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ON DEGREES IN THE HASSE DIAGRAM OF THE STRONG BRUHAT ORDER

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ABSTRACT. For a permutation π in the symmetric group S_n let the *total degree* be its valency in the Hasse diagram of the strong Bruhat order on S_n, and let the *down degree* be the number of permutations which are covered by π in the strong Bruhat order. The maxima of the total degree and the down degree and their values at a random permutation are computed. Proofs involve variants of a classical theorem of Turán from extremal graph theory.

1. THE DOWN, UP AND TOTAL DEGREES

Definition 1.1. For a permutation $\pi \in S_n$ let the *down degree* $d_-(\pi)$ be the number of permutations in S_n which are covered by π in the strong Bruhat order. Let the *up degree* $d_+(\pi)$ be the number of permutations which cover π in this order. The *total degree* of π is the sum

$$d(\pi) := d_-(\pi) + d_+(\pi),$$

i.e., the valency of π in the Hasse diagram of the strong Bruhat order.

Explicitly, for $1 \le a < b \le n$ let $t_{a,b} = t_{b,a} \in S_n$ be the transposition interchanging a and b, and for $\pi \in S_n$ let

$$\ell(\pi) := \min\{k \mid \pi = s_{i_1} s_{i_2} \cdots s_{i_k}\}$$

be the *length* of π with respect to the standard Coxeter generators $s_i = t_{i,i+1}$ $(1 \le i < n)$ of S_n. Then

$$d_-(\pi) = \#\{t_{a,b} \mid \ell(t_{a,b}\pi) = \ell(\pi) - 1\},$$
$$d_+(\pi) = \#\{t_{a,b} \mid \ell(t_{a,b}\pi) = \ell(\pi) + 1\},$$
$$d(\pi) = d_-(\pi) + d_+(\pi) = \#\{t_{a,b} \mid \ell(t_{a,b}\pi) = \ell(\pi) \pm 1\}.$$

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For the general definitions and other properties of the weak and strong Bruhat orders see , e . g . , [9 , Ex . 3 . 75] and [2 , §§2 . 1 , 3 . 1] .

We shall describe $\pi \in S_n$ by it s sequence of values $[\pi(1), ..., \pi(n)]$.

Observation 1.2. *π covers σ in the s trong Bruhat order on S_n if and only if there exist $1 \le i < k \le n$ such that*

$$(1) \quad b := \pi(i) > \pi(k) =: a.$$

$$(2) \quad \sigma = t_{a,b}\pi, \quad i.e., \pi = [..., b, ..., a, ...] \quad and \quad \sigma = [..., a, ..., b, ...].$$

(3) *There is no $i < j < k$ su ch that $a < \pi(j) < b$.*

Corollary 1 . 3 . *For every $\pi \in S_n$*

$$d_-(\pi) = d_-(\pi^{-1}).$$

Example 1 . 4 . In $S_3, d_-[123] = 0, d_-[132] = d_-[213] = 1$, and $d_-[321] = d_-[231] = d_-[312] = 2$. On the other hand , $d[321] = d[123] = 2$ and

$$d[213] = d[132] = d[312] = d[231] = 3.$$

Remark 1 . 5 . The classical descent number of a permutation π in the symmetric group S_n is the number of permutations in S_n which are covered by π in the (r ight) *weak* Bruhat order . Thus , the down degree may be considered as a " strong descent *number* ".

Definition 1 . 6 . For $\pi \in S_n$ denote

$$D_-(\pi) := \{t_{a,b} \mid \ell(t_{a,b}\pi) = \ell(\pi) - 1\},$$

the *s trong des cent s et* of π.

Example 1 . 7 . The strong descent set of $\pi = [7, 9, 5, 2, 3, 8, 4, 1, 6]$ i s

$$D_-(\pi) = \{t_{1,2}, t_{1,3}, t_{1,4}, t_{2,5}, t_{3,5}, t_{4,5}, t_{4,8}, t_{5,7}, t_{5,9}, t_{6,7}, t_{6,8}, t_{8,9}\}.$$

Remark 1 . 8 . Generalized pattern avoidance , involving strong descent sets , was applied by Woo and Yong [1 1] to determine which Schubert varieties are Gorenstein .

