exists an mset y which contains elements of elements of x, where the multiplicity of an element z in y is the

maximum multiplicity of z as an element of elements of x along with the above

stipulation on the existence of such a maximum . We denote this fact by $y = \cup x$ Clearly , $Dom(\cup x) = \cup\{Dom(A), A \in x\}$ and that the multipliticity of z in y is the maximum of its multiplicities as an element of elements of x, if it exists ;

otherwise , the minimum is taken .

For example , if $A = [2344], B = [1433]$ then $A \cup B = [123344]$.

Also , it follows that for a finite mset x, the maximum multiplicity of el - ements of elements of x always exists . However , for certain infinite sets like $x = \{\{y\}, [y]2, [y]3...\}$, the maximum multiplicity of elements of elements of x does not exist , and hence $\cup x = \{y\}$. It is obvious by definition of the union that multiplicity of any $y \in x \neq \varnothing$ is irrelevant to $\cup x$, and hence $\cup x = \cup x^*($ see [8] , pp . $48 - 49$, for details) .

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$$= \text{minimum } (m_A(x), m_B(x)), \text{ being the } intersecti \overset{notdef}{_{on}} \text{ of two numbers .}$$

That is , an object x o ccurring a times in A and b t imes in B, o ccurs minimum (a, b) times in $A\cap \Rightarrow_{\text{w}-notdef}$ i $-$ notdef $-$ h$notdef - c_{\text{h}\propto}notdef - anotdef - w$y $-$ element $-$ a$existential - notdef - snotdef$ e i $-$ existentials $-$ notdef $-$ notdef $-$ negationslash $-$ ts $-$ braceleft$universal - notdef - periodnotdef$

In general , for a given mset $x, Dom(\cap\text{parenright} - \text{arrowdblright} = \cap notdefnotdefproportional - D^{o-notdef}notdef - m(element - A^{\text{parenright}-notdef-\text{existential}}notdef - colonA\exists element - negationslash - notdef - notdefnotdef - negationslash - notdef - xnegationslash - notdef - braceright$ a n $-$ notdef $-$ notdef$notdef - dz - unionelement - arrowdblleftnotdef - intersection \quad \Rightarrow$ estha t them ut p $-$ l icity o f z i st h em ini mum of t $-$ i m ult p $-$ l icit e $-$ i$\{$s as el m n t $notdef$

m n tsof x.

For example , if $A = [33344], B = [1433]$, then $A\cap \Longrightarrow notdef3notdef3notdef - four.notdef$

Note that for any mset x, we have $\cap arrowdblright - reflexsubsetunion - notdefnotdef - x_{notdef\propto}$

3. $A \biguplus B$ is the mset defined by $m_{A\biguplus B}(x) = m_A(x) + m_B(x)$, direct sum of two numbers .

That is , an object x o ccurring a times in A and b times in B, o ccurs $a + b$ t imes in $A \biguplus B$. For example , if $A = [1, 1, 2, 2, 4, 4, 4], B = [1, 2, 3, 3]$, then

$$A \uplus B = [1, 1, 1, 2, 2, 2, 3, 3, 4, 4, 4].$$

Clearly, $C(A \uplus B) = C(A \cup B) + C(A \cap arrowdblright - parenright - periodnotdef$

Note that if x be an infinite mset , then the multi^{notdef}pli$^{\propto}$cit^{notdef} y of some mset $z \in \uplus x$ may not be finite . In that case , the multiplicity of z in $\cup x$ is used . For example , if $x = \{\{z\}, [z]2, [z]3, ...\}$ then $\uplus x = \cup x = \{z\}$(see ([8] , p . 5 1 , for details) .

4 . Some Properties holding for mset operations
(see [46] p . 636 ; [8] , p . 53 , in particular) .

$$A\cap \implies \underset{notdef}{B}\ intersection - notdef arrowdblright - notdef notdef - notdef_{notdef}$$

$$A\cap \Rightarrow^{intersection-notdef} notdef - parenright - arrowdblright\text{equal} - notdef - notdef - proportional parenleft - no$$

$_{def}^{A}$ $notdef_{\propto}$ t − notdef$_{notdef A}$ existential − notdef notdef − A notdef − notdef − negationslash − equal − negationslash − existentia

In fact , as it has been suggested recently (see [85] , p . 9 , in parti^{notdef} cular) $\overset{\Leftarrow}{_{,\,\mathrm{in}}}$ order

to obtain a linear combination of msets , kA may be interpreted to denote
the sum of k number of $A^{'}$ s , where k is a natural number .

