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SOLVABILITY OF DEGENERATED PARABOLIC EQUATIONS WITHOUT SIGN CONDITION AND THREE UNBOUNDED NONLINEA R – I TIES

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Abstract . In this article , we study the problem

$$\frac{\partial}{\partial t}b(x,u)-\operatorname{div}(a(x,t,u,Du))+H(x,t,u,Du)=f \quad \text{in}\Omega\times]0,T[,$$
$$b(x,u)(t=0)=b(x,u_0) \quad \text{in}\Omega,$$
$$u=0 \quad \text{in}\partial\Omega\times]0,T[$$

in the framework of weighted Sobolev spaces , with $b(x,u)$ unbounded function on u. The main contribution of our work is to prove the existence of a renor - malized solution without the sign condition and the coercivity condition on $H(x,t,u,Du)$. The critical growth condition on H is with respect to Du and

no growth condition with respect to u. The second term f belongs to $L^1(Q)$,

and$b(x,u_0)\in L^1(\Omega)$.

1 . Introduction

Let Ω be a bounded open set of $\mathbb{R}^N, p$ be a real number such that $2<p<\infty$, $Q=\Omega\times[0,T]$ and $w=\{w_i(x) : 0 \leq i \leq N\}$ be a vector of weight functions (i . e . , every component $w_i(x)$ is a measurable almost everywhere strictly positive function on Ω), satisfying some integrability conditions (see Section 2) . And let
$Au= -\operatorname{div}(a(x,t,u,Du))$ be a Leray - Lions operator defined from the weighted Sobolev space $L^p(0,T;W_0^{1,p}(\Omega,w))$ into its dual $L^{p'}(0,T;W^{-1,p'}(\Omega,w^*))$.

Now , we consider the degenerated parabolic problem associated for the differen - tial equation

$$\frac{\partial b(x,u)}{\partial t} +_u Au_= +0H_{\text{on}}(x,t\partial^{\ u,\ Du)}_{\Omega\times]0,\ T[,}{}^{=}f \quad \text{in}Q, \tag{1.1}$$
$$b(x,u)(t=0)=b(x,u_0) \quad \text{on}\Omega$$

where $b(x,u)$ is a unbounded function on u, H is a nonlinear lower order term . Problem (1 . 1) is studied in [2] with $f\in L^{p'}(0,T;W^{-1p'_,}(\Omega,w^*))$ and under the strong hypothesis relatively to H, more precisely they supposed that $b(x,u)=u$

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 and the nonlinearity H satisfying the sign condition

$$H(x,t,s,\xi)s \geq 0 \tag{1.2}$$

and the growth condition of the form

$$| H(x,t,s,\xi) | \leq b(s)(\sum_{i=1}^{N} w_i(x) | \xi i |^p + c(x,t)). \tag{1.3}$$

In the case where the second membre $f \in L^1(Q)$, (1.1) is studied in [3] .

It is our purpose to prove the existence of renormalized solution for (1 . 1) in the setting of the weighted Sobolev space without the sign condition (1 . 2) , and without the following coercivity condition

$$| H(x,t,s,\xi) | \geq \beta \sum_{i=1}^{N} w_i(x) | \xi i |^p \quad \text{for } | s | \geq \gamma, \tag{1.4}$$

our growth condition on H is simpler than (1 . 3) it is a growth with respect to Du and no growth condition with respect to u(see assumption (H 3) below) , the second term f belongs to $L^1(Q)$. Note that our paper generalizes [2 , 3] . The case $H(x,t,u,Du) = \text{div } (\phi(u))$ is studied by Redwane in the classical Sobolev spaces $W^{1,p}(\Omega)$ and in Orlicz spaces ; see [1 5 , 1 6] .

The notion of renormalized solution was introduced by Diperna and Lions [8] in their study of the Boltzmann equation . This notion was then adapted to an elliptic

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version of (1 . 1) by Boccardo et al [5] when the right hand side is in $W^{-1,p}(\Omega)$, by Rakotoson [1 4] when the right hand side is in $L^1(\Omega)$, and finally by Dal Maso , Murat , Orsina and Prignet [7] for the case of right hand side is general measure data .

Our article can be see as a continuation of [4] in the case where $b(x,u) = u$, $a(x,t,s,\xi)$ is independent of s and $H = 0$. The plan of the article is as follows . In Section 2 we give some preliminaries and the definition of weighted Sobolev spaces . In Section 3 we make precise all the assumptions on $b, a, H, f, b(x,u_0)$. In section 4 we give some technical results . In Section 5 we give the definition of a renormalized solution of (1 . 1) and we est ablish the existence of such a solution (Theorem 5 . 3) . Section 6 is devoted to an example which illustrates our abstract result , and finally an appendix in section 7 .

2 . Preliminaries

Let Ω be a bounded open set of $\mathbb{R}^N$, p be a real number such that $2 < p < \infty$ and $w = \{w_i(x), 0 \leq i \leq N\}$ be a vector of weight functions ; i . e . , every component $w_i(x)$ is a measurable function which is strictly positive a . e . in Ω. Further , we suppose in all our considerations that , there exits

$$r_0 > \max\ (N,p) \quad \text{such that } w_i^{\frac{-r_0}{r_0-p}} \in L^1_{\text{loc}}(\Omega), \tag{2.1}$$

$$w_i \in L^1_{\mathrm{loc}}(\Omega), \tag{2.2}$$

$$w_i^{-\frac{1}{p-1}} \in L^1_{\mathrm{loc}}(\Omega), \tag{2.3}$$

 for any $0 \leq i \leq N$. We denote by $W^{1,p}(\Omega, w)$ the space of real - valued functions
$u \in L^p(\Omega, w_0)$ such that the derivatives in the sense of distributions fulfill

$$\frac{\partial u}{\partial x_i} \in L^p(\Omega, w_i) \quad \text{for} i = 1, ..., N.$$

Which is a Banach space under the norm

$$\| u \| 1, p, w = [\int_\Omega | u(x) |^p w_0(x) dx + \sum_N^{i=1} \int_\Omega | \frac{\partial u(x)}{\partial x_i} |^p w_i(x) dx] 1/p. \tag{2.4}$$

Condition (2 . 2) implies that $C_0^\infty(\Omega)$ is a space of $W^{1,p}(\Omega, w)$ and consequently , we can introduce the subspace $V = W_0^{1,p}(\Omega, w)$ of $W^{1,p}(\Omega, w)$ as the closure of $C_0^\infty(\Omega)$ with respect to the norm (2 . 4) . Moreover , condition (2 . 3) implies that $W^{1,p}(\Omega, w)$ as well as $W_0^{1,p}(\Omega, w)$ are reflexive Banach spaces .

We recall that the dual space of weighted Sobolev spaces $W_0^{1,p}(\Omega, w)$ is equivalent to $W^{-1p',}(\Omega, w^*)$, where $w^* = \{w^* i = 1 i_w{}^{p'}{}_-, i = 0, ..., N\}$ and where p' is the conjugate of p; i . e ., $p' = \frac{p}{p-1}{}_,$ (see [1 1]) .

3 . Basic assumptions

Assumption (H 1) . For $2 \leq p < \infty$, we assume that the expression

$$||| u ||| V = (\sum_N^{i=1} \int_\Omega | \frac{\partial u(x)}{\partial x_i} |^p w_i(x) dx) 1/p \tag{3.1}$$

is a norm defined on V which is equivalent to the norm (2 . 4) , and there exists a weight function σ on Ω such that ,

$$\sigma \in L^1(\Omega) \quad \text{and} \sigma^{-1} \in L^1(\Omega).$$

We assume also the Hardy inequality ,

$$(\int_\Omega | u(x) |^p \sigma dx) 1/q \leq c(\sum_N^{i=1} \int_\Omega | \frac{\partial u(x)}{\partial x_i} |^p w_i(x) dx) 1/p, \tag{3.2}$$

holds for every $u \in V$ with a constant $c > 0$ independent of u, and moreover , the imbedding

$$W^{1,p}(\Omega, w) \to\to L^p(\Omega, \sigma), \tag{3.3}$$

expressed by the inequality (3 . 2) is compact . Notice that $(V, ||| \cdot ||| V)$ is a uniformly convex (and thus reflexive) Banach space .

Remark 3 . 1 . If we assume that $w_0(x) \equiv 1$ and in addition the integrability con - dition : There exists $\nu \in]\frac{N}{p}{}_, + \infty[\cap[\frac{1}{p-1}{}_, + \infty[$ such that

$$N$$

$w^{-\nu} i \in L^1(\Omega)$ and $wN - 1^i \underline{\qquad} \in L^1_{\text{loc}}(\Omega)$ for all $i = 1, ..., N$. (3.4) Notice that the assumptions (2 . 2) and (3 . 4) imply

$$||| u |||= (\sum_{i=1}^N \int_\Omega | \frac{\partial u}{\partial x_i} |^p w_i(x) dx) 1/p, \tag{3.5}$$

which is a norm defined on $W_0^{1,p}(\Omega, w)$ and its equivalent to (2 . 4) and that , the imbedding

$$W_0^{1,p}(\Omega, w) \to L^p(\Omega), \tag{3.6}$$

is compact for all $1 \leq q \leq p_1^*$ if $p\nu < N(\nu+1)$ and for all $q \geq 1$ if $p\nu \geq N(\nu+1)$ where $p1 = \frac{p\nu}{\nu+1}$ and p_1^* is the Sobolev conjugate of $p1$; see [1 0 , pp 30 - 3 1] .

Assumption (H 2) .

$b : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carath é odory function . (3 . 7) such that for every $x \in \Omega, b(x,.)$ is a strictly increasing C^1- function with $b(x,0) = 0$.

Next , for any $k > 0$, there exists $\lambda_k > 0$ and functions $A_k \in L^1(\Omega)$ and $B_k \in L^p(\Omega)$ such that

$$\lambda_k \leq \frac{\partial b(x,s)}{\partial s} \leq A_k(x) \quad \text{and} \quad |D_x(\frac{\partial b(x,s)}{\partial s}^{)} | \leq B_k(x) \tag{3.8}$$

for almost every $x \in \Omega$, for every s such that $| \ s \ | \leq k$, we denote by $D_x(\frac{\partial b(x,s)}{\partial s}^{)}$ the gradient of $\frac{\partial b(x,s)}{\partial s}$ defined in the sense of distributions . For $i = 1, ..., N$,

$$| \ a_i(x,t,s,\xi) \ | \leq \beta 1 {i_w}^{/p}(x)[k(x,t) + \sigma^{1/p'} \ | \ s \ |^{q/p'} + \sum_{j=1}^{N} 1 {j_w}^{/p'}(x) \ | \ \xi_j \ |^{p-1}], \tag{3.9}$$

for a . e . $(x,t) \in Q$, all $(s,\xi) \in \mathbb{R} \times \mathbb{R}^N$, some function $k(x,t) \in L^{p'}(Q)$ and $\beta > 0$. Here σ and q are as in (H 1) .

$$[a(x,t,s,\xi) - a(x,t,s,\eta)](\xi - \eta) > 0 \quad \text{forall}(\xi,\eta) \in \mathbb{R}^N \times \mathbb{R}^N, \tag{3.10}$$

$$a(x,t,s,\xi).\xi \geq \alpha \sum_{i=1}^{N} w_i \ | \ \xi i \ |^{p}, \tag{3.11}$$

Where α is a strictly positive constant .