Proposition 1 . 9 . *The strong descent s et $D_-(\pi)$ uniquely determines the permutation π.*

Proof . By induction on n. The claim clearly holds for $n = 1$.

Let π be a permutation in S_n, and let $\bar{\pi} \in S_{n-1}$ be the permutation obtained by deleting the value n from π. Note that , by Observation 1 . 2 ,

$$D_-(\bar{\pi}) = D_-(\pi) \setminus \{t_{a,n} \mid 1 \le a < n\}.$$

By the induction hypothesis $\bar{\pi}$ is uniquely determined by this set . Hence it suffices to determine the position of n in π.

Now, if $j := \pi^{-1}(n) < n$ then clearly $t_{\pi(j+1),n} \in D_-(\pi)$. Moreover, by Observation 1.2, $t_{a,n} \in D_-(\pi) \implies a \geq \pi(j+1)$. Thus $D_-(\pi)$ determines

$$\bar{\pi}(j) = \pi(j+1) = \min\{a \mid t_{a,n} \in D_-(\pi)\},$$

and therefore determines j. Note that this set of a's is empty if and only if $j = n$. This completes the proof. □

2. Maximal Down Degree

In this section we compute the maximal value of the down degree on S_n and find all the permutations achieving the maximum. We prove

Proposition 2.1. *For every positive integer* n

$$\max\{d_-(\pi) \mid \pi \in S_n\} = \lfloor n^2/4 \rfloor.$$

Remark 2.2. The same number appears as the order dimension of the strong Bruhat poset [7]. An upper bound on the maximal down degree for finite Coxeter groups appears in [4, Prop. 3.4].

For the proof of Proposition 2.1 we need a classical theorem of Turán.

Definition 2.3. Let $r \leq n$ be positive integers. The *Turán graph* $T_r(n)$ is the complete $r-$partite graph with n vertices and all parts as equal in size as possible, i.e., each size is either $\lfloor n/r \rfloor$ or $\lceil n/r \rceil$. Denote by $t_r(n)$ the number of edges of $T_r(n)$.

Theorem 2.4 (Turán's Theorem; [10], [3, IV, Theorem 8])**.**

(1) *Every graph of order* n *with more than* $t_r(n)$ *edges contains a complete subgraph of order* $r+1$.

(2) $T_r(n)$ *is the unique graph of order* n *with* $t_r(n)$ *edges that does not contain a complete subgraph of order* $r+1$.

We shall apply the special case $r = 2$ (due to Mantel) of Turán's theorem to the following graph.

Definition 2.5. The *strong descent graph* of $\pi \in S_n$, denoted $\Gamma_-(\pi)$, is the undirected graph whose set of vertices is $\{1, ..., n\}$ and whose set of edges is

$$\{\{a, b\} \mid t_{a,b} \in D_-(\pi)\}.$$

By definition, the number of edges in $\Gamma_-(\pi)$ equals $d_-(\pi)$.

Remark 2.6. Permutations for which the strong descent graph is connected are called *indecomposable*. Their enumeration was studied in [5]; see [6, pp. 7−8]. The number of components in $\Gamma_-(\pi)$ is equal to the number of *global descents* in πw_0 (where $w_0 := [n, n-1, ..., 1]$), which were introduced and studied in [1, Corollaries 6.3 and 6.4].