(iv) Identity laws : $A \cup \phi = A, A\cap \implies \underset{notdef}{\phi}\ notdef_A notdef phi - notdef$equal − element$existential - notdef A existential - notdef$

$$A\cap \Rightarrow^{union-notdef} notdef - C notdef - parenright - proportional = \underset{notdef}{(}\ existential - notdef^{intersection-notde}$$

The proof of all these identi^{B} ties follows from the int$^{\propto}$erpretati$^{\Leftarrow}$ on of $\cup, \cap \Rightarrow notdef notdef \propto$
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tive l − y. is ea sy t − o   s e t at $\uplus$ si sst rongerth an b th $\cup$ a d $\cap \Rightarrow notdef notdef notdef$ $\propto$
$notdef \exists notdef - existential notdef \exists negationslash - notdef - notdef \{notdef - existential notdef \forall negationslash -$
$notdef - notdef notdef - notdef notdef \cup$ r $\cap \Rightarrow$ $notdef notdef \propto$ e$_{notdef \exists}$h$_{existential - notdef}$ notdef $\exists notdef -$
$notdef - negationslash - comma\{$w − notdef − notdef h $notdef - r$ e $notdef - notdef - e$s − notdef − notdef$notdef -$
$notdef notdef \cup$ d i − arrowdblleft s $angbracketright - i - notdef$ r   u − notdef$_{notdef - tes -}$angbracketright$_{notdef notdef - o}$
v e − notdef$notdef - r_{notdef b}$notdef − ot − notdef − h$notdef \cup$ and $\cap \Rightarrow notdef notdef notdef \propto$
$notdef \exists existential - notdef$ $notdef$

$reflexsubset - proportional_{notdef} x - notdef \in notdef - existential notdef_{\exists}\{negationslash - notdef - notdef notde$

It is promising to observe that multiset operations form a   " e − notdef alm "   [85] , p .
9) .

Definition 8 . *Difference and complementation*
Let A and B be two msets over D, and $B \subseteq A$, then $m_{A-B}(x) = m_A(x) - m_{A\cap}notdef -$
$parenleft notdef - x) notdef_{\mathrm{f}}$ ora l − l$notdef - notdef - notdef x \in notdef - notdef._{notdef}$ ti

s∪$notdef$ ometimesc a l − l e dt hea rt hmeticd ifferenceo f $\quad B$ rom A. otet $hate$ $\overset{\text{I}}{\underset{\text{venif}}{}}$ B i

ssn otc ontainedi n A, t h i − s d efin t − i ionh oldsg ood .

tc anb es eenq uicklyt hats omeo ft hec onsequenceso ft hea foresaidd efin i − t ion red i − s turbing ([37] , p . 2 14) . For example , if $A = \quad [a, b]4, 5; B = \quad [a, b]2, 3$ then $A − B = [ab]2, 2 \subseteq B$, contradicting the classical law : $\quad (A − B) \cap \implies \phi notdef notdef \propto$

In order to define the complement , we follow Petrovsky [67] :

Let $\Im = \{A_1, A_2, ...\}$ be a family of multisets composed of the elements of the generic set D. Then , the maximum multiset z is defined by $m_z(x) = \quad \max A \in \Im\ m_A(x)$ for all $x \in D$ and all

$$A \in \Im.$$

Now , the complement of an mset A, denoted by $\bar{A}$, is defined as fallows :

$$\bar{A} = Z − A = \{m_{\bar{A}}(x).x / m_{\bar{A}}(x) = m_z(x) − m_A(x), \quad forall x \varepsilon D\}.$$

It is understood that some new operations like arithmetic multiplication , raising to the arithmetic power , direct product , raising to the direct power , defined by Petrovsky , can be gainfully exploited for further research .