Assumption (H 3) . Furthermore , let $H(x,t,s,\xi) : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ be a Carath é odory function such that for a . e $(x,t) \in Q$ and for all $s \in \mathbb{R}, \xi \in \mathbb{R}^N$, the growth condition

$$| \ H(x,t,s,\xi) \ | \leq \gamma(x,t) + g(s) \sum_{i=1}^{N} w_i(x) \ | \ \xi i \ |^p \tag{3.12}$$

is satisfied , where $g : \mathbb{R} \to \mathbb{R}^+$ is a continuous positive positive function that belongs to $L^1(\mathbb{R})$, while $\gamma(x,t)$ belongs to $L^1(Q)$.

We recall that , for $k > 1$ and s in $\mathbb{R}$, the truncation is defined as

$$T_k(s) = \begin{cases} s & \text{if } | \ s \ | \leq k \\ k\dfrac{s}{| \ s \ |} & \text{if } | \ s \ | > k. \end{cases}$$

4 . Some technical results

Characterization of the t ime mollification of a function u. To deal with time derivative , we introduce a time mollification of a function u belonging to a some weighted Lebesgue space . Thus we define for all $\mu \geq 0$ and all $(x,t) \in Q$,

$$u_\mu = \mu \int_\infty^t \tilde{u}(x,s) \exp(\mu(s-t)) ds$$

$$\text{where} \tilde{u}(x,s) = u(x,s) \chi_(0,T)^{(s)}_.$$

 Proposition 4 . 1 ([2]) . *(1) If* $u \in L^p(Q, w_i)$ *then* u_μ *is measurable in* Q *and*

$$\frac{\partial u_\mu}{\partial t} = \mu(u - u_\mu) and,$$
$$\| u_\mu \| Lp(Q, w_i) \leq \| u \| Lp(Q, w_i)^{\cdot}$$

(2) If $u \in W_0^{1,p}(Q, w)$, *then* $u_\mu \to u$ *in* $W_0^{1,p}(Q, w)$ *as* $\mu \to \infty$.
(3) If $u_n \to u$ *in* $W_0^{1,p}(Q, w)$, *then* $(u_n)_\mu \to u_\mu$ *in* $W_0^{1,p}(Q, w)$.

Some weighted embedding and compactness results . In this section we es - tablish some embedding and compactness results in weighted Sobolev spaces , some trace results , Aubin ' s and Simon ' s results [1 7] . Let $V = W_0^{1,p}(\Omega, w), H = L^2(\Omega, \sigma)$ and let $V^* = W^{-1,p'}$, with $(2 \leq p < \infty)$. Let $X = L^p(0, T; W_0^{1,p}(\Omega, w))$. The dual space of X is $X^* = L^{p'}(0, T, V^*)$ where $\frac{p}{1} + \frac{1}{p'} = 1$ and denoting the space $W_p^1(0, T, V, H) = \{v \in X : v' \in X^*\}$ endowed with the norm

$$\| u \| W_p^1 = \quad \| u \| X + \| u' \| X*,$$

which is a Banach space . Here u' stands for the generalized derivative of u; i . e . ,

$$\int_0^T u'(t)\varphi(t)dt = -\int_0^T u(t)\varphi'(t)dt \quad \text{forall} \varphi \in C_0^\infty(0, T).$$

Lemma 4 . 2 ([1 8]) **.** *(1) The evolution triple* $V \subseteq H \subseteq V^*$ *is satisfied .*
(2) The imbedding $W_p^1(0, T, V, H) \subseteq C(0, T, H)$ *is continuous .*
(3) The imbedding $W_p^1(0, T, V, H) \subseteq L^p(Q, \sigma)$ *is compact .*

Lemma 4 . 3 ([2]) **.** *Let* $g \in L^r(Q, \gamma)$ *and le t* $gn \in L^r(Q, \gamma)$, *with* $\| gn \| Lr(Q, \gamma) \leq C$, $1 < r < \infty$. *If* $gn(x) \to g(x)$ *a . e in* Q, *then* $gn \rightharpoonup g$ *in* $L^r(Q, \gamma)$ *where* $n \to \infty$.

Lemma 4 . 4 ([2]) **.** *Assume that*

$$\frac{\partial v_n}{\partial t} = \alpha_n + \beta_n \quad in D'(Q)$$

where α_n *and* β_n *are bounded respectively in* X^* *and in* $L^1(Q)$. *If* v_n *is bounded in* $L^p(0, T; W_0^{1,p}(\Omega, w))$, *then* $v_n \to u$ *in* $L^p_{\text{loc}}(Q, \sigma)$. *Further* $v_n \to v$ *s trongly in* $L^1(Q)$

$$where n \to \infty.$$

Lemma 4 . 5 ([2]) **.** *Assume that* (H 1) *and* (H 2) *are satisfied and le t* (u_n) *be a s equence in* $L^p(0, T; W_0^{1,p}(\Omega, w))$ *such that* $u_n \rightharpoonup u$ *weakly in* $L^p(0, T; W_0^{1,p}(\Omega, w))$ *and*

$$\int_Q [a(x, t, u_n, Du_n) - a(x, t, u, Du)][Du_n - Du] dxdt \to 0. \quad (4.1)$$

$$Then, u_n \to u in L^p(0, T; W_0^{1,p}(\Omega, w)).$$

Definition 4 . 6 . A monotone map $T : D(T) \to X^*$ is called maximal monotone if its graph

$$G(T) = \{(u, T(u)) \in X \times X^* \quad \text{forall} u \in D(T)\}$$

is not a proper subset of any monotone set in $X \times X^*$. Let us consider the operator $\frac{\partial}{\partial t}$ which induces a linear map L from the subset $D(L) = \{v \in X : v' \in X^*, v(0) = 0 \}$ of X into X^* by

$$\langle Lu, v\rangle X = \int_0^T \langle u'(t), v(t) V^{dt}\rangle \quad u \in D(L), v \in X$$

Lemma 4.7([18]). *L is a closed linear maximal monotone map .*

In our study we deal with mappings of the form $F = L + S$ where L is a given linear densely defined maximal monotone map from $D(L) \subset X$ to X^* and S is a bounded demicontinuous map of monotone type from X to X^*.

Definition 4 . 8 . A mapping S is called pseudo - monotone with $u_n \rightharpoonup u, Lu_n \rightharpoonup Lu$ and $\lim_{n\to\infty}$ sup $\langle S(u_n), u_n - u\rangle \le 0$, we have

$$\lim_{n\to\infty} \sup\langle S(u_n), u_n - u\rangle = 0$$
$$\text{and} S(u_n) \rightharpoonup S(u) \text{as} n \to \infty.$$

5 . Main results Consider the problem

$$b(x, u_0) \in L^1(\Omega), \quad f \in L^1(Q)$$
$$\frac{\partial b(x,u)}{\partial t} - \operatorname{div}(a(x,t,u,Du)) + H(x,t,u,Du) = f \quad \text{in} Q \tag{5.1}$$
$$u = 0 \quad \text{on} \partial\Omega \times]0, T[,$$
$$b(x,u)(t = 0) = b(x, u_0) \quad \text{on} \Omega.$$

Definition 5 . 1 . Let $f \in L^1(Q)$ and $b(x, u_0) \in L^1(\Omega)$. A real - valued function u defined on Q is a renormalized solution of problem 5 . 1 if

$T_k(u) \in L^p(0, T; W_0^{1,p}(\Omega, w))$ for all $k \ge 0$ and $b(x, u) \in L^\infty(0, T; L^1(\Omega));$, (5.2)

$$\int_{\{m \le |u| \le m+1\}} a(x,t,u,Du)Dudxdt \to 0 \quad \text{as} m \to +\infty; \tag{5.3}$$

$$\frac{\partial B_S(x,u)}{+\partial^t_{S''(u)a}}^{-}{}_{(x}, \operatorname{div}^{(\;\; S'(u)a(x,}_{t,u,Du)Du} + t, uH^{;\;\; Du))}_{(x,\;\; t,u,} Du) S'(u) \tag{5.4}$$
$$= fS'(u) \quad \text{in} D'(Q);$$

for all functions $S \in W^{2,\infty}(\mathbb{R})$ which is piecewise C^1 and such that S' has a compact support in $\mathbb{R}$, where $B_S(x, z) = \int_0^z \frac{\partial b(x,r)}{\partial r} S'(r)dr$ and

$$B_S(x, u)(t = 0) = B_S(x, u_0) \quad \text{in} \Omega. \tag{5.5}$$

Remark 5 . 2 . Equation (5 . 4) is formally obtained through pointwise multiplication of (5 . 1) by $S'(u)$. However , while $a(x, t, u, Du)$ and $H(x, t, u, Du)$ does not in general make sense in (5 . 1) , all the terms in (5 . 1) have a meaning in $D'(Q)$. Indeed , if M is such that $suppS' \subset [-M, M]$, the following identifications are made in (5 . 4) :

- $S(u)$ belongs to $L^\infty(Q)$ since S is a bounded function .
- $S'(u)a(x, t, u, Du)$ identifies with $S'(u)a(x, t, T_M(u), DT_M(u))$ a . e . in Q. Since $| T_M(u) | \le M$ a . e . in Q and $S'(u) \in L^\infty(Q)$, we obtain from (3 . 9) and (5 . 2) that

$$S'(u)a(x, t, T_M(u), DT_M(u)) \in \prod_{i=1}^{N} L^{p'}(Q, w^* i)$$

- $S''(u)a(x, t, u, Du)Du \text{identifies with} S''(u)a(x, t, T_M(u), DT_M(u))DT_M(u)$

and

$$S''(u)a(x,t,T_M(u),DT_M(u))DT_M(u) \in L^1(Q).$$

• $S'(u)H(x,t,u,Du)$ identifies with $S'(u)H(x,t,T_M(u),DT_M(u))$ a . e in Q. Since $\mid T_M(u) \mid \leq M$ a . e in Q and $S'(u) \in L^\infty(Q)$, we obtain from (3 . 9) and (3 . 1 2) that

$$S'(u)H(x,t,T_M(u),DT_M(u)) \in L^1(Q).$$

• $S'(u) f \text{belongsto} L^1(Q)$.

The above considerations show that (5 . 4) holds in $D'(Q)$ and that

$$\frac{\partial B_S(x,u)}{\partial t} \in L^{p'}(0,T;W^{-1,}p'_{(\Omega,}w^*i)) + L^1(Q).$$

Due to the properties of S and (5.4), $\frac{\partial S(u)}{\partial t} \in L^{p'}(0,T;W^{-1,}p'_{(\Omega,}w^*i)) + L^1(Q)$, which implies that $S(u) \in C^0([0,T];L^1(\Omega))$ so that the initial condition (5 . 5) makes sense , since , due to the properties of $S($ increasing) and (6 . 1) , we have

$$|B_S(x,r) - B_S(x,r')| \leq A_k(x)|S(r) - S(r')| \quad \text{forall} r,r' \in \mathbb{R}. \tag{5.6}$$

Theorem 5 . 3 . *Let* $f \in L^1(Q)$ *and* $b(x,u_0) \in L^1(\Omega)$. *Assume that* (H 1) – (H 3) *ho ld .*
Then , there exists at least one renormalized s o lution u *of problem* (5 . 1) *(in the*
s ense of Definition 5 . 1) .

