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AN UNBOUNDED BERGE'S MINIMUM THEOREM WITH APPLICATIONS TO DISCOUNTED MARKOV DECISION PROCESSES

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This paper deals with a certain class of unbound e − d o ptimizat o − i n problems T h eoptimizatio problems taken into account depend on a pa r − a me te r Firstly ther a r establishe condition which permit to guarantee the con t i − n u t − iy wi t − h r e p − s e c t − o th p a a − r mete o th eminim u − m o fth optimization problems under considera t i − o n , and h − te upp e semicont i − nui t o th multifunctio which applies each parameter into i − ts s t − e of minmizers Besi de s w i t − h th eadd itiona lconditio no uniqueness of the minimizer , i − ts con tnu t − iy i − s igv − i e n Some e − x amp le o nonconve xoptimizatio problems that satisfy the conditi ons of the a rtic l − e a e − r s − u p plie d Se o − c n f dly th theor ydevelope is applied to discounted Markov decis i − o n proc esse wi h − t u n − bu − o n d e − d cos function an wit possibly noncompact actions sets in order to o bta i − n c ont n − iu − o u o pt m − i a policie s T shi par o the paper is i llustrated with two examp l − e s of h − te co ntroll e − d sLn − i dle y ' rand o − m walk On o these examples has nonconstant ac t i − o n se t s

Keywords : Berge ' s minimum theorem , moment f u − n ct o − i n disc o − u nt d − e Ma rko vdecisio npro cess , uniqueness of the optimal po l i − c y c ont n − iu − o u o pt m − i a polic

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1. INTRODUCTION

Let X and A be nonempty Borel spaces . F o − r each $x \in X$ le $\gamma(x$ b a n onempt ysubse of A. Let $Gr(\gamma) := \{(x,a) : x \in X, a \in \gamma(x)\}$ and le G $Gr(\gamma \to \mathbb{R}$ be a nonnegativ function . Consider the following minim i − z a t o − i n p r − o b e − l m

$$\inf_{\in_a \gamma(x)} G(xa) \quad x \in X \qquad (1$$

Let f^* $: X \to A$, such that $f^*(x) \in \gamma(x) x \in X$ be a minimize o (1 assuming of course , that such a minim i − z er ex s − it − s, and le G^* be h − te corresp rdin opt ima valu function , i . e .

$$G^*(x) = G(x, f^*(x) = \in_a^{\mathrm{i-n}} {}^{\mathrm{f}}_{\gamma} (x G(x,a) \quad x \in X \quad (2$$

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The first part of this paper i − s concerned wi h − t es t − a blish n − i g a v ersi o − n o th Berg e ' Minimum Theorem (see [5] p . 1 1 6) wh i − c h permit o − t o bta i − n th cont n − i ui t − y o G^* , an dth

upper semicontinuity of the mu lt i − f_{unc} t o − i n $x \to \gamma^*(x \stackrel{:}{=} \{a \in \gamma(x \quad G(xa \quad = G*(x)\}$

The main condition that has been impo s − e d n − i thi pa r o th articl i tha G ha to be a moment function i . e . , that G g r − o ws wi h − t ou a bo u − n d n − o th complemen o compact sets . Additionally , for the con t i − n u t − i_y of f^* it u n i − q uenes i r q − e uire d

These conditions permit to deal w i − t h unbound d − e p r − o blm s i . e w i h − t minimizatio problems with possibly unbounded func t i − o n G and p ossib y − l n oncompac restriction sets $\gamma(x), x \in X$, and they also work for minm i − z a t o − i n p r − o blms fo w hi h − cG and / o rth

restrictions sets $\gamma(x), x \in X$, are nonconvex i (s − e e Se ct o − i n 3 e be l − o w)

It is important to mention that the moment fun ct i − o n ha v be n − e u s e − d i differen classes of sto chastic control problems (s − e e [1 0 20 2 1 2 4 and three − bracketleft e 0])

The first antecedent in the study of the con t i − n ui y − t of G^* a n − d th up pe rsemicontinuit of $x \to \gamma^*(x)$ requiring the compactne s − s of h − t_e restrict i − o n set $\gamma(x) x \in X$ i know a the Minimum Theorem (and related resu l − t s due o − t B e r − g e (se [5 p p . 1 15 − 1 17) (I nfac t Berge in [5] works with maximiza t i − o n pro b e − l ms and i − n hi b o − o k he n aturall referre dt the result related to the continu i − t_y of G^* and h − t_e upper semicont i − n ui y − t o $x \to \gamma \quad *(x$ a the Maximum Theorem) .