Definition 9 . *Functions between msets*

The underlying assumption in defining a function between msets has been invariably not to allow mapping of identical elements to non - identical elements

and hence , it amounts to defining the function between their root sets , which is j ust the classical definition of a function .

The function $f : A \to B$ is an inj ection iff
(i) $f : A^* \to B^*$ is an injection , and

(ii) $\forall z(z \in A^* \implies m_A(z) \leq m_B(f(z)))$.

The function $f : A \to B$ is a surjection iff (i) $f : A^* \to B^*$ is a surjection , and

(ii) $\forall z(z \in A^* \implies m_A(z) \geq m_B(f(z)))$.

The function $f : A \to B$ is a bij ection iff (i) $f : A^* \to B^*$ is a bijection and

(ii) $\forall z(z \in A^* \implies m_A(z) = m_B(f(z)))$.

For example , $f : [x]3 \to [x]10$ is an inj ection , $f : [x]5 \to [y]4$ is a surj ection , and
$f : [x, y]6, 2 \to [x, y]3, 4$ is neither an inj ection nor a surj ection . For various other details , see [8] and [37] . Note that some of the consequences of the above definitions are conflicting with some fundamental theorems of the classical set theory .

1 .   Having defined functions between msets as above , it can be proved that Cantor ' s theorem does not hold , there is no injection from

$$A \to \wp(A), \text{see}([37])\text{p.215}.$$

2 .   Msets of equal cardinality need not have a bij ection between them . For example , $[a, b]1, 2$ and $[a, b, c]$ both contain three elements , but there can be no bij ection between them because the objects and their multiplicities are
different in the domain and the range of any such function .   In other words , there is no bijection between their root sets , viz ; $f : \{a, b\} \to \{a, b, c\}$ can not be a bij ection .

3 .   Schr $\ddot{o}$ der – Bernstein theorem fails (see [37]) , p . 2 1 5] and ([8] , p . 47) .

$$\text{Let } A = [x_1, x_2, ...]2, 4, 6, ... \text{and} B = [y0, y1, y2, ...]1, 3, 5, ...$$

The function $f : A^* \to B^*$ defined as $f(x_n) = yn$ makes $f : A \to B$ an inj ection so that $A \leq B$. The function $g : B^* \to A^*$ defined by $g(yn) = x_{n+1}$
makes $g : B \to A$ an inj ection so that $B \leq A$. But there cannot be a bij ection $h : A \to B -$ since all multiplications in A are even and that in B are odd . Note that $\leq$ is the standard dominance relation .

Definition 10 .   *Multiset Ordering*
It seems really surprising that the seminal work of Knuth ([45] , pp . 2 1 3 – 2 1 4 , 241 – 242) , related to multiset orderings and their applications , has escaped the attention of most of us until quite recently . According to Knuth :

 " *Multiset μ_1 dominates μ_2 if both μ_1 and μ_2 contain the same number of e lements and the K th largest e lement of μ_1 is greater than or equal to th e K th largest e lement of μ_2 for all K* " (p . 214) .

$$\Lambda$$

 " If a and b are multisets of m numbers each , we say that $a \ll$ b iff $a\Lambda b = a$

(equivalently , $a \vee \vee b = b$, the largest element of a is less than or equal to the smallest of b). Thus $\Lambda\Lambda_a b \quad \ll a \vee \vee b$(p . 241) .

" *An n th le vel ' cas cade distribution ' is a multiset . . .* " *(p . 299) .*

However , [28] is the earliest reference known to introducing multiset ordering and using it for proving termination of programs and term rewriting systems . In fact , it has served as a basis for host of orderings introduced in this context . We endeavour to outline the Dershowitz - Manna multiset ordering as follows :

Let S be a set equipped with a partial ordering $<$ (irreflexive and transitive relation or , equivalently , a transitive but not an equivalence relation) . Let $M(S)$ be the set of all finite msets M on S, and let $\ll$ be the associated (induced by $<$) mset ordering on $M(S)$. It is easy to see that each M is an mset with a finite carrier , viz ; $\{x \in S : M(x) \neq 0\}$.