The proof of this theorem is done in four steps .

Step 1 : Approximate problem and a priori estimates . For $n > 0$, let us define the following approximation of b, H, f and u_0;

$$b_n(x,r) = b(x,T_n(r)) + \frac{1}{n} r \quad \text{for} n > 0, \tag{5.7}$$

In view of (5.7), b_n is a Carath é odory function and satisfies (6 . 1) , there exist $\lambda_n > 0$ and functions $A_n \in L^1(\Omega)$ and $B_n \in L^p(\Omega)$ such that

$$\lambda_n \leq \frac{\partial b_n(x,s)}{\partial s} \leq A_n(x) \quad \text{and} \quad |D_x(\frac{\partial b_n(x,s)}{\partial s}^{)}| \leq B_n(x)$$

a . e . in $\Omega, s \in \mathbb{R}$.

$$H_n(x,t,s,\xi) = \frac{H(x,t,s,\xi)}{1 + \frac{1}{n} \mid H(x,t,s,\xi) \mid} \chi\Omega_n.$$

Note that Ω_n is a sequence of compacts covering the bounded open set Ω and $\chi\Omega_n$ is its characteristic function .

$f_n \in L^{p'}(Q)$, and $f_n \to f$ a . e . in Q and strongly in $L^1(Q)$ as $n \to +\infty$, (5 . 8)

$$u_{0n} \in D(\Omega), \quad \| b_n(x,u_{0n}) \| L^1 \leq \| b(x,u_0) \| L^1, \tag{5.9}$$

$b_n(x,u_{0n}) \to b(x,u_0)$ a . e . in Ω and strongly in $L^1(\Omega)$. (5.10) Let us now consider the approximate problem :

$$\frac{\partial b_n(x,u_n)}{\partial t} - \text{div}(a(x,t,{}^{u_n,}_{u}{}_{n=}Du_{0^n})_{\text{in}}) + ({}^{Hn(x,}_{0\ ,}{}_{T)} \times t, u_n \partial_{\Omega,} Du_n) = f_n \quad \text{in} D'(Q), \tag{5.11}$$

$$b_n(x,u_n(t=0)) = b_n(x,u_{0n}).$$

Note that $H_n(x,t,s,\xi)$ satisfies the following conditions

$$| H_n(x,t,s,\xi) | \leq H(x,t,s,\xi) \quad \text{and} \quad | H_n(x,t,s,\xi) | \leq n.$$

$$\begin{aligned}
&\text{Forall} u, v \in L^p(0,T;W_0^{1,p}(\Omega,w)),\\
&|\int_Q H_n(x,t,u,Du)vdxdt|\\
&\leq (\int_Q \mid H_n(x,t,u,Du)\mid^{q'} \sigma^{-\frac{q'}{q}} dxdt)1/q'(\int_Q \mid v\mid^q \sigma dxdt)1/q\\
&\leq n\int_0^T(\int_{\Omega_n}\sigma^{1q'_-}dx)1/q'_{dt\|}v\parallel Lq(Q,\sigma)\\
&\leq C_n\parallel v\parallel_{Lp(0,T;W_0}1,p_{(\Omega,w))}
\end{aligned}$$

Moreover , since $f_n \in L^{p'}(0,T;W^{-1,p'}(\Omega,w^*))$, proving existence of a weak solution $u_n \in L^p(0,T;W_0^{1,p}(\Omega,w))$ of (5 . 1 1) is an easy task (see e . g . [1 3] , [2]) .

Let $\varphi \in L^p(0,T;W_0^{1,p}(\Omega,w)) \cap L^\infty(Q)$ with $\varphi > 0$, choosing $v = \exp\,(G(u_n))\varphi$ as test function in 5 . 1 1 where $G(s) = \int_0^s \frac{g(r)}{\alpha}dr($ the function g appears in (3 . 1 2)) . We have

$$\begin{aligned}
&\int_Q \frac{\partial b_n(x,u_n)}{\partial t}\exp(G(u_n))\varphi dxdt + \int_Q a(x,t,u_n,Du_n)D(\exp(G(u_n))\varphi)dxdt\\
&= \int_Q H_n(x,t,u_n,Du_n)\exp(G(u_n))\varphi dxdt + \int_Q f_n\exp(G(u_n))\varphi dxdt.
\end{aligned}$$

In view of (3 . 1 2) , we obtain

$$\begin{aligned}
&\int_Q \frac{\partial b_n(x,u_n)}{\partial t}\exp(G(u_n))\varphi dxdt\\
&+\int_Q a(x,t,u_n,Du_n)Du_n^{g(u_n)}\overline{}_{\alpha}\exp(G(u_n))\varphi dxdt\\
&+\int_Q a(x,t,u_n,Du_n)\exp(G(u_n))D\varphi dxdt\\
&\leq \int_Q \gamma(x,t)\exp(G(u_n))\varphi dxdt + \int_Q g(u_n)\sum_N^{i=1}|\frac{\partial u_n}{\partial x_i}|w_i\exp(G(u_n))\varphi dxdt\\
&+\int_Q f_n\exp(G(u_n))\varphi dxdt.
\end{aligned}$$

By (3 . 1 1) , we obtain

$$\begin{aligned}
&\int_Q \frac{\partial b_n(x,u_n)}{\partial t}\exp(G(u_n))\varphi dxdt + \int_Q a(x,t,u_n,Du_n)\exp(G(u_n))D\varphi dxdt\\
&\leq \int_Q \gamma(x,t)\exp(G(u_n))\varphi dxdt + \int_Q f_n\exp(G(u_n))\varphi dxdt,
\end{aligned}\tag{5 . 1 2}$$

for all $\varphi \in L^p(0,T;W_0^{1,p}(\Omega,w)) \cap L^\infty(Q), \varphi > 0.$ On the other hand , taking $v = \exp\,(-G(u_n))\varphi$ as test function in (5 . 1 1) we deduce , as in (5 . 1 2) , that

$$\begin{aligned}
&\int_Q \frac{\partial b_n(x,u_n)}{\partial t}\exp(-G(u_n))\varphi dxdt + \int_Q a(x,t,u_n,Du_n)\exp(-G(u_n))D\varphi dxdt\\
&+\int_Q \gamma(x,t)\exp(-G(u_n))\varphi dxdt
\end{aligned}$$

$$\geq \int_Q f_n \exp(-G(u_n))\varphi dxdt, \tag{5.13}$$

for all $\varphi \in L^p(0,T;W_0^{1,p}(\Omega,w)) \cap L^\infty(Q), \varphi > 0$. Let $\varphi = T_k(u_n)^+\chi_(0,\tau)$, for every $\tau \in [0,T]$, in (5 . 1 2) we have ,

$$\begin{aligned}\int_\Omega B_k^n(x,u_n(\tau))\exp(G(u_n))dx + \int_{Q_\tau} a(x,t,u_n,Du_n)\exp(G(u_n))DT_k(u_n)^+dxdt \\ \leq \int_{Q_\tau} \gamma(x,t)\exp(G(u_n))T_k(u_n)^+dxdt + \int_{Q_\tau} f_n \exp(G(u_n))T_k(u_n)^+dxdt \\ + \int_\Omega B_k^n(x,u_{0n})dx,\end{aligned}$$

(5 . 14) where $B_k^n(x,r) = \int_0^r T_k(s)^+ \frac{\partial b}{}\,{}_n\frac{(x,s)}{\partial s} ds$. Due to this definition , we have

$$0 \leq \int_\Omega B_k^n(x,u_{0n})dx \leq k \int_\Omega \mid b_n(x,u_{0n}) \mid dx \leq k \parallel b(x,u_0) \parallel L^1(\Omega)^\cdot \tag{5.15}$$

Using this inequality , $B_k^n(x,u_n) \geq 0$ and $G(u_n) \leq \frac{\|g\|_{L^1(\mathbb{R})}}{\alpha}$, we deduce

$$\begin{aligned}\int_{Q_\tau} a(x,t,u_n,DT_k(u_n)^+)DT_k(u_n)^+\exp(G(u_n))dxdt \\ \leq k\exp(\frac{\parallel g \parallel L^1(\mathbb{R})}{\alpha}^) (\parallel u_{0n} \parallel L^1(\Omega) + \parallel f_n \parallel L^1(Q) + \parallel \gamma \parallel L^1(Q) + \parallel b_n(x,u_{0n}) \parallel L^1(\Omega)) \\ \leq c_1 k.\end{aligned}$$

Thanks to (3 . 1 1) , we have

$$\alpha \int_{Q_\tau} \sum_N^{i=1} w_i(x) |\frac{\partial T_k(u_n)^+}{\partial x_i}|^p \exp(G(u_n))dxdt \leq c_1 k. \tag{5.16}$$

We deduce that

$$\alpha \int_Q \sum_N^{i=1} w_i(x) |\frac{\partial T_k(u_n)^+}{\partial x_i}|^p dxdt \leq c_1 k. \tag{5.17}$$

Similarly to (5 . 1 7) , we take $\varphi = T_k(u_n)^-\chi_(0,\tau)$ in (5 . 1 3) we deduce that

$$\alpha \int_Q \sum_N^{i=1} w_i(x) |\frac{\partial T_k(u_n)^-}{\partial x_i}|^p dxdt \leq c_2 k \tag{5.18}$$

where c_2 is a positive constant . Combining (5 . 1 7) and (5 . 1 8) we conclude that

$$\parallel T_k(u_n) \parallel^p_{Lp(0,T;W_0^{1,p}(\Omega,w))} \leq ck. \tag{5.19}$$

We deduce from the above inequality , (5 . 1 4) and (5 . 1 5) , that

$$\int_\Omega B_k^n(x,u_n)dx \leq k(\| f \| L^1(Q)+ \| b(x,u_0) \| L^1(\Omega)^) \equiv Ck. \tag{5.20}$$

Then , $T_k(u_n)$ is bounded in $L^p(0,T;W_0^{1,p}(\Omega,w))$, and $T_k(u_n) \rightharpoonup v_k$ in the space $L^p(0,T;W_0^{1,p}(\Omega,w))$, and by the compact imbedding (3 . 6) gives

$$T_k(u_n) \to v_k \quad \text{strongly in } L^p(Q,\sigma) \text{ and a . e . in } Q.$$