Lemma 2.7. *For every $\pi \in S_n$, the strong descent graph $\Gamma_-(\pi)$ is triangle-free.*

Proof. Assume that $\Gamma_-(\pi)$ contains a triangle. Then there exist $1 \le a < b < c \le n$ such that $t_{a,b}, t_{a,c}, t_{b,c} \in D_-(\pi)$. By Observation 1.2,

$$t_{a,b}, t_{b,c} \in D_-(\pi) \Longrightarrow \pi^{-1}(c) < \pi^{-1}(b) < \pi^{-1}(a) \Longrightarrow t_{a,c} \notin D_-(\pi).$$

This is a contradiction. □

Proof of Proposition 2.1. By Theorem 2.4 (1) together with Lemma 2.7,

$$d_-(\pi) \le t_2(n) = \lfloor n^2/4 \rfloor \quad \text{for every } \pi \in S_n.$$

Equality holds since

$$d_-([\lfloor n/2 \rfloor + 1, \lfloor n/2 \rfloor + 2, ..., n, 1, 2, ..., \lfloor n/2 \rfloor]) = \lfloor n^2/4 \rfloor.$$

□

Next we classify (and enumerate) the permutations which achieve the maximal down degree.

Lemma 2.8. *Let $\pi \in S_n$ be a permutation with maximal down degree. Then π has no decreasing subsequence of length 4.*

Proof. Assume that $\pi = [...d...c...b...a...]$ with $d > c > b > a$ and $\pi^{-1}(a) - \pi^{-1}(d)$ minimal. Then $t_{a,b}, t_{b,c}, t_{c,d} \in D_-(\pi)$ but, by Observation 1.2, $t_{a,d} \notin D_-(\pi)$. It follows that $\Gamma_-(\pi)$ is not a complete bipartite graph, since $\{a,b\}$, $\{b,c\}$, and $\{c,d\}$ are edges but $\{a,d\}$ is not. By Lemma 2.7, combined with Theorem 2.4 (2), the number of edges in $\Gamma_-(\pi)$ is less than $\lfloor n^2/4 \rfloor$. □

Proposition 2.9. *For every positive integer n*

$$\#\{\pi \in S_n \mid d_-(\pi) = \lfloor n^2/4 \rfloor\} = \begin{cases} n, & \text{if } n \text{ is odd;} \\ n/2, & \text{if } n \text{ is even.} \end{cases}$$

Each such permutation has the form

$$\pi = [t+m+1, t+m+2, ..., n, t+1, t+2, ..., t+m, 1, 2, ..., t],$$

where $m \in \{\lfloor n/2 \rfloor, \lceil n/2 \rceil\}$ and $1 \le t \le n-m$. Note that $t = n-m$ (for m) gives the same permutation as $t = 0$ (for $n-m$ instead of m).

Proof . It is easy to verify the claim for $n \le 3$. Assume $n \ge 4$.

Let $\pi \in S_n$ with $d_-(\pi) = \lfloor n^2/4 \rfloor$. By Theorem 2.4(2), $\Gamma_-(\pi)$ i s is omorphic to the complete bipartite graph $K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$. S ince $n \ge 4$, each s ide of the graph contains at least two vertices . Let $1 = a < b$ be two vertices on one s ide , and $c < d$ two vertices on the other s ide of the graph . S ince $t_{b,c}, t_{b,d} \in D_-(\pi)$, there are three possible cases :

$$(1) \quad b < c, \text{andthen} \pi = [...c...d...b...]$$

(s ince $\pi = [...d...c...b...]$ contradicts $t_{b,d} \in D_-(\pi)$).

$$(2) \quad c < b < d, \text{andthen} \pi = [...d...b...c...].$$

$$(3) \quad d < b, \text{andthen} \pi = [...b...c...d...]$$

(s ince $\pi = [...b...d...c...]$ contradicts $t_{b,c} \in D_-(\pi)$).

The same also holds for $a = 1$ instead of b, but then cases 2 and 3 are impossible s ince $a = 1 < c$. Thus necessarily c appears before d in π, and case 2 i s therefore impossible for *any* b on the same s ide as $a = 1$. In other words : no vertex on the same s ide as $a = 1$ i s intermediate , either in position (in π) or in value , to c and d.

Assume now that n i s even . The vertices not on the s ide of 1 form (in π) a block of length $n/2$ of numbers which are consecutive in value as well in position . They also form an increasing subsequence of π, s ince $\Gamma_-(\pi)$ is bipartite . The numbers preceding them are all larger in value , and are increasing ; the numbers succeeding them are all smaller in value , are increasing , and contain 1 . It is easy to check that each permutation π of this form has maximal $d_-(\pi)$. Finally , π is com - pletely determined by the length $1 \le t \le n/2$ of the last increasing subsequence .