In Lemmas 6 . 1 1 . 8 and 6 . 1 1 . 9 of [27 the con t i − n ui y − t of G^* a n − df^* ha be n − e analyze

under the assumption that $\gamma(x) = A$ for a l $x \in X$ p r − o vid d − e t − h a A i a compac se t

Also in [1 1] a result concerning the con t i − n ui y − t of G^* and f^* i presented bu th convexity of G is assumed . Neverthe l − es − s, i i − s impo r a − t n t − o o − c nside th nonconve cas as well , regarding not only the econom i − c al app l c − i at i − o n t [1 5 17 b u a ls it importanc

when the action set is finite or d i − s connec e − t d and h e − n c e n on o − c n v e − x [18]

The major bulk of the research on Be r − g e s − quoteright The orm as s − u m e b u − o n dednes o th reward (or cost) function and compactne s − s of h − $t_e \gamma(x) x \in X$ s_{See}, [4], [8], [9], [14], [16] [25] , [32] , [33] , [34] . The importance of the unbounde d n on c − o mp ac cas i apparen i such a work as [1 5] or [1 7] in econom i − c s , and one − bracketleft0] wh e r − e larg bibliograph relate t

Markov decision processes (MDPs) can be found on thi s − u b jec t Henc e presentin a version of the Berge ' s Theorem for the unbounded n o − no − c mp ac cas c − a n n − i ou ropinion considerably extend its usefulness .

On the other hand , correspondences or mu lt i − f_{un} ct o − i n a r b asi tool i − n contempo rary economic mathematical mode llng i − n such p o − r b l − e ms a o − c n s − u m e theor o game theoretical modelling of economic in i_{t-erac} t o − i n s wh e r − e th y − e n atural l − y appea fr m − o th st art as basic building blocks of the cor r − espon' d n − i g ma t − h_{ema} t ica model Berg e ' Max imum Theorem provides extremely va l − u a b e − l i − n foma t o − i n a b − o u l th continuit o th optimal actions of the agents involved , f o − rn − i sanc e un de a d e − q u at c n − o dition s y_i guar antees the continuity of the ind i − r ect u tili y − t func t i − o n n − i c − o n u − s m e theory a − n d th uppe semi - continuity of the best responses co r − rs − e pond e − n c e n − i game r theory Furthe infor mation on the application of Berge quoteright − s Theo e − r m t − o Dy a − n mi Prog^{r-a} m min a d − nr_t th problems mentioned above can be found i − n two − bracketleft5 and bracketleft − three1]

The second part of the paper dea l − s w t − i h t − h_e a ppl c − i a t i − o n of th result o th firs part to MDPs on Borel spaces , w i − t h (po ssibly unboun d d − e cos f − u nctio n wit (possi

bly) noncompact action sets , and w i − t h the expe c e − t d tota disc o − u nt d − e cos a ob jectiv function (see [1 0]) .

For this type of MDPs the ex i − s tence of sta t o − i na r − y o pt m − i a p olicie a minimizer o the Optimality Equation (OE) i − s assumed (e − s e [1 0]) And fo su c − h a s_{kn-i} d o discounte Markov decision process , denote by f^* i − t_s op tmal po licy a d − n by V^* it optima valu function [1 0] .