Let $k>0$ be large enough and B_R be a ball of Ω, we have

$$k\text{meas}(\{| u_n |> k\} \cap B_R \times [0,T])$$

$$= \int_0^T \int_{\{|u_n|>k\}\cap B_R} | T_k(u_n) | \,dxdt$$

$$\leq \int_0^T \int_{B_R} | T_k(u_n) | \,dxdt$$

$$\leq (\int_Q | T_k(u_n) |^p \sigma dxdt)1/p(\int_0^T \int_{B_R} \sigma^{1p'}_- dxdt)1/p'$$

$$\leq Tc_R(\int_Q \sum_N^{i=1} w_i(x) |\frac{\partial T_k(u_n)}{\partial x_i}|^p dxdt)1/p$$

$$\leq ck^{1/p},$$

which implies

$$\text{meas}(\{| u_n |> k\} \cap B_R \times [0, T]) \leq \frac{c_1}{k^{1-\frac{1}{p}}}, \quad \forall k \geq 1.$$

So, we have

$$\lim_{\to_k+\infty} (\text{meas}(\{|u_n |> k\} \cap B_R \times [0, T])) = 0.$$

Now we turn to prove the almost every convergence of u_n and $b_n(x, u_n)$. Consider now a function non decreasing $gk \in C^2(\mathbb{R})$ such that $gk(s) = s$ for $| s | \leq \frac{k}{2}$ and $gk(s) = k$ for $| s |\geq k$. Multiplying the approximate equation by $g'_k(b_n(x, u_n))$, we obtain

$$\begin{aligned} &\frac{\partial gk(b_n(x, u_n))}{\partial t} - \text{div}(a(x, t, u_n, Du_n)g'_k(b_n(x, u_n))) \\ &+a(x, t, u_n, Du_n)g''_k(b_n(x, u_n))D_x(\frac{\partial b_n(x, u_n)}{\partial s}) Du_n \\ &+H_n(x, t, u_n, Du_n)g'_k(b_n(x, u_n)) \\ &= f_n g'_k(b_n(x, u_n)) \end{aligned} \tag{5.21}$$

in the sense of distributions, which implies that

$gk(b_n(x, u_n))$ is bounded in $L^p(0, T; W_0^{1,p}(\Omega, w))$, (5.22)

$\frac{\partial gk(b_n(x,u_n))}{\partial t}$ is bounded in $X^* + L^1(Q)$, (5.23) independent of n as long as $k < n$. Due to Definition (3.7) and (5.7) of b_n, it is clear that

$$\{| b_n(x, u_n) |\leq k\} \subset \{| u_n |\leq k^*\}$$

as long as $k < n$ and k^* is a constant independent of n. As a first consequence we have

$Dgk(b_n(x, u_n)) = g'_k(x, b_n(u_n))D_x(\frac{\partial b_n(x,T_{k*}(u_n))}{\partial s})DT_{k*}(u_n)$ a.e in Q (5.24) as long as $k < n$. Secondly, the following estimate holds

$$\|g'_k(b_n(x, u_n))D_x(\frac{\partial b_n(x, T_{k*}(u_n))}{\partial s}) \|L\infty(Q)$$

$$\leq \| g'_k \| L\infty(Q)(| \max_{r|\leq k*}(D_x(\frac{\partial b_n(x, s)}{\partial s})) + 1).$$

 As a consequence of (5 . 1 9) , (5 . 24) we then obtain (5 . 22) . To show that (5 . 23) holds , due to (5 . 2 1) we obtain

$$\frac{\partial gk(b_n(x,u_n))}{\partial t} = \operatorname{div}(a(x,t,u_n,Du_n)g_k'(b_n(x,u_n))) \\ -a(x,t,u_n,Du_n)g_k''(b_n(u_n))D_x(\frac{\partial b_n(x,u_n)}{\partial s}^{)} \\ +H_n(x,t,u_n,Du_n)g_k'(b_n(x,u_n)) + f_n g_k'(b_n(x,u_n)). \tag{5.25}$$

Since support of g_k' and support of g_k'' are both included in $[-k,k]$, u_n may be replaced by $T_{k*}(u_n)$ in each of these terms . As a consequence , each term on the right - hand side of (5 . 25) is bounded either in $L^{p'}(0,T;W^{-1p'_,}(\Omega,w^*))$ or in $L^1(Q)$. Hence lemma 4 . 4 allows us to conclude that $gk(b_n(x,u_n))$ is compact in $L^p_{\rm loc}(Q,\sigma)$. Thus , for a subsequence , it also converges in measure and almost every where in Q, due to the choice of gk, we conclude that for each k, the sequence $T_k(b_n(x,u_n))$ converges almost everywhere in $Q($ since we have , for every $\lambda > 0,)$

$$\operatorname{meas}(\{|b_n(x,u_n) - b_m(x,u_m)| > \lambda\} \cap B_R \times [0,T]) \\ \leq \operatorname{meas}(\{| b_n(x,u_n) |> k\} \cap B_R \times [0,T]) + \operatorname{meas}(\{| b_m(x,u_m) |> k\} \cap B_R \times [0,T]) \\ +\operatorname{meas}(\{|gk(b_n(x,u_n)) - gk(b_m(x,u_m))| > \lambda\}).$$

Let $\varepsilon > 0$, then there exist $k(\varepsilon) > 0$ such that

$$\operatorname{meas}(\{|b_n(x,u_n) - b_m(x,u_m)| > \lambda\} \cap B_R \times [0,T]) \leq \varepsilon$$

for all $n,m \geq n_0(k(\varepsilon),\lambda,R)$. This proves that $(b_n(x,u_n))$ is a Cauchy sequence in measure in $B_R \times [0,T]$, thus converges almost everywhere to some measurable function v. Then for a subsequence denoted again u_n,
$u_n \to u$ a . e . in Q, (5.26) $b_n(x,u_n) \to b(x,u)$ a . e . in Q. (5.27)
We can deduce from (5 . 1 9) that

$$T_k(u_n) \rightharpoonup T_k(u) \quad \text{weaklyin} L^p(0,T;W_0^{1,p}(\Omega,w)) \tag{5.28}$$

and then , the compact imbedding (3 . 3) gives
$T_k(u_n) \to T_k(u)$ strongly in $L^q(Q,\sigma)$ and a . e . in Q. Which implies , by using (3 . 9) , for all $k > 0$ that there exists a function $h_k \in$

$$\prod_{i=1}^{N} L^{p'}(Q,w^*i), \text{suchthat}$$

$$a(x,t,T_k(u_n),DT_k(u_n)) \rightharpoonup h_k \quad \text{weaklyin} \prod_{i=1}^{N} L^{p'}(Q,w^*i). \tag{5.29}$$

We now establish that $b(x,u)$ belongs to $L^\infty(0,T;L^1(\Omega))$. Using (5 . 26) and passing to the limit - inf in (5 . 20) as n tends to $+\infty$, we obtain that

$$\frac{1}{k}\int_\Omega B_k(x,u)(\tau)dx \leq [\| f \| L^1(Q) + \quad \| u_0 \| L^1(\Omega)^] \equiv C,$$

for almost any τ in $(0,T)$. Due to the definition of $B_k(x,s)$ and the fact that $\frac{1}{k}B_k(x,u)$ converges pointwise to $b(x,u)$, as k tends to $+\infty$, shows that $b(x,u)$

belongto$L^{\infty}(0,T;L^1(\Omega))$.

Lemma 5.4. *Let u_n be a solution of the approximate problem (5.11). Then*

$$\lim_{m\to\infty}\limsup_{n\to\infty}\int_{\{m\le|u_n|\le m+1\}} a(x,t,u_n,Du_n)Du_n\,dxdt=0 \tag{5.30}$$

Proof. Considering the function $\varphi=T_1(u_n-T_m(u_n))^-:=\alpha_m(u_n)$ in (5.13) this function is admissible since $\varphi\in L^p(0,T;W_0^{1,p}(\Omega,w))$ and $\varphi\ge 0$. Then, we have

$$\begin{aligned}\int_Q \frac{\partial b_n(x,u_n)}{\partial t}\alpha_m(u_n)dxdt+\int_{\{-(m+1)\le u_n\le -m\}} a(x,t,u_n,Du_n)Du_n\alpha_m'(u_n)dxdt\\ +\int_Q f_n\exp(-G(u_n))\alpha_m(u_n)dxdt\\ \le\int_Q \gamma(x,t)\exp(-G(u_n))\alpha_m(u_n)dxdt.\end{aligned}$$

Which, by setting $B_n^m(x,r)=\int_0^r \frac{\partial b_n(x,s)}{\partial s}\alpha_m(s)ds$, gives

$$\begin{aligned}\int_\Omega B_n^m(x,u_n)(T)dx+\int_{\{-(m+1)\le u_n\le -m\}} a(x,t,u_n,Du_n)Du_n\alpha_m'(u_n)dxdt\\ +\int_Q f_n\exp(-G(u_n))\alpha_m(u_n)dxdt\\ \le\int_Q \gamma(x,t)\exp(-G(u_n))\alpha_m(u_n)dxdt+\int_\Omega B_n^m(x,u_{0n})dx.\end{aligned}$$

Since $B_n^m(x,u_n)(T)\ge 0$ and by Lebesgue's theorem, we have

$$\lim_{m\to\infty}\lim_{n\to\infty}\int_Q f_n\exp(-G(u_n))\alpha_m(u_n)dxdt=0. \tag{5.31}$$

Similarly, since $\gamma\in L^1(\Omega)$, we obtain

$$\lim_{m\to\infty}\lim_{n\to\infty}\int_Q \gamma\exp(-G(u_n))\alpha_m(u_n)dxdt=0. \tag{5.32}$$

We conclude that

$$\lim_{m\to\infty}\limsup_{n\to\infty}\int_{\{-(m+1)\le u_n\le -m\}} a(x,t,u_n,Du_n)Du_n\,dxdt=0. \tag{5.33}$$

On the other hand, let $\varphi = T_1(u_n-T_m(u_n))^+$ as test function in (5.12) and reasoning as in the proof of (5.33) we deduce that

$$\lim_{m\to\infty}\limsup_{n\to\infty}\int_{\{m)\le u_n\le m+1\}} a(x,t,u_n,Du_n)Du_n\,dxdt=0. \tag{5.34}$$

Thus (5.30) follows from (5.33) and (5.34). □

Step 2: Almost everywhere convergence of the gradients. This step is devoted to introduce for $k\ge 0$ fixed a time regularization of the function $T_k(u)$ in order to perform the monotonicity method. This kind of regularization has been first introduced by R. Landes (see Lemma 6 and proposition 3, p. 230, and proposition 4, p. 231, in [12]).