For n odd one obtains a s imilar classification , except that the length of the s ide not containing 1 i s either $\lfloor n/2 \rfloor$ or $\lceil n/2 \rceil$. This completes the proof . □

3 . Maximal Total Degree

Obviously , the maximal value of the total degree $d = d_- + d_+$ cannot exceed $\binom{n}{2}$, the total number of transpositions in S_n. This is s lightly better than the bound $2\lfloor n^2/4 \rfloor$ obtainable from Proposition 2 . 1 . The actual maximal value i s smaller .

Theorem 3 . 1 . *For $n \ge 2$, the maximal to tal degree in the Hasse dia - gram of the s trong Bruhat order on S_n is*

$$\lfloor n^2/4 \rfloor + n - 2.$$

In order to prove this result , associate with each permutation $\pi \in S_n$ a graph $\Gamma(\pi)$, whose s et of vertices i s $\{1, ..., n\}$ and whose set of edges is

$$\{\{a,b\} \mid \ell(t_{a,b}\pi) - \ell(\pi) = \pm 1\}.$$

This graph has many properties ; e . g . , it i s K_5– free and is the edge - disj oint union of two triangle - free graphs on the same set of vertices . However , these properties are not strong enough to imply the above result . A property which does imply it i s the fo llowing bound on the minimal degree .

Lemma 3 . 2 . *There exists a vertex in* $\Gamma(\pi)$ *with degree at most*

$$\lfloor n/2 \rfloor + 1.$$

Proof . Assume , on the contrary , that each vertex in $\Gamma(\pi)$ has at least $\lfloor n/2 \rfloor$ $+2$ neighbors . This applies , in particular , to the vertex $\pi(1)$. Being the first value of π, the neighborhood of $\pi(1)$ in $\Gamma(\pi)$, viewed as a subsequence of $[\pi(2), ..., \pi(n)]$, consists of a shuffle of a decreasing sequence of numbers larger than $\pi(1)$ and an increasing sequence of numbers smaller than $\pi(1)$. Let a be the r ightmost neighbor of $\pi(1)$. The intersection of the neighborhood of a with the neighborhood of $\pi(1)$ is of cardinality at most two . Thus the degree of a i s at most

$$n - (\lfloor n/2 \rfloor + 2) + 2 = \lceil n/2 \rceil \leq \lfloor n/2 \rfloor + 1,$$

which i s a contradiction . □

Proof of Theorem 3 . 1 . First note that , by definition , the total degree of $\pi \in S_n$ in the Hasse diagram of the strong Bruhat order i s equal to the number of edges in $\Gamma(\pi)$. We will prove that this number $e(\Gamma(\pi)) \leq$ $\lfloor n^2/4 \rfloor + n - 2$, by induction on n.

The claim i s clearly true for $n = 2$. Assume that the claim holds for $n-1$, and let $\pi \in S_n$. Let a be a vertex of $\Gamma(\pi)$ with minimal degree , and let $\bar{\pi} \in S_{n-1}$ be the permutation obtained from π by deleting the value a(and decreasing by 1 all the values larger than a). Then

$$e(\Gamma(\bar{\pi})) \geq e(\Gamma(\pi) \setminus a),$$

where the latter is the number of edges in $\Gamma(\pi)$ which are not incident with the vertex a. By the induction hypothesis and Lemma 3 . 2 ,

$$\begin{aligned} e(\Gamma(\pi)) = e(\Gamma(\pi) \setminus a) + d(a) &\leq e(\Gamma(\bar{\pi})) + d(a) \\ \leq \quad \lfloor (n-1)^2/4 \rfloor + (n-1) - 2 + \lfloor n/2 \rfloor + 1 \\ = \quad \lfloor n^2/4 \rfloor + n - 2. \end{aligned}$$

Equality holds s ince , lett ing $m := \lfloor n/2 \rfloor$,

$$e(\Gamma([m+1, m+2, ..., n, 1, 2, ..., m])) = \lfloor n^2/4 \rfloor + n - 2.$$

□

Theorem 3 . 3 .