The Dershowtiz - Manna Ordering :

$M \quad \ll N$ if there exist two msets X and Y in $M(S)$ satisfying :

$$(i) \quad \{\} = X \subseteq N,$$
$$(ii) \quad M = (N - X) + Y,$$
$$(iii) \quad (\forall y \in Y)(\exists x \in X)[y < x].$$

In other words , $M \quad \ll N$ if M is obtained from N by removing none or at least one element (those in X) from N, and replacing each such element x by zero or any finite number of elements (those in Y), each of which is strictly less than (in the ordering $<$) one of the elements x that have been removed . Informally , we say that M is smaller than N in this case . Similarly , $\gg$ on $M(S)$ with $(S, >)$ can be defined . For example , let $S = (\{0, 1, 2, ...\} = \aleph)$, then under the corresponding multiset ordering $\gg$ over $\aleph$, the mset $[3\ 3\ 4\ 0]$ is greater than each of the following msets : $[3\ 4]$, $[3\ 2\ 2\ 1\ 1\ 1\ 4\ 0]$ and $[3\ 3\ 3\ 3\ 2\ 2]$. The empty set $\{ \}$ is smaller than any multiset . It is also easy to observe that : $[\forall y$

$$\gg N. \qquad\qquad y \in N \Longrightarrow \exists x x \in M \wedge x > y] \Longrightarrow M$$

For various ramifications of the Dershowitz - Manna Ordering , see [4 1] and [56] , in particular .

3 . Applications

Over the years , besides sporadic evidence of the applications of mulisets in philosophy , Logic , Linguistics and Physics , a good number of them witnessed in mathematics and computer science , which have led to the formulation of a comprehensive theory of multisets . In this section , a modest attempt is made to present a comprehensive survey of various applications of multisets which

is arranged under two major headings : Mathematics (especially , combinatorial and computational aspects) and computer science , along with some overlapping results placed appropriately .

3 . 1 . Applications in Mathematics

An early elaborate reference is ([46] , pp . 441 – 466 ; 636 - 667) . Some of the Knuth ' s findings are as follows : The prime factoriz ation of an integer $n > 0$

is a multiset N whose elements are primes , where $\pi_{P \in N} = N$.   Accordingly , as every positive integer can be uniquely factored into primes , one can obtain a

bij ection between the positive integers and the finite msets of prime numbers .

For example , if $n = 2^2.3^3.17$, the corresponding mset is $N = [2, 2, 3, 3, 3, 17]$. A simple algebra of msets is also developed . The natural correspondence between

a monic polynomial over the complex numbers and the unique mset containing its roots exists . That is , every monic polynomial $f(z)$ over the complex numbers corresponds in a natural way to the mset F of its roots .   Hence , if $f(z)$ and $g(z)$ are the polynomials corresponding to the finite msets F and G of complex numbers , then $F + G = f(z)g(z), F \cup G = lcm(f(z), g(z)),$ and $F \cap \implies g notdefnotdef - cd \propto$

$$f \text{parenleft} - \text{z}), gz)).$$

Zeros and poles of meromorphic functions , invariants of matrices in canonical form and invariants of finite Abelian groups do correspond to multiset represen -

t ations .   Generating functions and nonnegative integer coefficients correspond one - to - one with msets of nonnegative integers .   Thus , if A and B are msets of nonnegative integers , and if $G(z) = \sum_{n \in A} z^n$ and $H(z) = \sum_{n \in B} z^n$ be gen -

erating functions corresponding to A and B, then $G(z) + H(z)$ corresponds to $A \propto B$, etc . Also , for msets A and B, the product of Dirichlet generating func - t ions $g(z) = \sum_{n \in A} 1_{n^z}$ and $H(z) = \sum_{n \in B} 1_{n^z}$ corresponds to the mset product

$$AB.$$

Not much is known about the history of permutations of an mset .   Knuth ([45] , pp . 22 – 34) in line with some early references cited by himself , expounds the area of permutations of msets .