Let $\psi i \in D(\Omega)$ be a sequence which converge strongly to u_0 in $L^1(\Omega)$. Set $\mu_w^i = (T_k(u))_\mu + e^{-\mu t}T_k(\psi i)$ where $(T_k(u))_\mu$ is the mollification with respect to time of $T_k(u)$. Note that μ_w^i is a smooth function having the following properties :

$$\frac{\partial i_{w_\mu}}{\partial t} = \mu(T_k(u) - w_\mu^i), \quad \mu_w^i(0) = T_k(\psi i), \quad |\mu_w^{i|} \le k, \tag{5.35}$$

$$\mu_w^i \to T_k(u) \quad \text{in} L^p(0,T;W_0^{1,p}(\Omega,w)), \tag{5.36}$$

as $\mu \to \infty$. We introduce the following function of one real :

$$h_m(s) = \begin{cases} 0_1^m + 1 - s & \text{if}_{\text{if}}^{\text{if}} \mid s \mid_{|s|}^{m} \ge_{\le}^{\le} m + 1_m^{s \le m} + 1 \\ m + 1 + s & \text{if} - (m+1) \le s \le -m \end{cases}$$
$$\text{where} m > k.$$

Let $\varphi = (T_k(u_n) - \mu_w^i)^+ h_m(u_n) \in L^p(0,T;W_0^{1,p}(\Omega,w)) \cap L^\infty(Q)$ and $\varphi \ge 0$, then we take this function in (5 . 1 2) , we obtain

$$\begin{aligned} &\int_{\{T_k(u_n) - i_{w_\mu} \ge 0\}} \frac{\partial b_n(x,u_n)}{\partial t} \exp(G(u_n))(T_k(u_n) - \mu_w^i)h_m(u_n)dxdt \\ &\quad + \int_{\{T_k(u_n) - i_{w_\mu} \ge 0\}} a(x,t,u_n,Du_n)D(T_k(u_n) - w_\mu^i)h_m(u_n)dxdt \\ - &\int_{\{m \le u_n \le m+1\}} \exp(G(u_n))a(x,t,u_n,Du_n)Du_n(T_k(u_n) - w_\mu^i)^+ dxdt \\ &\le \int_Q \gamma(x,t)\exp(G(u_n))(T_k(u_n) - w_\mu^i)^+ h_m(u_n)dxdt \\ &\quad + \int_Q f_n \exp(G(u_n))(T_k(u_n) - \mu_w^i)^+ h_m(u_n)dxdt. \end{aligned} \tag{5.37}$$

Observe that

$$\begin{aligned} &\int_{\{m \le u_n \le m+1\}} \exp(G(u_n))a(x,t,u_n,Du_n)Du_n(T_k(u_n) - w_\mu^i)^+ dxdt \\ &\qquad \le 2k \int_{\{m \le u_n \le m+1\}} a(x,t,u_n,Du_n)Du_n dxdt. \end{aligned}$$

Thanks to (5 . 30) the third integral tend to zero as n and m tend to infinity , and by Lebesgue ' s theorem , we deduce that the right hand side converge to zero as n, m and μ tend to infinity . Since

$$(T_k(u_n) - w_\mu^i)^+ h_m(u_n) \rightharpoonup (T_k(u) - w_\mu^i)^+ h_m(u) \quad \text{weakly} * \text{ in } L^\infty(Q), \text{ as } n \to \infty,$$
$$\text{and } (T_k(u) - w_\mu^i)^+ h_m(u) \rightharpoonup 0 \quad \text{weakly} * \text{ in } L^\infty(Q) \text{ as } \mu \to \infty.$$

Let $\varepsilon_l(n,m,\mu,i) l = 1,...,n$ various functions tend to zero as n, m, i and μ tend to infinity .

The definition of the sequence $w^i\mu$ makes it possible to establish the following lemma , which will be proved in the Appendix .

 Lemma 5 . 5 . [14] *For* $k \geq 0$ *we have*

$$\int_{\{T_k(u_n)-\mu_w^i \geq 0\}} \frac{\partial b_n(x,u_n)}{\partial t} \exp(G(u_n))(T_k(u_n)-\mu_w^i)h_m(u_n)dxdt \geq \varepsilon(n,m,\mu,i)$$

(5 . 38) On the other hand , the second term of left hand side of (5 . 37) reads as follows

$$\begin{aligned} &\int_{\{T_k(u_n)-\mu_w^i \geq 0\}} a(x,t,u_n,Du_n)D(T_k(u_n)-w_\mu^i)h_m(u_n)dxdt \\ &= \int_{\{T_k(u_n)-\mu_w^i \geq 0, |u_n| \leq k\}} a(x,t,T_k(u_n),DT_k(u_n))D(T_k(u_n)-\mu_w^i)h_m(u_n)dxdt \\ &\quad - \int_{\{T_k(u_n)-\mu_w^i \geq 0, |u_n| \geq k\}} a(x,t,u_n,Du_n)Dw_\mu^i h_m(u_n)dxdt. \end{aligned}$$

Since $m > k, h_m(u_n) = 0$ on $\{| u_n | \geq m+1\}$, One has

$$\begin{aligned} &\int_{\{T_k(u_n)-\mu_w^i \geq 0\}} a(x,t,u_n,Du_n)D(T_k(u_n)-w_\mu^i)h_m(u_n)dxdt \\ &= \int_{\{T_k(u_n)-\mu_w^i \geq 0\}} a(x,t,T_k(u_n),DT_k(u_n))D(T_k(u_n)-w_\mu^i)h_m(u_n)dxdt \\ &- \int_{\{T_k(u_n)-\mu_w^i \geq 0, |u_n| \geq k\}} a(x,t,T_{m+1}(u_n),DT_{m+1}(u_n))Dw_\mu^i h_m(u_n)dxdt \\ &= J_1 + J_2 \end{aligned}$$

(5 . 39) In the following we pass to the limit in (5 . 39) : first we let n tend to $+\infty$, then μ and finally m, tend to $+\infty$. Since $a(x,t,T_{m+1}(u_n),DT_{m+1}(u_n))$ is bounded in
$\prod_{i=1}^N L^{p'}(Q,w^*i)$, we have that

$$a(x,t,T_{m+1}(u_n),DT_{m+1}(u_n))h_m(u_n)\chi_{\{|u_n|>k\}} \to h_m h_m(u)\chi_{\{|u|>k\}}$$

strongly in $\prod_{i=1}^N L^{p'}(Q,w^*i)$ as n tends to infinity , it follows that

$$\begin{aligned} J_2 &= \int_{\{T_k(u_n)-\mu_w^i \geq 0\}} h_m Dw_\mu^i h_m(u)\chi_{\{|u|>k\}}dxdt + \varepsilon(n) \\ &= \int_{\{T_k(u_n)-\mu_w^i \geq 0\}} h_m(DT_k(u)_\mu - e^{-\mu t}DT_k(\psi i))h_m(u)\chi_{\{|}u |> k\}dxdt + \varepsilon(n). \\ &\text{Byletting}\mu \to +\infty, \\ J_2 &= \int_{\{T_k(u_n)-\mu_w^i \geq 0\}} h_m DT_k(u)dxdt + \varepsilon(n,\mu). \end{aligned}$$

 Using now the term J_1 of (5 . 39) one can easily show that

$$\int_{\{T_k(u_n)-\mu_w^i\geq 0\}} a(x,t,T_k(u_n),DT_k(u_n))D(T_k(u_n)-w_\mu^i)h_m(u_n)dxdt$$
$$=\int_{\{T_k(u_n)-i_{w_\mu}\geq 0\}}[a(x,t,T_k(u_n),DT_k(u_n))-a(x,t,T_k(u_n),DT_k(u))]$$
$$\times[DT_k(u_n)-DT_k(u)]h_m(u_n)dxdt$$
$$+\int_{\{T_k(u_n)-\mu_w^i\geq 0\}} a(x,t,T_k(u_n),DT_k(u))(DT_k(u_n)-DT_k(u))h_m(u_n)dxdt$$
$$+\int_{\{T_k(u_n)-\mu_w^i\geq 0\}} a(x,t,T_k(u_n),DT_k(u_n))DT_k(u)h_m(u_n)dxdt$$
$$-\int_{\{T_k(u_n)-\mu_w^i\geq 0\}} a(x,t,T_k(u_n),DT_k(u_n))Dw_\mu^i h_m(u_n)dxdt$$
$$=K_1+K_2+K_3+K_4.$$

(5 . 40) We shall go to the limit as n and $\mu\to+\infty$ in the three integrals of the right - hand side . Starting with K_2, we have by letting $n\to+\infty$,

$$K_2=\varepsilon(n). \tag{5.41}$$

About K_3, we have by letting $n\to+\infty$ and using (5 . 29) ,

$$K_3=\int_{\{T_k(u_n)-\mu_w^i\geq 0\}} h_k DT_k(u)h_m(u)\chi_{\{|}u\ |>k\}dxdt+\varepsilon(n)$$
$$\text{Byletting}\mu\to+\infty,$$
$$K_3=\int_{\{T_k(u_n)-\mu_w^i\geq 0\}} h_k DT_k(u)dxdt+\varepsilon(n,\mu). \quad (5.42)$$

For K_4 we can write

$$K_4=-\int_{\{T_k(u_n)-\mu_w^i\geq 0\}} h_k Dw_\mu^i h_m(u)dxdt+\varepsilon(n),$$
$$\text{Byletting}\mu\to+\infty,$$
$$K_4=-\int_{\{T_k(u_n)-i_{w_\mu}\geq 0\}} h_k DT_k(u)dxdt+\varepsilon(n,\mu). \quad (5.43)$$

We then conclude that

$$\int_{\{T_k(u_n)-\mu_w^i\geq 0\}} a(x,t,T_k(u_n),DT_k(u_n))D(T_k(u_n)-w_\mu^i)h_m(u_n)dxdt$$
$$=\int_{\{T_k(u_n)-\mu_w^i\geq 0\}}[a(x,t,T_k(u_n),DT_k(u_n))-a(x,t,T_k(u_n),DT_k(u))]$$
$$\times[DT_k(u_n)-DT_k(u)]h_m(u_n)dxdt+\varepsilon(n,\mu).$$

 On the other hand, we have

$$\begin{aligned}
&\int_{\{T_k(u_n)-\mu_w^i\geq 0\}} [a(x,t,T_k(u_n),DT_k(u_n)) - a(x,t,T_k(u_n),DT_k(u))] \\
&\qquad\qquad \times [DT_k(u_n) - DT_k(u)]dxdt \\
&= \int_{\{T_k(u_n)-\mu_w^i\geq 0\}} [a(x,t,T_k(u_n),DT_k(u_n)) - a(x,t,T_k(u_n),DT_k(u))] \\
&\times + a(x,t,T_k(u_n),DT_k^{0\}}{}_{i^{\mu}}{}_{[DT_k\int_{\{T_k(u_n)-w}^{(u_n)}-\geq}{}^{DT_k(u)]} h_m(u_n)dxdt^{(u_n))(DT_k(u_n)} - DT_k(u)) \qquad (5.44) \\
&\qquad\qquad \times (1-h_m(u_n))dxdt \\
&\quad - \int_{\{T_k(u_n)-i_{w_\mu}\geq 0\}} a(x,t,T_k(u_n),DT_k(u))(DT_k(u_n) - DT_k(u)) \\
&\qquad\qquad \times (1-h_m(u_n))dxdt.
\end{aligned}$$

Since $h_m(u_n) = 1$ in $\{|u_n| \leq m\}$ and $\{|u_n| \leq k\} \subset \{|u_n| \leq m\}$ for m large enough, we deduce from (5.44) that

$$\begin{aligned}
&\int_{\{T_k(u_n)-\mu_w^i\geq 0\}} [a(x,t,T_k(u_n),DT_k(u_n)) - a(x,t,T_k(u_n),DT_k(u))] \\
&\qquad\qquad \times [DT_k(u_n) - DT_k(u)]dxdt \\
&= \int_{\{T_k(u_n)-\mu_w^i\geq 0\}} [a(x,t,T_k(u_n),DT_k(u_n)) - a(x,t,T_k(u_n),DT_k(u))] \\
&\qquad\qquad \times [DT_k(u_n) - DT_k(u)]h_m(u_n)dxdt \\
&\quad + \int_{\{T_k(u_n)-\mu_w^i\geq 0, |u_n|>k\}} a(x,t,T_k(u_n),DT_k(u))DT_k(u)(1-h_m(u_n))dxdt.
\end{aligned}$$