In this part , the function G gven in (1) i − s h − t_e r i − g ht h − hyphena − n d sid o th OE an th main conditions which ensure the con t i − n ui y − t of f^* and V^* ar th u n i − q uenes o f d (se [6] for conditions for the uniqueness of op tmal po l icie of edisco − u nt d − e MD P s an th fact that the cost function c is a moment func t i − o n

The theory presented in this part of the a rtic l − e i app l e − i d t − o th eve y − r importan tmodel proposed by Lindley [22] that in the paper wil be r − e ferr d − e t − o a L n − i dle y ' srand o − m wal k useful in queueing and dam management t − h_{eo} r e − i s f o th c − o ntroll d − e cas eo fth eLindley ' random walk see , for inst ance , [35]

When dealing with MDPs in econom i − c app l c − i a t i − o n s th cont n − i ui t − y o th optima policy f^* greatly simplifies or clarifies the ana l − y s s − i of th corre s − p n − o din stabil[e] it o th model [1 6 , 1 7 , 23] . However , th i − ss − i not alwa y s tress $e - d$ e n − iam or *theoretica* researc h becoming then an important research area . T s_{hi-s} paper u − s ggest a lin o researc o the uniqueness of the optimal policy based on t − h_e w o k − r start e − d n − i [6]

It is interesting to note that research on d i − so − c un t − e d MD P s n − i a n o − n− Berg e ' sTheor m − e approach , usually analyzes contin u i − t_y of the v a u − l e fu nct i − o n se fo x − e ampl [12]

The continuity of the optimal po l i − c y f^*s − i e s t − a blish d − e i − n [13 fo linea m odel s wit finite horizon , constant mul ti func t i − o n $x \to \gamma($ $x)$ und e c n − o vexi t − y a s u − s mption s Th nonlinear case , with infinite hor i − z on , a pos sib y − l no n − c ons a − t n m ulti f − u nctio $x \to$ $\gamma(x$ and nonconvexity restrictions i − s, of cour e − s of t − h_e g e − r a interes t p recise l − y motivate b [1 3] , and constitutes an important por t i − o n of thi pape r

The paper is organized as fo l − $l_{ows.}$ Sec t o − i n 2 p resent th minimizatio problem Section 3 gives the version of the Minmum The o r − e m and o − s m n onc o − n ve examples Section 4 applies Section 3 to d i − s coun e − t d MDP s and Se ct i − o n 5 present tw oexample st i llustrate the theory developed in the prev i − o us e − s ct i − o n T − period he fi na (Conclusions Sectio is followed by two appendixes which cont a i − n h − t_e comp let deta[l] il o th proof o th examples in the article .

2 . PRELIMINARIES

For short , throughout the paper , u period − sc − period wil be u s − e d f o u ppe semicont n − i uou s an l . s . c for lower semicontinuous .

Let X and A be nonempty Borel spac e − s (i . e me asu a − r bl s − u bset o complet an separable metric spaces) .

Now , some basics on multifunc t i − o ns are supp l e − i d i − n thi sect o − i n fo mor informatio see [1] .

A *multifunction* γ from X to A i − s a func t o − i n f r − o m X o − tA w hos v alu $\gamma(x)$ fo reac $x \in X$, is a nonempty subset of A.

The *graph* of the multifunc t i − o n γs − i a subs e of $X \times A$ d e − fi n d − e as $Gr(\gamma \stackrel{:}{=} \{(x, a$

$$x \in X, a \in \gamma(x)\}.$$

Definition 2 . 1 . A multifunction γ from X to A i sa i − d o − t be

 (a) *Borel - measurable* if $\{x \in X : \gamma(x) \cap O \neq \varnothing\}$ i a B ore s − u bse o X fo ever ope

set$O \subset A$;

(b) *upper s emicontinuous* if $\{x \in X : \gamma(x) \cap F \neq \varnothing\}$ i clos e − d n − iX fo ever close

$F \subset A$;

(c) *lower s emicontinuous* if $\{x \in X : \gamma(x) \cap O \neq \varnothing\}$ i o p e − n fo eve y − r op^{e-n} se $O \subset$ A

(d) *continuous* if it is both u period − s. c . and l s − periodperiod − c.