Goguen ([33] , pp . 5 1 3 – 56 1) , in course of developing a category – theoretic foundation of fuzzy set theory , investigates properties of semiring sets , as in [30] , along with various applications to msets .   He also discusses usefulness of msets in the study of combinatorics and formal languages . Chapin ([1 9] pp . 6 1 9 – 634 ; [20] , pp . 255 – 267) , while formalizing the theory of fuzzy sets and Boolean - lattice – valued sets , seems to have mset - model in mind . The interpretation of the atomic formula $\in (x, y, z) a - is - s a$ "$_{n-parenright}$e − T l slash − m F n − onetwo − t$_{nine-notdeff-nine-period-o6y}$T−notdef w − two − notdef − it − fourh − onee − dT − notdefg − De − bracke e mb e s − r h i − p st z " re fl − e c tssemi al t − i y , (se es ecti n 3.e3, d e fi   n t − i on 3,$^{notdef-zero}$ of h i sp ap e , c period − f[80]

7]) .   Mathematics of multisets proves a potential tool to study computing of Grobner Bases and straightening laws in polynomial rings (see [1 5] and [3] , in particular) . Brink ([1 7] , pp . 1 − 1 3) outlines an algebra of msets and shows that it could model relevant structures to a great extent , yet falls short of DeMorgan

monoid .

Anderson [2] can be considered as the first sustained development of conbi - natorics of msets , in particular . The area of concentration includes the study of sizes , numbers and properties of ' chains ' (collections of pairwise $\subseteq$ comparable subsets) and ' antichains ' (collections of pairwise $\subseteq$ comparable subsets) in the subset and msubset lattices (see [83] , [73] , [5] , [2 1 , 22] , [9] , in particular) . Martin

([56] , pp . 37 − 54) observes that many well - founded partial orderings of the set of finite msets on a given set have appeared that played significant roles in the study of invariant theory

(see [62] and [3 1] for early references) , ring theory (see [3] , and [68] for some recent developments) and the theory of partitions (see [5] ,

[54] , [7] and [1 1] for various recent developments) . It is also mentioned (p . 37) that msets can be gainfully exploited in the study of measure theory which , however , would require tools of functional analysis , especially when the msets considered are infinite .

Singh and Singh [79] point out a fundamental problem . It is observed that the best known efficient formula $nH_r =^{r+n-l} C_r$, designed to compute the number of combinations of n distinct objects taken r at a time when objects may o ccur with repetitions , includes terms of the type $[a_i]_n$, for each i, that are also msubsets of an mset in which each distinct element o ccurs unlimited

number of times (hence , help determining the cardinality of the power set of

an mset) , turns out to be unworkable and computationally inefficient even if adjusted to be applicable , if the mset in consideration is finite or infinite , but with finite multiplicities . For example , for a 6 – element mset $\alpha = [x, y, z]1, 2, 3$ the

number of all combinations of size $-6($ repetitions allowed and consequently $[x]6$

– like combinations included which is not an msubset of α) is $3_{H_6} =^8 C_6 = 28$ whereas the number of $6-$ msubsets of α is only one and that is α itself . Also , besides discussing computational inefficiency of Inclusion – Exclusion principle , the paper puts forward a reasonably efficient and workable formula .

Petrovsky [66] outlines some very innovative mathematical applications of msets . An axiomatic foundation for metrization of multiset spaces is developed . Poplin [68] develops a multiset algebra (Chapter 8 , pp . 58 – 1 29) in his Ph . D . thesis . The thesis , titled " The semiring of multisets " , explicates that msets provide a connection between eigen values / eigen vectors equations for the Max - Plus and the nonnegative real number systems . It is shown that Max - Plus , Max - Times and the nonnegative real numbers can be viewed as a special case of msets . The guiding factors for undertaking research in the area of mathematics abound . For example , the $(Max, +)$ algebra has been extensively used in discrete event systems , transportation networks , parallel computations , project management , machine scheduling , to name a few (see [68] , pp . 1 – 2) .

The interest in multisets and subsets of commutative monoids has increased in the recent years (see [24] for various deliberations) .

Hegarty and Larson ([38] , pp . 1 – 25) study permutations π of the natural numbers for which the numbers $\pi(n) - n$ belongs to a given (multi) subset M of $Z($ the set of integers) , for all $n \in S($ a given subset of the set of natural numbers) .