It is easy to see that the last terms of the last equality tend to zero as $n \to +\infty$, which implies

$$\begin{aligned}
&\int_{\{T_k(u_n)-\mu_w^i\geq 0\}} [a(x,t,T_k(u_n),DT_k(u_n)) - a(x,t,T_k(u_n),DT_k(u))] \\
&\qquad\qquad \times [DT_k(u_n) - DT_k(u)]dxdt \\
&= \int_{\{T_k(u_n)-\mu_w^i\geq 0\}} [a(x,t,T_k(u_n),DT_k(u_n)) - a(x,t,T_k(u_n),DT_k(u))] \\
&\qquad\qquad \times [DT_k(u_n) - DT_k(u)]h_m(u_n)dxdt + \varepsilon(n)
\end{aligned}$$

Combining (5.38), (5.40), (5.41), (5.42), (5.43) and (5.44), we obtain

$$\begin{aligned}
&\int_{\{T_k(u_n)-i_{w_\mu}\geq 0\}} [a(x,t,T_k(u_n),DT_k(u_n)) - a(x,t,T_k(u_n),DT_k(u))] \qquad (5.45) \\
&\qquad\qquad \times [DT_k(u_n) - DT_k(u)]dxdt \leq \varepsilon(n,\mu,m)
\end{aligned}$$

Passing to the limit in (5.45) as n and m tend to infinity, we obtain

$$\begin{aligned}
&\lim_{n\to\infty} \int_{\{T_k(u_n)-\mu_w^i\geq 0\}} [a(x,t,T_k(u_n),DT_k(u_n)) - a(x,t,T_k(u_n),DT_k(u))] \qquad (5.46) \\
&\qquad\qquad \times [DT_k(u_n) - DT_k(u)]dxdt = 0.
\end{aligned}$$

 On the other hand , taking $\varphi = (T_k(u_n) - w_\mu^i)^- h_m(u_n)$ in (5 . 1 3) , we deduce as in (5 . 46) that

$$\lim_{n\to\infty} \int_{\{T_k(u_n)-\mu_w^i\le 0\}} [a(x,t,T_k(u_n),DT_k(u_n)) - a(x,t,T_k(u_n),DT_k(u))] \times [DT_k(u_n) - DT_k(u)]dxdt = 0. \tag{5.47}$$

Combining (5 . 46) and (5 . 47) , we conclude

$$\lim_{n\to\infty} \int_Q [a(x,t,T_k(u_n),DT_k(u_n)) - a(x,t,T_k(u_n),DT_k(u))] \times [DT_k(u_n) - DT_k(u)]dxdt = 0. \tag{5.48}$$

Which , by lemma (4 . 5) , implies

$T_k(u_n) \to T_k(u)$ strongly in $L^p(0,T;W_0^{1,p}(\Omega,w))$ for all k. (5.49) Now , observe that for every $\sigma > 0$,

$$\begin{aligned} \text{meas}\{(x,t) \in \Omega\times[0,T] :| Du_n - Du |> \sigma\} \\ \le \text{meas}\{(x,t) \in \Omega\times[0,T] :| Du_n |> k\} \\ +\text{meas}\{(x,t) \in \Omega\times[0,T] :| u |> k\} \\ +\text{meas}\{(x,t) \in \Omega\times[0,T] : |DT_k(u_n) - DT_k(u)| > \sigma\} \end{aligned}$$

then as a consequence of (5 . 49) we have that Du_n converges to Du in measure and therefore , always reasoning for a subsequence ,

$Du_n \to Du$ a . e . in Q. (5.50) Which implies

$$a(x,t,T_k(u_n),DT_k(u_n)) \rightharpoonup a(x,t,T_k(u),DT_k(u)) \quad \text{in} \prod_{i=1}^{N} L^{p'}(Q,w^*i). \tag{5.51}$$

Step 3 : Equi - integrability of the nonlinearity sequence . We shall now prove that $H_n(x,t,u_n,Du_n) \to H(x,t,u,Du)$ strongly in $L^1(Q)$ by using Vitali ' s theo - rem . Since $H_n(x,t,u_n,Du_n) \to H(x,t,u,Du)$ a . e . in Q, Consider a function
$\rho h(s) = \int_0^s g(\nu)\chi_{\{\nu>h\}}d\nu$, take $\varphi = \rho h(u_n) = \int_0^{u_n} g(s)\chi_{\{s>h\}}ds$ as test function in (5 . 1 2) , we obtain

$$\begin{aligned} T_0 + \int_Q a(x,t,u_n,Du_n)Du_n g(u_n)\chi_{\{u_n>h\}}dxdt \\ \le (\int_h^\infty g(s)\chi_{\{s>h\}}ds)\exp(\frac{\| g \| L^1(\mathbb{R})}{\alpha}{}^{)}(\| \gamma \| L^1(Q)+ \| f_n \| L^1(Q)), \end{aligned}$$

where $B_h^n(x,r) = \int_0^r \frac{\partial b_n(x,s)}{\partial s}\rho h(s)ds$, which implies , since $B_h^n(x,r) \ge 0$,

$$\begin{aligned} \int_Q a(x,t,u_n,Du_n)Du_n g(u_n)\chi_{\{}u_n > h\}dxdt \\ \le (\int_h^\infty g(s)ds)\exp(\frac{\| g \| L^1(\mathbb{R})}{\alpha}{}^{)}(\| \gamma \| L^1(Q)+ \| f_n \| L^1(Q)^{)} + \int_\Omega B_h^n(x,u_{0n})dx. \end{aligned}$$

 Using (3 . 1 1) , we have

$$\int_{\{u_n>h\}} g(u_n)\sum_{N}^{i=1} w_i|\frac{\partial u_n}{\partial x_i}|^p dxdt \le C\int_h^\infty g(s)ds.$$

Since $g \in L^1(\mathbb{R})$, we have

$$\lim_{h\to\infty}\sup_{n\in\mathbb{N}}\int_{\{u_n>h\}} g(u_n)\sum_{N}^{i=1} w_i|\frac{\partial u_n}{\partial x_i}|^p dxdt = 0.$$

Similarly , let $\varphi = \int_{u_n}^0 g(s)\chi_{\{s<-h\}}ds$ as a test function in (5 . 1 3) , we conclude that

$$\lim_{h\to\infty}\sup_{n\in\mathbb{N}}\int_{\{u_n<-h\}} g(u_n)\sum_{N}^{i=1} w_i|\frac{\partial u_n}{\partial x_i}|^p dxdt = 0.$$

Consequently ,

$$\lim_{\to_h+\infty}\sup_{n\in\mathbb{N}}\int_{\{|u_n|>h\}} g(u_n)\sum_{N}^{i=1} w_i|\frac{\partial u_n}{\partial x_i}|^p dxdt = 0,$$

which , for h large enough , implies

$$\begin{aligned}\int_Q g(u_n)\sum_{N}^{i=1} w_i|\frac{\partial u_n}{\partial x_i}|^p dxdt &\le \int_{\{|u_n|<h\}} g(u_n)\sum_{N}^{i=1} w_i|\frac{\partial u_n}{\partial x_i}|^p dxdt + 1\\ &\le \int_Q g(T_k(u_n))\sum_{N}^{i=1} w_i|\frac{\partial T_k(u_n)}{\partial x_i}|^p dxdt + 1.\end{aligned}$$

Then by (5 . 49) and Vitali ' s theorem , we can deduce that $g(u_n)\sum_{i=1}^N w_i|\frac{\partial u_n}{\partial x_i}|^p$ converges to $g(u)\sum_{i=1}^N w_i|\frac{\partial u}{\partial x_i}|^p$ strongly in $L^1(Q)$. Consequently , using (3 . 1 2) , we conclude that

$$H_n(x,t,u_n,Du_n) \to H(x,t,u,Du) \quad \text{stronglyin} L^1(Q). \tag{5.52}$$

Step 4 . In this step we prove that u satisfies (5 . 3) , (5 . 4) and (5 . 5) . **Lemma 5 . 6 .** *The limit u of the approximate s o lution u_n of (5 . 1 1) satisfies*

$$\lim_{\to_m+\infty}\int_{\{m\le|u|\le m+1\}} a(x,t,u,Du)Dudxdt = 0.$$

Proof . Note that for any fixed $m \ge 0$,

$$\begin{aligned}&\int_{\{m\le|u_n|\le m+1\}} a(x,t,u_n,Du_n)Du_n\\ &= \int_Q a(x,t,u_n,Du_n)(DT_{m+1}(u_n) - DT_m(u_n))\\ &= \int_Q a(x,t,T_{m+1}(u_n),DT_{m+1}(u_n))DT_{m+1}(u_n)\\ &\quad - \int_Q a(x,t,T_m(u_n),DT_m(u_n))DT_m(u_n).\end{aligned}$$

 According to (5 . 5 1) and (5 . 49) , one is alloed to pass to the limit as $n \to +\infty$ for fixed $m \geq 0$, and to obtain

$$\begin{aligned} &\lim_{\to n+\infty} \int_{\{m\leq|u_n|\leq m+1\}} a(x,t,u_n,Du_n)Du_n dxdt \\ &= Q^{a(x, \quad Tm+1(u),}_{\int_{-\int_Q}^{a(x,} t, \quad t,T_m(u),} DT_m^{DT_m+1(u))DT_{m+1}(u)}_{(u))DT_m(u_n)\ dx} dt^{dx}_{.} dt \\ &= \int_{\{m\leq|u_n|\leq m+1\}} a(x,t,u,Du)Dudxdt. \end{aligned} \tag{5.53}$$

Taking the limit as $m \to +\infty$ in (5 . 53) and using the estimate (5 . 30) show that u satisfies (5 . 4) and the proof is complete . $\square$

Now , we show that u satisfies (5 . 4) and (5 . 5) . Let S be a function in $W^{1,\infty}(\mathbb{R})$ such that S has a compact support . Let M be a positive real number such that support of (S') is a subset of $[-M, M]$. Pointwise multiplication of the approximate equation (5 . 1 1) by $S'(u_n)$ leads to

$$\begin{aligned} &\frac{\partial B^n_S(x,u_n)}{+\prime^{\partial t}_{S(u_n)H}}^{-}{}_{\ n}\ {}^{\operatorname{div}[S'(u_n)a(u_n,}_{(u\ n,} {}_{Du_n)} Du_n)] + S''(u_n)a(u_n,Du_n)Du_n \\ &\qquad = fS'(u_n) \quad \text{in} D'(Q). \end{aligned} \tag{5.54}$$

Passing to the limit , as n tends to $+\infty$, we have

• Since S is bounded and continuous , $u_n \to u$ a . e . in Q implies that $B^n_S(x,u_n)$ converges to $B_S(x,u)$ a . e . in Q and L^∞ weak $- *$. Then $\frac{\partial B^n_S(x,u_n)}{\partial t}$ converges to $\frac{\partial B_S(x,u)}{\partial t}$ in $D'(Q)$ as n tends to $+\infty$.