The terms correspondence inst (ead of mu l t i − f$_{unc}$ t o − i n a n − d hem icont n − i uou (instea do semicontinuous) are more conven i − e nt n − i gen e − r a l but n − i o rde t st a − y clos t Berg e ' original terminology (see [5]) , they wil not be ado p t − e d n − i thi p ape r

Remark 2 . 1 . It is well - known that Defi nit i − o n 2one − period(b) i − n th cas o a compact - value γ, is equivalent to : if $x_n \to x$ in X and $a_n \in \gamma$parenleft − x$_n$) th n − e e ther exis^f t a subsequenc $\{a_{n(k)}\}$ of $\{a_n\}$ and $a \in \gamma(x)$, such that $a_n(k \to a$ (se T heor e − m 17.2 p . 56 i [1])

Besides , Definition 2 . 1 (c) is also equ i − v a e − l nt t − o f − i$x_n \to x$ i − nX th n − e fo eac $a \in$ $\gamma(x$ there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ and $a_k \in \gamma$parenleft − x $n(k)$ fo ea h − ck su c − h tha t $a \to a$

(see Theorem 1 7 . 2 1 p . 565 in [1bracketright − parenright

Now , the minimization problem wil be e s a bli s − h$_e$ d

Throughout the remainder of Sec t i − o ns 2 and 3 le X and A be fi x d − e nonempt yBore spaces and γ denotes a given Borel - measurab l − e mu lt^t i u − f n ct i − o n f o − r m X t A $\mathbb{F}$ denote the set of measurable functions $f : X \to A$ such t − h$_@$$f(x \in \gamma(x$ fo al $x \in X$ $f -$ *parenleft* $\in \mathbb{F}$ is called a *s e lector* for the mul t i − f$_{unc}$ t o − i n γ) F ur t − h emo r e G $)_G r(\gamma \to \mathbb{R}$ i a give nonnegative (or bounded below) and measurab l − e func r$_{to-i}$ n a n − d

$$G^*(x) :=\in_a^{i-n} {}_\gamma^f(xGx - parenlefta) \tag{3}$$

$x \in X$. If $G(x, \cdot)$ attains its minimum at some p oin i − n$\gamma(x)$ ther w il b writte " min instead of inf " " in (3) . **Remark 2 . 2 .** In Rieder [28] i − ts − i proved that i − f$Grgamma - parenleft)$ i a Bo re subse o $X \times$ AG i lower semicontinuous , bounded be l − o w , and n − i f hyphen − c ompa c on $Gr(\gamma$ (i . e t$_{fo}$ ever $x \in X$ and $r \in \mathbb{R}$, the set $\{a \in \gamma(x)$: $G(x, a) \leq r\}$ i − s compa c t) th n − e) ther exist a selecto $f^* \in \mathbb{F}$ such that for each $x \in X, G(x, \cdot$ a ttain it min i − m um n − i$f^*(x)$ **Lemma 2 . 1 .** If γ is closed - valued (i. e . $\gamma(x)$ i − s c lo e − s d fo ea c − h$x \in X$ n − a d u . s . c . the $Gr(\gamma)$ is closed in $X \times A$.

P r o o f . This is a consequence of Propo sit i − o n 7 p 110 [3] ☐

Lemma 2 . 2 . If γ is l . s . c . and G i − s u s − periodperiod − c, then G^* i u . s . c

P r o o f . See the proof of Lemma 17.29 i − n [1] ☐

3 . THE MINIMUM THEOREM

The Moment Condition (MC) . The r − es − i a s e − q u n − e c $\{\mathbb{K}_n\}$ o compac set suc

$$\text{that}\mathbb{K}_n \uparrow Gr(\gamma), \text{and}$$

$$\lim_{n\to\infty}((x\text{n}-\text{i}^{\text{f}}_{comma-a)element-negationslash\mathbb{K}_n}G(xa) = +\infty \quad (4$$