3 . 2 . Applications in Computer Science

References [45] and [46] are the early known references to the applications of msets in computer science . Knuth notes :

The t erminal s tring of a non - circular context free grammar form an mset that is a s e t if and only if the grammar is unambiguous ([45] , p . 636) .

He introduces multisets into algorithms that compute values of x^n where x is a real quantity and n is a large positive integer ([46] , pp . 441 - 466) . Multisets and permutations of multisets are applied in a variety of *s earch* and *s o rt* procedures ([45] , p . 7 1 7 for various page numbers) .

Eilenberg ([30] , chapter *vi*) describes a general theory of msets and applies it to automata . Various semiring structures are exploited and also , extensions to the cases of msets and field structures are studied . Shoesmith and Smiley ([78] ,

pp . 66 − 69 ,   1 1 3 − 1 1 4 ,   1 64 , 2 1 1 , 224) , while studying multiple − conclusion logic , find the application of msets quite useful to account for " arrays " of formulae . Dersowitz and Manna [28] introduce mset ordering and ingeniously exploit it for proving termination of certain programs , which has served a basis for many alternative and equivalent orderings subsequently proposed to date . The major intent of this seminal work can be seen in the following :   " . . . the mset ordering . . .   permits the use of relatively simple and intuitive termination functions in otherwise difficult termination proofs "   see   [28] .

Huet and Oppen [39] , and Jouannuad and Lescanne [4 1] introduced signif - i cant refinements of Dersowitz - Manna mset ordering .   In [4 1] , it is shown that the standard mset ordering is a maximal extension function .   ([41] , pp . 6 1 − 62) also provides an efficient implementation of Dersowitz – Manna mset ordering .

Peterson ([65] ,   pp . 237 − 240) shows that the very foundation of Petri net theory , introduced by C . A . Petri in 1 962 , needs the use of msets (see also , [34]

and[69]).

Mayer and McRobbie   [57] find use of msets quite appropriate to account for how often a premise is repeated in characterizing relevance aspect of an argument . Thistlewaite , McRobbie and Meyer [82] make use of algebra of msets developed in [57] in explicating automated theorem proving for relevance logics , especially in the implementation while using the program KRIPKE . They also

recognize that msets have been taken as *datatypes*   in a number of programming languages [82] p . 26) .

Bundy ([1 6] , Chapter 1 3 , pp . 225 − 240) exploits mset of numbers to illustrate the definition - and - conjecture - formation program of Lenat . The work of Manna and Waldinger [6 1] can be considered as a sustained exposition to substantiate the fact that mathematical logic plays a fundamental role in the realm of the - oretical computer science .   Chapter II , pp . 505 − 527 of [6 1] is solely devoted to explicating fundamentals of msets .   A theory of BAG is developed by way of

introducing a novel binary primitive operation , O. : if an atom u has multiplicity

$n \geq 0$ in a bag x, then u has multiplicity $n + 1$ in the bag uOx..

Reisig   [74]   uses msets to define   " e − notdef lation   ets " .   nterestingly ,   xploiting mathematical intent of the definition of msets as generalized (integer - valued) characteristic functions given in [84] , the novel idea of   " e − notdef gativem u l − t ip i − l ci t y " in an mset is introduced ([74] , p . 1 26 ; see [9] for a detailed exposition) .   Reisig also outlines the concept of   " multirelations "   – an mset whose domain is the Cartesian product of a set of sorts ([74] , pp . 1 26 − 1 3 1) .

Yager ([86] , pp . 23 − 27 ;   [87] , pp . 441 − 446) ,   after developing an algebra of msets , introduces the notion of   " notdef − u zzy   b − a$_{g''}$ s   ollows :   ") - 3 0 (. . to eachele e − x − m in a fuzzy bag A is associated a multiset contai^{f} ning elements α(real number in the interval $[0, l]$) with multiplicities n(non - negative integers) .   The number n indicates the number of times the element x appears with membership grade α in the fuzzy bag A "   ([86] , p . 33) . Yager notes that Zadeh fuzzy sets are special

cases of fuzzy bags ([86] , p . 35) .