$\#\{\pi \in S_n \mid d(\pi) = \lfloor n^2/4 \rfloor + n - 2\} = braceex-braceex-braceex-braceleftmid-braceex-braceex-braceex-b$

The extremal permutations have one of the fol lowing forms :

$$\pi_0 := [m+1, m+2, ..., n, 1, 2, ..., m] \quad (m \in \{\lfloor n/2 \rfloor, \lceil n/2 \rceil\}),$$

and the permutations ob tained from π_0 *by one or more of the fol lowing*

operations :

$$\pi \mapsto \pi^r := [\pi(n), \pi(n-1), ..., \pi(2), \pi(1)] \quad (\text{reversing}\pi),$$
$$\pi \mapsto \pi^s := \pi \cdot t_{1,n} \quad (\text{interchanging}\pi(1)\text{and}\pi(n)),$$

$$\pi \mapsto \pi^t := t_{1,n} \cdot \pi \quad (\text{ interchanging 1 and } n \text{ in } \pi).$$

Proof . It is not difficult to see that all the specified permutations are indeed extremal , and their number is as claimed (for all $n \geq 2$).

The claim that there are no other extremal permutations will be proved by induction on n. For small values of n(say $n \leq 4$) this may be verified directly . Assume now that the claim holds for some $n \geq 4$,
and let $\pi \in S_{n+1}$ be extremal . Following the proof of Lemma 3 . 2 , let a be a vertex of $\Gamma(\pi)$ with degree at most $\lfloor (n+1)/2 \rfloor + 1$, which i s either $\pi(1)$ or it s r ightmost neighbor . As in the proof of Theorem 3 . 1 , let $\bar{\pi} \in S_n$ be the permutation obtained from π by deleting the value a (and decreasing by 1 all the values larger than a). All the inequalities in the proof of Theorem 3 . 1 must hold as equalities , namely : $e(\Gamma(\pi) \setminus a) = e(\Gamma(\bar{\pi}))$, $d(a) = \lfloor (n+1)/2 \rfloor + 1$, and $\bar{\pi}$ i s extremal in S_n. By the induction hypothesis ,$\bar{\pi}$ must have one of the prescribed forms . In all of them , $\{\bar{\pi}(1), \bar{\pi}(n)\} = \{m, m+1\}$ i s an edge of $\Gamma(\bar{\pi})$. Therefore the corresponding edge $\{\pi(1), \pi(n+1)\}$ (or $\{\pi(2), \pi(n+1)\}$ i f $a = \pi(1)$, or $\{\pi(1), \pi(n)\}$ i f $a = \pi(n+1)$) is an edge of $\Gamma(\pi) \setminus a$, namely of $\Gamma(\pi)$. If $a \neq \pi(1), \pi(n+1)$ then $\pi(n+1)$ i s the r ightmost neighbor of $\pi(1)$, contradicting the choice of a. If $a = \pi(n+1)$ we may use the operation $\pi \mapsto \pi^r$. Thus we may assume from now on that $a = \pi(1)$.

Let $N(a)$ denote the set of neighbors of a in $\Gamma(\pi)$. Assume first that

$$\bar{\pi} = \pi_0 = [m+1, m+2, ..., n, 1, 2, ..., m] \quad (m \in \{\lfloor n/2 \rfloor, \lceil n/2 \rceil\}).$$

Noting that $\lceil n/2 \rceil = \lfloor (n+1)/2 \rfloor$ and keeping in mind the decrease in certain values during the transition $\pi \mapsto \bar{\pi}$, we have the following cases :
(1) $a > m+1$: in this case $1, ..., m \notin N(a)$, so that

$$d(a) \leq n - m \leq \lceil n/2 \rceil = \lfloor (n+1)/2 \rfloor < \lfloor (n+1)/2 \rfloor + 1.$$

Thus π i s not extremal .