Remark 3 . 1 . (a) In case $x \to \gamma(x)$ i − s con s an t h − t$_{@}$ i $\gamma(x =$ A fo al $x \in X$ wit A as a compact set , and X wh i − c h s − iσhyphen − c ompa c s (i . e) ther exis a increasin sequence of compact sets $\{W_n\}$ such that $W_n \uparrow X)G$ i e triviall a momen t because , if $\mathbb{K}_n = W_n \times A$, $n = 12,$. t − hn − e$\mathbb{K}_n \uparrow Gr,(\gamma$ a n − d (4 hold becaus $Gr(\gamma) \setminus \mathbb{K}_n$ is empty , for each n, and the minmum o ve th em p t − y se i equa t

$$+\infty.$$

(b) A nonnegative measurable func t i − o n H on a B ore p − s ac Y i sa i − d t b a *mom en*

on Y(see , [1 0 , 20 , 2 1] [24] and [30] i − f h − t$_{\text{e}}$r − e i an increas n − i g s e − q uenc o compac sets $Y_n \uparrow Y$ such that

$$\text{li}_{\to_n}\text{m}_\infty(y\text{i}-\text{nf}element-negationslash Y_n H(y) = +\infty$$

Thus , the MC states that G i − s a moment on $Grgamma-parenleft)$

In many important optimization prob l − e ms co erciv n − e es of th objectiv functio i observed . It is remarkable that th i − s simp e − l and u s − e fu c oncep ei sometime ignored being so natural in the context of the prob l − e ms s u − t d i − ed $\frac{\text{l}}{\text{n}-\text{i}}$. *thi* p ape r Se [26s$_{\text{fo}}$ mor information .

Define , for each $x \in X$, $\gamma^*(x) := \{a \in \gamma($ x $G(xa) \leq G*(x) = \{\in \gamma(x$ $G(x,a) = G^*(x)\}$, and for each $\zeta \subset X$, whe r − eζ : i a no nmp y − t compac se t $\Omega_\zeta :=$ $\{(x,a) \in Gr(\gamma):$ $x \in \zeta, a \in \gamma^*(x)\}$. Obs e − re − v t − h$_{@}$ fo a − ec − h$x \in X$ γ $*(x$ i nonempt and compact if G is lower semicon t i − n uous and n − i f hyphen − c ompa c on $Gr(\gamma$ (se) Remar 2 . 2)

Now the version of the Minimum Theo r − e m wil be pre e − s nte d

Theorem 3 . 1 . Suppose that the mu lt i − f$_{\text{unc}}$ t o − i n γ i c los e − dv − hyphen alu e − d a d − n cont i − n uou s G i continuous and inf - compact on $Gr(\gamma)$, and t − h$_{\text{e}}$ MC h old s T h n − eG^* i continuou an the multifunction $x \to \gamma^*(x)$ i − s u s − period c .

P r o o f . Fix $\zeta \subset X$, where ζ is a nonempty compa c se t Le $(x_k, ak \in$ $\Omega\zeta k =$ $1, 2, ..$ and $(x,a) \in Gr(\gamma)$, such that $(x_k, a_k) \to (x,a)$ No t − e h − t$_{@}x \in$ ζ M oreove r $G(x_k, ak \leq G^*(x_k)$, for all k, and as from Lemma 2period − two G^* i u . s . c . i result h − t a $G(x, a \leq G*(x$ (recall that G is continuous) . Then $a \in \gamma^*(x)$ i . e Ω_ζ i c t$_{\text{los}}$e − d n − is$_G r(\gamma)$ a − n d)$_{\text{b}}$ yL m − e m 2 . 1 , it is also closed in $X \times A$.

Now , suppose that for each $m = 1, 2, .$ t − h$_{\text{e}}$e − r i parenleft − w$_m a_m)$ $\in$ Ω n − a d parenleft − w $m, am \notin$ $\mathbb{K}_m$ (here $\mathbb{K}_m, m = 1, 2, ...,$ are the compact set i − n th MC) Sinc $w_m \in$ ζ fo al $m = 1, 2, ..., \zeta$ is compact , and G^* i − s u . s . c . i f ol o − l ws t − h a

$$G(w_m, a_m) \leq G^*(w_m) \leq \sup_{x\in\zeta} G*(x < +\infty$$

for all m(recall that an u period − s. c . func t o − $\mathrm{i_n}$ attain it ma x i − m um ove a compac se t se Theorem 2 . 4 . 3 p . 44 [1]) .