Grzymala – Busse ([35] , pp . 325 – 332) extends the notion of " o − notdef ugh " et (introduced by Pawlak to rough mset (see also , [58 , 59])) .

Martin [55] constructs a number of extension functions for mset orderings . Also , Martin [56] introduces various well - founded partial orderings of the set of all finite msets whose elements are taken from a given set . Martin exploits the notion of *cone* in $R^n (n-$ dimensional real space) and provides a systematization of the construction and classification of various well founded partial orderings underlying in proofs of program termination (see [28] , [39] , [4 1] , et c .) , in term rewriting systems (see [26] , [42] , [63] , [55] , [23] , etc .) , and computer algebra (see [30] , [1 5] , [29] , [68] , etc .) .

Pratt ([69] , pp . 33 – 7 1) , besides giving several arguments for the use of *pom - s e ts* , specially showing how pomsets can be used to represent *parallel processes* . He also describes how Petri nets can be modelled as pomsets .

Gischer ([32] pp . 1 99 – 224) exploits the notion of a partial string and a partial language introduced in [34] to show how pomsets can be used as a model of concurrency .

Applications of msets in Logic Programming languages is found to over - come " computational inefficiency " inherent in otherwise situation , especially in solving a sweep of real - life problems where multiple o ccurrences of an identical element are persistent . In fact , both the usual options , viz . " attaching different levels " or " assigning numbers " to account for multiplicity turn out to be compu - tationally inefficient . Representation of msets in a logic programming language can be effected , for example , by introducing a binary associative and commu - tative function symbol *o*, which also admits a unit element . In a constraint logic programming language , constraints are built up from mset operations and relations . Kizil- tan ([43] , Chapter 7 , pp . 1 65 – 202) has recently dealt with multi - set and strict multiset ordering constraints . Rule - based multiset programming paradigm is recently exploited to study synthetic biology (see [4] , [24] and [47] for details) . Basically , multisets are interpreted to represent biological systems , such as molecules in a biochemical system . A host of simulators for biological systems has been developed and found useful to several fields of biology such as biochemistry , microbiology , and evolutionary biology . For example , a system could be a multiset $[A, A, B, B, C]$, where A, B and C are elements that evolve by means of the application of a set of rewriting rules (say , of the biological systems) viz . :

$\{\{A, B\} \to \{C\}, \{C\} \to \{A, B\}\}$. If we choose the first rule to apply to the multiset in the example , the multiset $\{A, B, C, C\}$ is obtained (see [1 2] , for details) . The interaction among elements in a multiset object space , which includes the environment , are like chemical reactions and the evolution of multiset can mimic the biological evolution leading to plausibility of DNA computing and programmable living machines (see [47] , for details) .

The seminal work [28] has dominated research in termination , especially in Term Rewrite Systems (TRS) , during the last two decades or so . Full surveys , particularly of division orderings on structures such as strings , partitions or terms , given in [27] and [8 1] [52] , provide a complete development of the topic

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with the help of Coq Proof Assistant .

As for the present , a typical method to prove termination of a particular TRS involves finding a well - founded ordering or equivalently , finding a quantity which decreases at each step of computation . A widely accepted outcome of research undertaken in this direction dictates that the choice of ordering may have a surprising effect on efficiency (see [55 , 56] , and endeavours of the research group , University of St . Andrews , led by Martin *e t al* , especially on classification of orderings) . In the course of time , Multiset Path Orderings (i . e . Recursive

Path Orders with mulitset status only) [25 , 27] , Simplification Orders [53] and [8 1] , Lexicographic Path Orders [1 3] , Decomposition Orders [4 1] , etc . including Higher - order Path Orderings , have been experimented (see [1 8] , [1 3] , and [40] for various details) .

In [40] , it is shown that a termination proof for a TRS using multiset path orderings yields a primitive recursive bound on the length of derivations . The striking point is that this result holds for a great variety of path orderings , including AC – Path Orderings described in [14] , [44] , [49] and [48] if lexico - graphic status is not incorporated . The reference [50] is a comprehensive work on Termination , $AC-$ Termination and Dependency Pairs (introduced in [1]) .