(2) $a < m$: in this case $m+3,...,n+1,m+1 element - negationslash N(a)$, so that

$$d(a) \leq 1+(m-1) \leq \lceil n/2 \rceil < \lfloor (n+1)/2 \rfloor + 1.$$

Again $,\pi$ is not extremal . (3) $a \in \{m, m+1\}$: in this case

$$d(a) = 1+m \leq \lfloor (n+1)/2 \rfloor + 1,$$

with equality iff $m = \lfloor (n+1)/2 \rfloor$. This gives $\pi \in S_{n+1}$ of the required form (either π_0 or $\pi^s 0$).

A s imilar analysis for $\bar{\pi} = \pi^s 0$ gives extremal permutations only for $a \in \{m+1, m+2\}$ and $d(a) = 3$, so that $n = 4$ and $\bar{\pi} = [2413] \in S_4$. The permutations obtained are $\pi = [32514]$ and $\pi = [42513]$, which are

$$\pi^{rt} 0, \pi^{rst} 0 \in S_5, \text{respectively}.$$

The other possible values of $\bar{\pi}$ are obtained by the $\pi \mapsto \pi^r$ and $\pi \mapsto \pi^t$ operations from the ones above , and yield analogous results . □

4 . Expectation

In this subsection we prove an exact formula for the expectation of the down degree of a permutation in S_n. **Theorem 4 . 1 .** *For every positive integer n, the expected down degree of a random permutation in S_n is*

$$E_{\pi \in S_n}[d_-(\pi)] = \sum_{n}^{i=2} \sum_{i}^{j=2} \sum_{j}^{k=2} \frac{1}{i \cdot (k-1)} = (n+1) \sum_{n}^{i=1} \frac{1}{i} - 2n.$$

It fo llows that

Corollary 4 . 2 . *As $n \to \infty$,*

$$E_{\pi \in S_n}[d_-(\pi)] = n \ln n + O(n)$$

and

$$E_{\pi \in S_n}[d(\pi)] = 2n \ln n + O(n).$$

To prove Theorem 4 . 1 we need some notation . For $\pi \in S_n$ and $2 \leq i \leq n$ let $\pi_{|i}$ be the permutation obtained from π by omitting all letters which are larger than or equal to i. For example , i f $\pi =$

$$\text{then} \pi_{|9} = [6,1,4,8,3,2,5,7], \pi_{|7} = [6,1,4,3,2,5],$$
$$\text{and} \pi_{|4} = [1,3,2].$$

Also , denote by $\pi^{|j}$ the suffix of length j of π. For example , if

$$\pi = [6,1,4,8,3,2,5,9,7] \text{then} \pi^{|3} = [5,9,7] \text{and} |_{\pi}^{|} {}^{2}_{4} = [3,2].$$

 Let $l.t.r.m.(\pi)$ be the number of left - to - r ight maxima in π :

$$l.t.r.m.(\pi) := \#\{i \mid \pi(i) = 1^{\max} \leq j \leq i\pi(j)\}$$

Lemma 4 . 3 . *For every* $\pi \in S_n$, *if* $\pi^{-}|i1+1^{(i)} = j$ *then*

$$d_-(\pi_{|i+1}) - d_-(\pi_{|i}) = l.t.r.m.(|^{|}_{\pi}\ {}^{i-j}_{i}).$$

Proof of Theorem 4 . 1 . Clearly , for every $\pi \in S_n$

$$d_-(\pi) = \sum_{i=2}^{n} [d_-(\pi_{|i+1}) - d_-(\pi_{|i})].$$

Thus , by Lemma 4 . 3 ,

$$d_-(\pi) = \sum_{i=2}^{n} l.t.r.m.(|^{|}_{\pi}\ {}^{i-ji}_{i}),$$

where ji i s the position of i in $\pi_{|i+1}$, i . e .,$ji := \pi^{-}|i1+1(i)$.