Consequently ,

$$\limsup_{m \to \infty} G(w_m, a_m) < +\infty \tag{5}$$

Now , as

$$\inf_{(x,a)\notin \mathbb{K}_m} G(x, a) \leq G(w_m a_m)$$

$m = 1, 2, ...,$ then

$$+\infty = \lim_{m\to\infty} ((x \inf_{,a\notin\mathbb{K}_m} G(x a \text{parenright} - \text{parenrightBig} \leq \quad \mathrm{lm}^{\mathrm{u-s}}_{m\to\infty} \mathrm{p}G\text{parenleft} - \mathrm{w}_m a_m)$$

which is a contradiction to (5) . Theref o − r e h − $\mathrm{t_e}$r − e e xist a p ositiv intege η suc tha $\Omega_\zeta \subset \mathbb{K}_\eta$. As Ω_ζ is closed , it fo lows that i i − s compa c t

Since ζ is arbitrary , the conclusion i − s that Ω_ζ i compa c fo ea h − c nonempt compac

$\Omega^{\zeta\prime}_{,\zeta_n\prime}$ Let= $\{\{^{z_n}_{z_n}\}\cup^{\mathrm{be}}\{za\}$ is=a1, compact$_{2,...,\mathrm{and}}$ set$^{\mathrm{and}}_{a}$ $_{*\in}$ sequence(noticethat in for$_{\gamma^*}$ $^{\mathrm{each}}_{(}$ $_z$), $x\in$such $\prime^{\zeta}_{X}$, and$_{\mathrm{i-sa}}$ compact e−ltzbe that^X $\gamma^*{}^{(x}_{\mathrm{f}\ \ \mathrm{o-r}}$)$_{\mathrm{a}}i_{\mathrm{c}}$no−neerta an$_{\mathrm{se}}$ee−lmeand mp$^{\mathrm{t}}$ty ssther exis $a \in \gamma$

$\{(z_n, a_n)\}, (z_{n(k)}, a_{n(k)}) \to (z, a^*), k \to \infty$ Now s − u p − p o s − e h − t a fo $r \geq$ $0G * (zn \leq r$ for all n. Hence , letting $n \to \infty$ in the la s − tn − i equa lity a pply i − n g T $\mathrm{r_{heorm-e}}$ 1.1 p . 1 i [3 1] (specifically the characteriza ton of lm sup f o − r a s q − e uenc given a n − d th 7_{fac} tha $(z, a^*) \in \Omega_{\zeta\prime}$, it results that

$$r \geq \limsup_{n\to\infty} G^*(z_n) = \quad \mathrm{lm\ sup}_{n\to\infty}\, G(z_n a_n) \geq G\text{parenleft} - \mathrm{z}a* = \quad G*(z)$$

i . e . for $r \geq 0$, $\{x \in X : G^*(x) \quad \leq r\}$ i − s c o − ls − e d n − iX On th othe h a − n d obviousl $\{x \in X : G^*(x) \leq r\}$ is empty for $r < 0$(r − e ca l h − $\mathrm{t_@}G$ i n o − n n g − e ative) Consequently G^* is l . s . c . Therefore , as from Lemma 2period − two$_,G^*$ i u . s . c . h − $\mathrm{t_e}$ c − o nt n − i ui t − y o G^* follow s

Besides , as the multifunction $x \to \gamma^*(x)$ i − s compa ct v − hyphen alu e − d a d − n $an(k \to a^*$ i tfollow

from Remark 2 . 1 that this mul t i − $\mathrm{f_{unc}}$ t o − i n i − s u . s . c □

Corollary 3 . 1 . Under the Assump t i − o ns of The o e − r m 3 . 1 i ther exist a uniqu $f^* \in \mathbb{F}$ such that

$$G^*(x) = G(xf \quad *(x)) \tag{6}$$

$x \in X$, then f^* is continuous .

