

A NEW INTEGRAL UNIVALENT OPERATOR

DANIEL Breaz¹, SHIGEYOSHI OWA AND NICOLETA Breaz²

ABSTRACT . In this paper the authors introduced an integral operator and proved its properties .

2000 Mathematics Subject Classification : 30C45

Keywords and phrases : analytic functions , univalent functions , integral operator , convex functions .

1 . INTRODUCTION

Let $U = \{z \in \mathbb{C}, |z| < 1\}$ be the unit disc of the complex plane and denote by $H(U)$, the class of the holomorphic functions in U .

Consider $A =$

$\{f \in H(U), f(z) = z + a_2z^2 + a_3z^3 + \dots, z \in U\}$ be the class of analytic functions in U and $S = \{f \in A : f \text{ is univalent in } U\}$.

Consider S^* , the class of starlike functions in unit disk , defined by

$$S^* = \{f \in H(U) : f(0) = f'(0) - 1 = 0, \operatorname{Re} \frac{zf'(z)}{f(z)} > 0, z \in U\}.$$

A function $f \in S$ is the starlike function of order $\alpha, 0 \leq \alpha < 1$ and denote this class by $S^*(\alpha)$ if f verify the inequality

$$\operatorname{Re} \frac{zf'(z)}{f(z)} > \alpha, z \in U.$$

Denote with K the class of convex functions in U , defined by

$$K = \{f \in H(U) : f(0) = f'(0) - 1 = 0, \operatorname{Re} \left\{ \frac{zf''(z)}{f'(z)} + 1 \right\} > 0, z \in U\}.$$

¹Supported by GAR 19 / 2008

²Supported by GAR 19 / 2008

A function $f \in S$ is convex function of order $\alpha, 0 \leq \alpha < 1$ and denote this class by $K(\alpha)$ if f verify the inequality

$$\operatorname{Re}\left\{\frac{zf''(z)}{f'(z)} + 1\right\} > \alpha, z \in U.$$

A function $f \in UCV$ if and only if

$$\operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} \geq \left|\frac{zf''(z)}{f'(z)}\right|, z \in U.$$

S . N . Kudryashov in 1973 investigated the maximum value of M such that the inequality

$$\left|\frac{f''(z)}{f'(z)}\right| \leq M$$

implies that f is univalent in U . He showed that if $M = 3, 5, \dots$ and

$$\left|\frac{f''(z)}{f'(z)}\right| \leq 3, 5, \dots$$

where M is the solution of the equation $8[M(M-2)^3]^{1/2} - 3(3-M)^2 = 12$, then f is univalent in U .

Also, if

$$\left|\frac{f''(z)}{f'(z)}\right| \leq 2, 8329, \dots,$$

then the function f is starlike in unit disk. This result is obtained by Miller and Mocanu.

Consider the new general integral operator defined by the formula :

$$F_{\alpha_1, \dots, \alpha_n}(z) = \int_0^z (f_1'(t))^{\alpha_1} \dots (f_n'(t))^{\alpha_n} dt. \quad (1)$$

2 . MAIN RESULTS

In the next theorem we proved the univalence for this operator .

Theorem 1 . Let $\alpha_i \in \mathbf{R}, i \in \{1, \dots, n\}$ and $\alpha_i > 0$. Consider $f_i \in S$ (univalent functions) and suppose that

$$\left| \frac{f_i''(z)}{f_i'(z)} \right| \leq M, \quad (2)$$

where $M = 3, 5, \dots$, for all $z \in U$.

If $\sum_{i=1}^n \alpha_i \leq 1$ the integral operator $F_{\alpha_1, \dots, \alpha_n}$ is univalent .