• Since supp $(S') \subset [-M, M]$, we have for $n \geq M$,

$$S'(u_n)a_n(u_n,Du_n) = S'(u_n)a(T_M(u_n),DT_M(u_n)) \quad \text{a . e . in } Q.$$

The pointwise convergence of $u_n tou$ and (5 . 5 1) as n tends to $+\infty$ and the bounded character of S' permit us to conclude that

$$S'(u_n)a_n(u_n,Du_n) \rightharpoonup S'(u)a(T_M(u),DT_M(u)) \quad \text{in} \prod_{i=1}^{N} L^{p'}(Q,w^*i), \tag{5.55}$$

as n tends to $+\infty$. $S'(u)a(T_M(u),DT_M(u))$ has been denoted by $S'(u)a(u,Du)$ in equation (5 . 4) .

• Regarding the ‘ energy ’ term , we have

$$S''(u_n)a(u_n,Du_n)Du_n = S''(u_n)a(T_M(u_n),DT_M(u_n))DT_M(u_n) \quad \text{a . e . in } Q.$$

The pointwise convergence of $S'(u_n)toS'(u)$ and (5 . 5 1) as n tends to $+\infty$ and the bounded character of S'' permit us to conclude that

$$S''(u_n)a_n(u_n,Du_n)Du_n \rightharpoonup S''(u)a(T_M(u),DT_M(u))DT_M(u) \quad \textit{weakly} \quad \text{in} L^1(Q). \tag{5.56}$$

Recall that

$S''(u)a(T_M(u), DT_M(u))DT_M(u) = S''(u)a(u, Du)Du$ a . e . in Q.

• Since supp $(S') \subset [-M, M]$, by (5 . 52) , we have

$$S'(u_n)H_n(x, t, u_n, Du_n) \to S'(u)H(x, t, u, Du) \quad \text{stronglyin} L^1(Q), \tag{5.57}$$

as n tends to $+\infty$.

• Due to (5 . 8) and ($u_n \to u$ a . e in Q), we have

$S'(u_n)f_n \to S'(u)f$ strongly in $L^1(Q)$ as $n \to +\infty$.

As a consequence of the above convergence result , we are in a position to pass to the limit as n tends to $+\infty$ in equation (5 . 54) and to conclude that u satisfies (5 . 4) . It remains to show that $B_S(x, u)$ satisfies the initial condition (5 . 5) . To this end , firstly remark that , S being bounded , $B^n_S(x, u_n)$ is bounded in $L^\infty(Q)$. Secondly , (5 . 54) and the above considerations on the behavior of the terms of this equation show that $\frac{\partial B^n_S(x,u_n)}{\partial t}$ is bounded in $L^1(Q) + L^{p'}(0, T; W^{-1p'_,}(\Omega, w^*))$. As a con - sequence , an Aubin ' s type lemma (see , e . g , [1 7]) implies that $B^n_S(x, u_n)$ lies in a

compact set of $C^0([0, T], L^1(\Omega))$. It follows that on the one hand , $B^n_S(x, u_n)(t = 0) = B^n_S(x, un_0)$ converges to $B_S(x, u)(t = 0)$ strongly in $L^1(\Omega)$. On the other hand , the smoothness of S implies that

$$B_S(x, u)(t = 0) = B_S(x, u_0) \quad \text{in}\Omega.$$

As a conclusion , steps 1 − 5 complete the proof of theorem 5 . 3 . 6 . Example

Let us consider the special case

$$b(x, r) = \sigma(x) \mid s \mid^{q(x)-2} s,$$

and $q : \Omega \to]1, +\infty[$ with $q(x) \leq - \mid x \mid^2 +2$. Then $b : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carath é odory function , Such that for every $x \in \Omega, b(x, .)$ is a strictly increasing C^1- function with $b(x, 0) = 0$. Next , for any $k > 0$, there exist $\lambda_k > 0$ and functions $A_k \in L^1(\Omega)$ and

$B_k \in L^p(\Omega)$suchthat

$$\lambda_k \leq \frac{\partial b(x, s)}{\partial s} \leq A_k(x), \quad |D_x(\frac{\partial b(x, s)}{\partial s}^{)} | \leq B_k(x), \tag{6.1}$$

$$H(x, t, s, \xi) = \rho \sin(s) \exp(s^{-2}) \sum_{i=1}^{N} w_i(x) \mid \xi i \mid^p, \quad \rho \in \mathbb{R}, \tag{6.2}$$

$$a_i(x, t, s, d) = w_i(x) \mid d_i \mid^{p-1} \operatorname{sgn}(d_i), \quad i = 1, ..., N, \tag{6.3}$$

with $w_i(x)$, $(i = 1, ..., N)$, a weight function strictly positive , $x \in Q$. Then , we can consider the Hardy inequality in the form

$$(\int_\Omega \mid u(x) \mid^p \sigma(x)dx)1/p \leq c(\int_\Omega \mid Du(x) \mid^p w(x)dx)1/p.$$

It is easy to show that the $a_i(t, x, s, d)$ are Caratheodory functions satisfying the growth condition (3 . 9) and the coercivity (3 . 1 1) . On the order hand the mono - tonicity condition is verified . In fact ,

$$\sum_{i=1}^{N} (a_i(x, t, d) - a(x, t, d'))(d_i - d'_i)$$

$$= w(x)\sum_{i=1}^{N-1}(|\ d_i\ |^{p-1}\ \mathrm{sgn}(d_i) -|\ d'_i\ |^{p-1}\ \mathrm{sgn}(d'_i))(d_i - d'_i) > 0,$$

for almost all $x \in \Omega$ and for all $d, d' \in \mathbb{R}^N$. This last inequality can not be strict , since for $d \neq d'$, since $w > 0$ a . e . in Ω.

While the Carath $\acute{e}$ odory function $H(x, t, s, \xi)$ satisfies the condition (3 . 1 2) indeed

$$|\ H(x,t,s,\xi)\ | \leq |\ \rho\ |\ \exp(s^{-2}) \sum_{i=1}^{N} w_i(x)\ |\ \xi i\ |^p = g(s) \sum_{i=1}^{N} w_i(x)\ |\ \xi i\ |^p$$

where $g(s) = |\ \rho\ |$ exp $(s^{-2}$ is a function positive continuous which belongs to $L^1(\mathbb{R})$. Note that $H(x, t, s, \xi)$ does not satisfy the sign condition (1 . 2) and the coercivity condition (1 . 4) .

In particular , let us use special weight function , w, expressed in terms of the distance to the bounded $\partial\Omega$. Denote $d(x) \quad = \mathrm{dist}\ (x, \partial\Omega)$ and set $w(x) \quad = d^\lambda(x)$,

$$\sigma(x) = d^\mu(x).$$

Finally , the hypotheses of Theorem 5 . 3 are satisfied . Therefore , for all $f \in L^1(Q)$, the problem

$$b(x, u) \in L^\infty([0, T]; L^1(\Omega)); \quad T_k(u) \in L^p(0, T; W_0^{1,p}(\Omega, w)),$$

$$\lim_{\to m+\infty} \int_{\{m \leq |u| \leq m+1\}} d^\lambda(x) \sum_{N}^{i=1} |\frac{\partial u}{\partial x_i}|^{p-1} \mathrm{sgn}(\frac{\partial u}{\partial x_i}) \frac{\partial u}{\partial x_i} dxdt = 0;$$

$$B_S(x, r) = \int_0^r \frac{\partial b(x, \sigma)}{\partial \sigma} S'(\sigma) d\sigma,$$

$$\begin{aligned}
&\int_\Omega B_S(x, u(T))\varphi(T)dx - \int_Q B_S(x, u)\frac{\partial \varphi}{\partial t} dxdt \\
&+ \int_Q S'(u) d^\lambda(x) \sum_{N}^{i=1} |\frac{\partial u}{\partial x_i}|^{p-1} \mathrm{sgn}(\frac{\partial u}{\partial x_i}) \frac{\partial \varphi}{\partial x_i} dxdt \\
&+ \int_Q S''(u) d^\lambda(x) \sum_{N}^{i=1} |\frac{\partial u}{\partial x_i}|^{p-1} \mathrm{sgn}(\frac{\partial u}{\partial x_i}) \frac{\partial u}{\partial x_i} \varphi dxdt \\
&+ \int_Q \rho S'(u) \sin(u) \exp(u^{-2}) \sum_{N}^{i=1} w_i |\frac{\partial u}{\partial x_i}|^{p-1} \varphi dxdt \\
&= \int_Q f S'(u) \varphi dxdt + \int_\Omega B_S(x, u_0)\varphi(0) dx,
\end{aligned}$$

$$B_S(x, u)(t = 0) = B_S(x, u_0) \quad \mathrm{in}\Omega,$$

for all $\varphi \in C_0^\infty(Q)$ and $S \in W^{1,\infty}(\mathbb{R})$ with $S' \in C_0^\infty(\mathbb{R})$, has at least one renor - malised solution .

7 . APPENDIX *Proof of Lemma 5 . 5 .* (see also [1 5]) Integration by parts and the use of the prop -
erties of $(w)_\mu^i$ yield

$$\begin{aligned}\int_0^T \int_{\{x\in\Omega; T_k(u_n)-i_{w_\mu}\geq 0\}} & \frac{\partial b_n(x,u_n)}{\partial t} h_m(u_n)\exp(G(u_n))(T_k(u_n)-\mu_w^i)dxdt \\ = \int_0^T \int_{\{x\in\Omega; T_k(u_n)-\mu_w^i\geq 0\}} & \frac{\partial b_n(x,u_n)}{\partial t} h_m(u_n)T_k(u_n)\exp(G(u_n)), dxdt \\ - \int_0^T \int_{\{x\in\Omega; T_k(u_n)-\mu_w^i\geq 0\}} & \frac{\partial b_n(x,u_n)}{\partial t} h_m(u_n)\exp(G(u_n))w_\mu^i dxdt \\ & = I_1^n + I_2^{n,\mu}.\end{aligned} \tag{7.1}$$

We denote

$$\begin{aligned}B_{m,k}^n(x,r) &= \int_0^r \frac{\partial b_n(x,s)}{\partial s} h_m(s)T_k(s)\exp(G(s))ds, \\ B_m^n(x,r) &= \int_0^r \frac{\partial b_n(x,s)}{\partial s} h_m(s)\exp(G(s))ds.\end{aligned}$$

By a standard argument we can write the first term on the right - hand side of (7 . 1) as

$$\begin{aligned}I_1^n = [\int_{\{x\in\Omega;} & T_k(u_n)-\mu_w^i\geq 0\} B_{m,k}^n(x,u_n)dx]T_0 \\ = \int_{\{x\in\Omega;} & T_k(u_n)(T)-\mu_w^i(T)\geq 0\} B_{m,k}^n(x,T_m(u_n)(T))dx \\ - \int_{\{x\in\Omega;} & T_k(u_n)(0)-\mu_w^i(0)\geq 0\} B_{m,k}^n(x,T_m(u_n)(0))dx.\end{aligned} \tag{7.2}$$