The study around finding bounds on derivation length does have an an - tecedent in associating ordinals to proofs that computations terminate *e la Tur - ing* (1 994) . This is now generally known as Floyd ' s method of analyzing program correctness . An ordinal technique (including ordinal powers) was exploited to prove termi - nation of an example which calculates factorial by repeated addition . Ordinals enable us to link termination proofs with classical proof and recursion theory .

The specification of an abstract data type (ADT) for msets has been studied in ([6] , pp . 1 – 29) . Dovier , Policriti and Rossi ([29] , pp . 208 – 234) , after devel - oping mathematics of msets , substantiate that msets are the fundamental data structures for P systems . In fact , in computational sense , an mset is j ust a data structure , which differs from a set in permitting repetition of some of its elements and that from a *lis t* in being unordered , and hence it turns out to be a suitable modelling tool for a large class of real - life phenomena . Ross and Stoyanovich [71] , study cardinality bounded msets in Database systems to over - come consistency and performance problems that conventional representations in relational database suffered from .

Petrovsky ([67] , pp . 1 74 – 1 84) explores a new dimension . He , after provid - ing an extensive treatment of mset operations , discusses " Cluster " analysis in mset spaces , and demonstrates their application in augmenting Decision Sup - port System (DSS) . It is shown that " a multi - aspect analysis of problem and structuring alternatives allow us to gain an insight into the problem nature and find better decisions . " - 3 7 (. . . sugg ste dthe too s f − o r struct r ng a col e − l ti objects represented by many qualitative attributes when a lot of copies of ob - j ects or values of attributes describing them exist . . . . approach is based on a theory of multiset metric spaces and indexes of difference / similarity between

multisets $\overset{''}{\cdot}$

A " o − notdef ugh " et pproacht o m − u lti – attributed ecisiona nalysiso f a − P wlaka nd

Slowinski ([70] , pp . 443 – 459) is a pioneering work in this direction .

4 . Concluding remarks and some future directions

Multiset theory , owing to its multitudinal applications , has o ccupied a cen - tral place among a number of non - classical (nonstandard) set theories developed during the last few decades .

Category theory is emerging as a strong alternative :   Epsilons are replaced by " Arrows " , but that time is not yet ripe .   Currently , all efforts seem to point to discovering some primitive theory of structures (a pre - set theoretic) to which all set theories can be alluded (see [1 0] , pp . 32 1 - 322 , for details) .

4 . 1 . Some future directions

1 . Pursuing the techniques followed in [82] ,   an extension of the program KRIPKE can be achieved by interpreting a multiset of formulae viz ., $[A, B^m, C^n, ...]$ as an ' Ordinal ' tree viz

., $[A.B \overset{m}{.} C^n....]$, where $A, B, C, ...$ stand for

' events ' and A^k for ' A o ccurs k t imes ' .   ([76] , p . 1 1 4)

2 .   A reasonable expectation is that a set theory based on ω – valued logic (" $element -$ $parenrighty - T /$ F c one $- n2nine - notdef - hnine - v_{e-sixT-notdefa-f}r - period - tu - five_{th-zero-notdef}T - notdef$ e $n \in \omega$) a d   a m l $- ut - i$ s etth eory b sed o nc assical $-$ alued l $-$ ogic^{a}quoteleft $-$ parenleft$x \in ny$ c n h ve o ly t wot uth v lues) a e e uivalentth eories

v 1 0] , pp . 337 – 338) . See Skolem ([77] , Chapter 1 8) :   " . . . it seems to be possible to obtain a consistent set theory with an unrestricted axiom of comprehension if all rational numbers ≥ 0 and ≤ 1 are allowed as truth values " (p . 69) .

3 $\overset{"}{.}$ *Multiset* as a model for Multi - Attribute objects " is expected to play a

significant role in the area of mathematical modelling , Discovery of Intelligent systems , control of Non - linear mechanical systems , just to mention a few (see

[67]).

4 .   Discovering competing simulators for biological systems seems to dom - inate the research in biotechnology for a foreseeable future (see [1 2] ,   [47] ,   [24]

and[4]).

Acknowledgement

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