$i = 1$

Proof . We have

$$\frac{F''_{\alpha_1, \dots, \alpha_n}(z)}{F'_{\alpha_1, \dots, \alpha_n}(z)} = \alpha_1 \frac{f_1''(z)}{f_1'(z)} + \dots + \alpha_n \frac{f_n''(z)}{f_n'(z)}. \quad (3)$$

that is equivalent with ,

$$\left| \frac{F''_{\alpha_1, \dots, \alpha_n}(z)}{F'_{\alpha_1, \dots, \alpha_n}(z)} \right| \leq \alpha_1 \left| \frac{f_1''(z)}{f_1'(z)} \right| + \dots + \alpha_n \left| \frac{f_n''(z)}{f_n'(z)} \right|.$$

Now , applying the inequality (2) , we obtain that

$$\left| \frac{F''_{\alpha_1, \dots, \alpha_n}(z)}{F'_{\alpha_1, \dots, \alpha_n}(z)} \right| \leq \alpha_1 M + \dots + \alpha_n M = M \sum_{i=1}^n \alpha_i \leq M.$$

The last inequality implies that the integral operator $F_{\alpha_1, \dots, \alpha_n}$ is univalent .
Theorem 2 . Let $\alpha_i \in \mathbf{R}$, $i \in \{1, \dots, n\}$ and $\alpha_i > 0$.
 Consider $f_i \in S$
 (univalent functions) and suppose that

$$\left| \frac{f_i''(z)}{f_i'(z)} \right| \leq M_1, \quad (4)$$

where $M_1 = 2, 8329, \dots$, is the smallest root of equation $x \sin x + \cos x = 1/e$.

n

If $\sum_{i=1}^n \alpha_i \leq 1$, the integral operator $F_{\alpha_1, \dots, \alpha_n}$ is starlike .

$i = 1$

Proof . We have

$$\frac{F''_{\alpha_1, \dots, \alpha_n}(z)}{F'_{\alpha_1, \dots, \alpha_n}(z)} = \alpha_1 \frac{f_1''(z)}{f_1'(z)} + \dots + \alpha_n \frac{f_n''(z)}{f_n'(z)}. \quad (5)$$

that is equivalent with ,

$$\left| \frac{F''_{\alpha_1, \dots, \alpha_n}(z)}{F'_{\alpha_1, \dots, \alpha_n}(z)} \right| \leq \alpha_1 \left| \frac{f''_1(z)}{f'_1(z)} \right| + \dots + \alpha_n \left| \frac{f''_n(z)}{f'_n(z)} \right|.$$

Now , applying the inequality (4) , we obtain that

$$\left| \frac{F''_{\alpha_1, \dots, \alpha_n}(z)}{F'_{\alpha_1, \dots, \alpha_n}(z)} \right| \leq \alpha_1 M_1 + \dots + \alpha_n M_1 = M_1 \sum_{n=1}^i \alpha_i \leq M_1.$$

The last inequality implies that the integral operator $F_{\alpha_1, \dots, \alpha_n}$ is starlike .
Theorem 3 . Let $\alpha_i \in \mathbf{R}$, $i \in \{1, \dots, n\}$ and $\alpha_i > 0$.
 Consider $f_i \in K$
 (convex functions) , for all $i \in \{1, \dots, n\}$. Then the integral operator $F_{\alpha_1, \dots, \alpha_n}$
 is convex .

Proof .

$$\operatorname{Re}\left\{ \frac{z F''_{\alpha_1, \dots, \alpha_n}(z)}{F'_{\alpha_1, \dots, \alpha_n}(z)} + 1 \right\} = \sum_{n=1}^i \alpha_i \operatorname{Re}\left(\frac{z f''_i(z)}{f'_i(z)} + 1 \right) \geq 0.$$

Since

$$\operatorname{Re}\left(\frac{z f''_i(z)}{f'_i(z)} + 1 \right) \geq 0$$

for all $i \in \{1, \dots, n\}$, we have the integral operator $F_{\alpha_1, \dots, \alpha_n}$ is convex .