Define a random variable X to be the down degree $d_-(\pi)$ of a random (uniformly distributed) permutation $\pi \in S_n$. Then , for each $2 \leq i \leq n$, $\pi_{|i+1}$ is a random (uniformly distributed) permutation in S_i, and therefore $j = \pi^{-}|i1+1(i)$ i s uniformly distributed in $\{1, ..., i\}$ and $|^{|}_{\pi}\ {}^{i-j}_{i+1}$ is essentially a random (uniformly distributed) permutation in S_{i-j} (after monotonically renaming its values) . Therefore , by l inearity of the expectation ,

$$E[X] = \sum_{n}^{i=2} \frac{1}{i} \sum_{i}^{j=1} E[X_{i-j}] = \sum_{n}^{i=2} \frac{1}{i} \sum_{i-1}^{t=0} E[X_t], \tag{1}$$

where $X_t := l.t.r.m.(\sigma)$ for a random $\sigma \in S_t$. Recall from [9 , Corollary 1 . 3 . 8] that

$$\sum_{\sigma \in S_t} q^{l.t.r.m.(\sigma)} = \prod_{k=1}^{t} (q + k - 1).$$

It follows that , for $t \geq 1$,

$$E[X_t] = \frac{1}{t!} \sum_{\sigma \in S_t} l.t.r.m.(\sigma) = \frac{1}{t!} \quad (\frac{d}{dq} \sum_{\sigma \in S_t} q^{l.t.r.m.(\sigma)})|_{q=1}$$

$$= \frac{1}{t!} \quad (\frac{d}{dq} \prod_{t}^{k=1} (q + k - 1))|_{q=1} = \frac{1}{t!} \sum_{t}^{r=1} 1 \prod_{\substack{\leq k \leq \\ k \neq r}} k_t = \sum_{t}^{r=1} \frac{1}{r}.$$

 Of course $, E[X_0]=0.$ Substituting these values into (1) gives

$$E[X]=\sum_{n}^{i=2}\sum_{i-1}^{t=1}\sum_{t}^{r=1}\frac{1}{i\cdot r}$$

and this i s equivalent (with $j=t+1$ and $k=r+1$) to the first formula in the statement of the theorem .

The second formula may be obtained through the following manip - ulations :

$$E[X]=\sum_{n}^{i=2}\sum_{i}^{j=2}\sum_{j}^{k=2}\frac{1}{i\cdot(k-1)}=\sum_{k\le_2\le j\le i\le} n\frac{1}{i\cdot(k-1)}$$

$$=\sum_{\le_2 k\le i\le n}\frac{i-k+1}{i\cdot(k-1)}=\sum_{\le_2 k\le i\le n}(\frac{1}{k-1}-\frac{1}{i})$$

$$=\sum_{\le_2 k\le n}\frac{n-k+1}{k-1}-\sum_{\le_2 i\le n}\frac{i-1}{i}$$

$$=n\sum_{n}^{k=2}\frac{1}{k-1}-(n-1)-(n-1)+\sum_{n}^{i=2}\frac{1}{i}$$

$$=n\sum_{n}^{i=1}\frac{1}{i}-2n+\sum_{n}^{i=1}\frac{1}{i}.$$

□

Proof of Corollary 4 . 2 . Notice that

$$\sum_{n}^{i=1}\frac{1}{i}=\ln n+O(1).$$

(The next t erm in the asymptotic expansion is Euler ' s constant .) Sub - st itute into Theorem 4 . 1 to obtain the desired result . □

5 . Generalized Down Degrees

Definition 5 . 1 . For $\pi\in S_n$ and $1\le r<n$ let

$$D_-^{(r)}(\pi):=\{t_{a,b}\mid \ell(\pi)>\ell(t_{a,b}\pi)>\ell(\pi)-2r\}$$

the $r-$ *th s trong descent s et* of $\pi.$

Define the $r-$ *th down degree* as

$$d_-^{(r)}(\pi):=\#D_-^{(r)}(\pi).$$

Example 5.2. The first strong descent set and down degree are those studied in the previous section; namely, $D_-^{(1)}(\pi) = D_-(\pi)$ and $d_-^{(1)}(\pi) =$