We observe that

$$\frac{\partial b_n(x,T_m(u_n))}{\partial s} h_m(u_n) = (\frac{\partial b_n(x,T_m(u_n))}{\partial s} + \frac{1}{n}) h_m(u_n)$$

for $n > m$ with supp $h_m \subset [-m;m]$. Passing to the limit in (7 . 2) as $n \to +\infty$, we deduce that

$$I_1^n =_{-} \int_{\{x\in\Omega;}^{\int_{\{x\in\Omega;}} T_{T_k}^{k(u)(T)-i_{w_\mu}(T} {}_{(u)(0)-\mu_w^i(0)} B_m \geq_{0\}}^{)\geq 0\}} B_{m,k(x,}^{,k(x} {}^{,\ T_m(u(T)))dx}_{T m(u_0))dx} + \varepsilon(n). \tag{7.3}$$

where $B_{m,k}(x,r) = \int_0^r \frac{\partial b(x,s)}{\partial s} h_m(s)T_k(s) \exp\, (G(s))ds$. Passing to the limit in (7 . 3) as $i \to +\infty$ and $\mu \to +\infty$, we have

$$I_1^n = \int_\Omega [B_{m,k}(x,u(T)) - B_{m,k}(x,u_0)]dx + \varepsilon(n,\mu,i). \tag{7.4}$$

 The second term on the right - hand side of (7 . 1) can be written as

$$
\begin{aligned}
I_2^{n,\mu} = -\int_0^T \int_{\{x\in\Omega;/T_k(u_n)-\mu_w^i\geq 0\}} & \frac{\partial b_n(x,u_n)}{\partial t} h_m(u_n)\exp(G(u_n))w_\mu^i dxdt \\
&= -[\int_{\{x\in\Omega;} T_k(u_n)-\mu_w^i \geq 0\} B_m^n(x,u_n)\mu_w^i dx]T_0 \\
&\int_0^T \int_{\{x\in\Omega;} T_k(u_n)-\mu_w^i \geq 0\} B_m^n(x,u_n)\frac{\partial i_{w_\mu}}{\partial t} dxdt \qquad (7.5)\\
= -\int_{\{x\in\Omega;} & T_k(u_n)(T)-\mu_w^i(T) \geq 0\} B_m^n(x,T_m(u_n(T)))\mu_w^i(T)dx \\
&+\int_{\{x\in\Omega;} T_k(u_n)(0)-\mu_w^i(0) \geq 0\} B_m^n(x,u_{0n})\mu_w^i(0)dx \\
+\mu\int_0^T \int_{\{x\in\Omega;} & T_k(u_n)-\mu_w^i \geq 0\} B_m^n(x,u_n)(T_k(u)-w_\mu^i)dxdt.
\end{aligned}
$$

By passing to the limit as n tends to infinity in (7 . 5) , we obtain

$$
\begin{aligned}
I_2^{n,\mu} = &-\int_{\{x\in\Omega;} T_k(u)-\mu_w^i \geq 0\}[B_m(x,u(T))\mu_w^i(T) - B_m(x,u_0)\mu_w^i(0)dx \\
&+\mu\int_{\{x\in\Omega;} T_k(u)-\mu_w^i \geq 0\} \int_0^T B_m(x,u)(T_k(u)-w_\mu^i)dxdt + \varepsilon(n),
\end{aligned}
$$

where $B_m(x,r) = \int_0^r \frac{\partial b(x,s)}{\partial s} h_m(s)\ \exp\ (G(s))ds.$ Therefore , passing to the limit , in i and μ, in the first terms on the right - hand side of the last equality , we deduce that

$$
\{x\in\Omega; T_k(u)-\mu_w^i \geq 0\}^{=\int_\Omega [B_m(x,u(T))(T_k(u(T))-)\mu_w^i(T) - B_m(x,u_0)w \ \ i_\mu\ (0)dx}_{\int \quad [B_m(x,u(T) \qquad B_m(x,\ u\ 0)T \quad k(u_0))dx+ \quad \varepsilon(n,\mu,} i). \tag{7.6}
$$

The second term on the right - hand side of (7 . 5) can be rewritten as

$$
\begin{aligned}
&\mu\int_0^T \int_{\{x\in\Omega;} T_k(u)-\mu_w^i \geq 0\} B_m(x,u)(T_k(u)-w_\mu^i)dxdt \\
&= \mu\int_0^T \int_{\{x\in\Omega;} T_k(u)-\mu_w^i \geq 0\}^{(B_m(x,}u) - B_m(x,T_k(u)))(T_k(u)-w_\mu^i)dxdt \\
&+\mu\int_0^T \int_{\{x\in\Omega;} T_k(u)-\mu_w^i \geq 0\}^{(B_m(x,}T_k(u)) - B_m(x,w_\mu^i)(T_k(u)-w_\mu^i)dxdt \qquad (7.7)\\
&\qquad +\mu\int_0^T \int_{\{x\in\Omega;} T_k(u)-\mu_w^i \geq 0\} B_m(x,w_\mu^i)(T_k(u)-\mu_w^i)dxdt \\
&\qquad = J_1+J_2+J_3,
\end{aligned}
$$

 where

$$J_1 = \mu \int_0^T \int_{\{x\in\Omega;} T_k(u) - \mu_w^i \geq 0; u > k\}^{(B_m(x,} u) - B_m(x,k))(k - w_\mu^i) dxdt$$
$$+\mu \int_0^T \int_{\{x\in\Omega;} T_k(u) - \mu_w^i \geq 0; u < -k\}^{(B_m(x,} u) - B_m(x,-k))(-k - w_\mu^i) dxdt$$
$$\geq 0.$$

(7 . 8) As $B_m(x, z)$ is non - decreasing for z and $-k \leq \mu_w^i \leq k$, it follows that

$$J_2 \geq 0. \tag{7.9}$$

Moreover ,

$$J_3 = \mu \int_0^T \int_{\{x\in\Omega;} T_k(u) - \mu_w^i \geq 0\} B_m(x, w_\mu^i)(T_k(u) - \mu_w^i) dxdt$$
$$== \overline{} B(x, \int_{\{x\in\Omega;\ T_k(u)(T)-\mu_w^i(T)\geq 0\}}^{\int_0^T \int_{\{x\in\Omega;\ T_k(u)-\mu_w^i\geq 0\}} B_m(x,} w_\mu^i) \frac{\partial (w)_\mu^i}{\partial t} dxdt \; w^i{}_{\mu(T))dx} \tag{7.10}$$
$$- \int_{\{x\in\Omega;} T_k(u)(0) - \mu_w^i(0) \geq 0\} \text{-}B_{((x,}\mu_w^i(0))dx,$$

where $\text{-}B_{(x,}z) = \int_0^z B_m(x,r)dr$. Also $\mu_w^i \to T_k(u)$ a . e . in Q as i and μ tends to $+\infty$ and $| \mu_w^i | \leq k$. Then Lebegue ' s convergence theorem shows that

$$J_3 = \int_\Omega (\overline{} B(x, T_k(u(T))) - \text{-}B_{(x,} T_k(u_0)))dx + \varepsilon(n, \mu, i). \tag{7.11}$$

In view of (7 . 6) - (7 . 1 1) , one has

$$I_2^{n,\mu} \geq -+\overline{} \int_\Omega^{\Omega[B_m(x,} {}^{T_k(u(T)))-B(x,T_k(u_0)))dx+} (\overline{} B(x, u(T)) T_k(u(T)) - B_m(x,u_0) T_k {}^{(u_0)]dx} \varepsilon(n,\ \mu, i). \tag{7.12}$$

As a consequence of (7 . 1) , (7 . 4) and (7 . 1 2) , we deduce that

$$\int_{\{(x,t)\in\Omega\times(0,T);} T_k(u) - \mu_w^i \geq 0\} \frac{\partial b_n(x,u_n)}{\partial t} h_m(u_n) \exp(G(u_n))(T_k(u_n) - w_\mu^i) dxdt \geq$$
$$\geq \int_\Omega [B_{m,k}(x, u(T)) - B_{m,k}(x, u_0)]dx$$
$$- \int_\Omega [B_m(x, u(T)) T_k(u(T)) - B_m(x, u_0) T_k(u_0)]dx$$
$$+ \int_\Omega (\overline{} B(x, T_k(u(T))) - \text{-}B_{(x,} T_k(u_0)))dx + \varepsilon(n, \mu, i).$$

(7 . 1 3) Observe that for any $z \in \mathbb{R}$ and for almost every $x \in \Omega$, we have

$$\text{-}B_{(x,} T_k(z)) = B_m(x, z) T_k(z) - B_{m,k}(x, z).$$

 Indeed ,

$$
\begin{aligned}
\text{-}B_{(x,}T_k(z)) &= \int_0^{T_k(z)} B_m(x,r)dr \\
&= [r\int_0^r \frac{\partial b(x,\sigma)}{\partial\sigma}h_m(\sigma)\exp(G(\sigma))d\sigma]T_0^k(z) \\
&\quad - \int_0^{T_k(z)} r\frac{\partial b(x,r)}{\partial r}h_m(r)\exp(G(r))dr \\
&= T_k(z)\int_0^{T_k(z)} \frac{\partial b(x,r)}{\partial r}h_m(r)\exp(G(r))dr \\
&\quad - \int_0^{T_k(z)} T_k(r)\frac{\partial b(x,r)}{\partial r}h_m(r)\exp(G(r))dr \\
&= T_k(z)B_m(x,T_k(z)) - B_{m,k}(x,T_k(z)).
\end{aligned} \tag{7.14}
$$

This is due to the fact that for $\mid r \mid< k$, we have

$$\text{-}B_{(x,}T_k(r)) = T_k(r)B_m(x,r) - B_{m,k}(x,r),$$

and if $r > k$ we have

$$
\begin{aligned}
&B_{m,k}(x,r) \\
&= \int_0^k \frac{\partial b(x,\sigma)}{\partial\sigma}h_m(\sigma)\sigma\exp(G(\sigma))d\sigma + k\int_k^r \frac{\partial b(x,\sigma)}{\partial\sigma}h_m(\sigma)\exp(G(\sigma))d\sigma, \\
&-T_k(r)B_m(x,r) \\
&= -k\int_0^k \frac{\partial b(x,\sigma)}{\partial\sigma}h_m(\sigma)\exp(G(\sigma))d\sigma - k\int_k^r \frac{\partial b(x,\sigma)}{\partial\sigma}h_m(\sigma)\exp(G(\sigma))d\sigma,
\end{aligned}
$$

and

$$\text{-}B_{(x,}k) = k\int_0^k \frac{\partial b(x,\sigma)}{\partial\sigma}h_m(\sigma)\exp(G(\sigma))d\sigma - k\int_0^k \frac{\partial b(x,\sigma)}{\partial\sigma}h_m(\sigma)\exp(G(\sigma))\sigma d\sigma.$$

The case $r < -k$ is similar to the previous one . This conclude the proof . □

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