Theorem 4 . Let $\alpha_i \in \mathbf{R}$, $i \in \{1, \dots, n\}$ and $\alpha_i > 0$. Consider
 $f_i \in K(\beta_i)$,
 $0 \leq \beta_i < 1$, for all $i \in \{1, \dots, n\}$. In these conditions , the integral
 operator

$$F_{\alpha_1, \dots, \alpha_n}$$

is convex of order $\sum \alpha_i(\beta_i - 1) + 1$, where $0 \leq \sum \alpha_i(\beta_i - 1) + 1 < 1$.

$$i = 1 \quad i = 1$$

Proof . We have

$$\operatorname{Re}\left\{ \frac{z F''_{\alpha_1, \dots, \alpha_n}(z)}{F'_{\alpha_1, \dots, \alpha_n}(z)} + 1 \right\} = \sum_{n=1}^i \alpha_i \operatorname{Re}\left(\frac{z f''_i(z)}{f'_i(z)} + 1 \right). \quad (6)$$

Since $f_i \in K(\beta_i)$, we have

$$\operatorname{Re}\left(\frac{z f''_i(z)}{f'_i(z)} + 1 \right) \geq \beta_i,$$

for all $i \in \{1, \dots, n\}$.

We apply this property in (6) and we obtain that :

$$\operatorname{Re}\left\{\frac{zF''_{\alpha_1, \dots, \alpha_n}(z)}{F'_{\alpha_1, \dots, \alpha_n}(z)} + 1\right\} = \sum_n^{i=1} \alpha_i \beta i + 1 - \sum_n^{i=1} \alpha_i = \sum_n^{i=1} \alpha_i (\beta i - 1) + 1. \quad (7)$$

The relation (7) implies that the integral operator $F_{\alpha_1, \dots, \alpha_n}$ is convex of

$$\text{order} \sum_{i=1}^n \alpha_i (\beta i - 1) + 1.$$

Theorem 5 . Let $\alpha_i \in \mathbf{R}$, $i \in \{1, \dots, n\}$ and $\alpha_i > 0$. Consider $f_i \in UCV$, for all $i \in \{1, \dots, n\}$. In these conditions, the integral operator $F_{\alpha_1, \dots, \alpha_n}$ is

convex of order $1 - \sum_{i=1}^n \alpha_i$, where $1 - \sum_{i=1}^n \alpha_i \geq 0$.

Proof . We have

$$\operatorname{Re}\left\{\frac{zF''_{\alpha_1, \dots, \alpha_n}(z)}{F'_{\alpha_1, \dots, \alpha_n}(z)} + 1\right\} = \sum_n^{i=1} \alpha_i \operatorname{Re}\left(\frac{zf''_i(z)}{f'_i(z)} + 1\right). \quad (8)$$

Because $f_i \in UCV$, we have

$$\operatorname{Re}\left\{1 + \frac{zf''_i(z)}{f'_i(z)}\right\} \geq \left|\frac{zf''_i(z)}{f'_i(z)}\right|, z \in U.$$

We apply this inequality in relation (8) and obtain that :

$$\operatorname{Re}\left\{\frac{zF''_{\alpha_1, \dots, \alpha_n}(z)}{F'_{\alpha_1, \dots, \alpha_n}(z)} + 1\right\} \geq \sum_n^{i=1} \alpha_i \left|\frac{zf''_i(z)}{f'_i(z)}\right| + 1 - \sum_n^{i=1} \alpha_i \geq 1 - \sum_n^{i=1} \alpha_i. \quad (9)$$

The relation (9) implies that the integral operator $F_{\alpha_1, \dots, \alpha_n}$ is convex of

$$\text{order} 1 - \sum_{i=1}^n \alpha_i.$$

REFERENCES

[1] S . S . Miller and P . T . Mocanu , *Differential Subordinations . Theory and Applications* , Marcel Dekker , INC . , New York , Basel , 2000 .

[2] F . Ronning , *A survey on uniformly convex and uniformly starlike functions* , Annales Universitatis Mariae Curie - Sklodowska , vol . XLVII , 1993 , 123 - 134 , Lublin - Polonia .

Authors :

Daniel Breaz Department of Mathematics
" 1 Decembrie 1918 " University
Alba Iulia Romania e - mail : *dbreaz @ uab . ro*
Shigeyoshi Owa
Department of Mathematics
Kinki University
Higashi - Osaka , Osaka 577 - 8502
Japan
e - mail : *owa @ math . kindai . ac . jp*
Nicoleta Breaz Department of Mathematics
" 1 Decembrie 1918 " University
Alba Iulia
Romania e - mail : *nbreaz @ uab . ro*