$$d_-(\pi).$$

The $(n-1)-$th strong descent set is the set of *inversions*:

$$D_-^{(n-1)}(\pi) = \{t_{a,b} \mid a < b, \pi^{-1}(a) > \pi^{-1}(b)\}.$$

Thus

$$d_-^{(n-1)}(\pi) = inv(\pi),$$

the inversion number of π. **Observation 5.3.** *For every* $\pi \in S_n$ *and* $1 \le a < b \le n, t_{a,b} \in D_-^{(r)}(\pi)$ *if and only if* $\pi = [..., b, ..., a, ...]$ *and there are less than* r *letters between the positions of* b *and* a *in* π *whose value is between* a *and* b.

Example 5.4. Let $\pi = [7,9,5,2,3,8,4,1,6]$. Then

$$D_-^{(1)}(\pi) = \{t_{6,7}, t_{6,8}, t_{1,4}, t_{1,3}, t_{1,2}, t_{4,8}, t_{4,5}, t_{8,9}, t_{3,5}, t_{2,5}, t_{5,9}, t_{5,7}\}$$

and

$$D_-^{(2)}(\pi) = D_-^{(1)}(\pi) \cup \{t_{6,9}, t_{1,8}, t_{4,9}, t_{4,7}, t_{3,9}, t_{3,7}, t_{2,9}, t_{2,7}\}.$$

Corollary 5.5. *For every* $\pi \in S_n$ *and* $1 \le r < n$

$$d_-^{(r)}(\pi) = d_-^{(r)}(\pi^{-1}).$$

Proof. By Observation 5.3, $t_{a,b} \in D_-^{(r)}(\pi)$ if and only if $t_{\pi^-1(a),\pi^-1(b)} \in$

$$D_-^{(r)}(\pi^{-1}). \quad \square$$

Definition 5.6. The $r-$*th strong descent graph* of $\pi \in S_n$, denoted $\Gamma_-^{(r)}(\pi)$, is the graph whose set of vertices is $\{1, ..., n\}$ and whose set of edges is $\{\{a,b\} \mid t_{a,b} \in D_-^{(r)}(\pi)\}$.

The following lemma generalizes Lemma 2.7.

Lemma 5.7. *For every* $\pi \in S_n$, *the graph* $\Gamma_-^{(r)}(\pi)$ *contains no subgraph isomorphic to the complete graph* K_{r+2}.

Proof. Assume that there is a subgraph in $\Gamma_-^{(r)}(\pi)$ isomorphic to K_{r+2}. Then there exists a decreasing subsequence

$$n \ge a_1 > a_2 > \cdots > a_{r+2} \ge 1$$

such that for all $1 \le i < j \le r+2, t_{a_i,a_j}$ are $r-$th strong descents of π. In particular, for every $1 \le i < r+2, t_{a_i,a_{i+1}}$ are $r-$th strong descents of π. This implies that, for every $1 \le i < r+2$, a_{i+1} appears to the right of a_i in π. Then, by Observation 5.3, $t_{a_1,a_{r+2}}$ is not an $r-$th strong descent. Contradiction. $\square$

Corollary 5.8. *For every* $1 \le r < n$,

$$\max\{d_-^{(r)}(\pi) \mid \pi \in S_n\} \le t_{r+1}(n) \le \binom{r+1}{2} (\frac{n}{r+1})2.$$

Proof. Combining Turán's Theorem together with Lemma 5.7. □

Note that for $r = 1$ and $r = n-1$ equality holds in Corollary 5.8.

Remark 5.9. For every $\pi \in S_n$ let $\bar{\pi}$ be the permutation obtained from π by omitting the value n. If j is the position of n in π then

$$d_-^{(r)}(\pi) - d_-^{(r)}(\bar{\pi})$$

equals the number of $(r-1)$-th almost left-to-right minima in the $(j-1)$-th suffix of $\bar{\pi}$, see e.g. [8]. This observation may be applied to calculate the expectation of $d_-^{(r)}(\pi)$.

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