P r o o f . Note that for each $x \in X, \gamma^*(x) = \{f^*(x)\}$ T h e − n fo $W \subset$ A $\{\in X\ \gamma^*(x) \cap W \neq \varnothing\} = \{x \in X : f^*(x) \in W\}$. Now a t − $\mathrm{h_e}$ mu lti u − f ncti o − n$x \to \gamma \quad *(x$ ^{x})isu.s.c. it results that for each closed $F \subset A, \{x \in X \quad \gamma^*(x \cap F \neq \quad \varnothing = \quad braceleft - x \in X \quad f \quad *(x \in F\}$ is closed in X. Consequentl y f^* i − s con t n − i uous : (s − e e T $)_{\mathrm{heorm}}$ 8.3 p .7 [7]) □

The following examples show that w i − t h h − t_e se of a s s − u mp t o − i n a − n d condition i
this article , it is possible to consider minm i − z a t o − i n p r − o blms w i h − t nonconve functio G and / or with nonconvex restriction sets $\gamma(x) x \in X$ as w el l

As it has been mentioned in Sec t i − o n 1, h − t_e p r − o o f relat d − e t th e − x ample i thi
Section and in Section 5 , wi l − l be given in Appen d x − i A and Ap p e − no_{dx-i} B belo w

Let k be a fixed positive integer , $k \geq 2$ Define

$$\varphi(x) := \quad bar - x^{1/3} + 1 \quad + \quad 1 \qquad (7$$

$x \in \mathbb{R}$(notice that $\varphi(x) > 0$, for all $x \in \mathbb{R}$) and

$$P(x, w) := (1/(2(k+1))) w^2 \text{parenleft} - \text{k}_+ 1 + \varphi(x w^2 + \varphi(x w - parenright + \varphi(x) \quad (8$$

$$x \in \mathbb{R} \text{and} w \in \mathbb{R}.$$

Lemma 3 . 1 . For each $x \in \mathbb{R}$, there e x i − ss − t a un i − q ue $h^*(x <$ 0 su h − c t − h a

$$P(x, h^*(x) = \mathrm{m}^{\mathrm{i-n}}_{w \in \mathbb{R}} P x - parenleftw)$$

$$\text{Moreover}, P(x, h^*(x)) < P(x, 0) = \varphi(x) x \in \mathbb{R}$$

Remark 3 . 2 . The polynomial $P(x, \cdot$ (w i − t h r e p − s e c t − o th sec n − o d v ariable define above will have a degree greater or equal than 6 f o $k \geq$ 2 a n − d henc e th minimize ca not , in general , be explicitly given , becau s − e i s − t d er i − v a t v − i e w il be o a $e_{\text{degre}} \geq$ 5 an so its roots will not be determined by ra d i − c a s − l i − n m os of th case s f_{Fo} a clea genera
exposition on this subject , aimed at non hyphen − s pe cialist s e − s e [19]

Example 3 . 1 . Take $X = A = \gamma(x) = \mathbb{R}$ $x \in \mathbb{R}$ and

$$G(x, a) = a^1 slash - three + \varphi(x) \quad (9$$

$$a > 0, x \in \mathbb{R} \text{and}$$

$$G(x, a) = P(x a) \quad (10$$

$$a \leq 0, x \in \mathbb{R}.$$

Lemma 3 . 2 . Example 3 . 1 sa t i − s fies the Assump t o − i n i − n Co rolla y − r 3.1 a n − dG i noncon
vex . **Example 3 . 2 .** Take $X = \mathbb{R}, A = \gamma(x) = (-\infty 0 \quad \cup \quad [1 + \infty) x \in$ $\mathbb{R}$ a n − d

$$G(x, a) = a^1 slash - three + \varphi(x) \quad (11$$

$$a \geq 1, x \in \mathbb{R}, \text{and}$$

$$G(x, a) = P(x a) \quad (12$$

$$a \leq 0, x \in \mathbb{R}.$$

Lemma 3 . 3 . Example 3 . 2 sa t i − s fies the Assump t o − i n i − n C orolla y − r 3.1 a − n d bot G an the restriction set $A = (-\infty, 0] \cup [1, +\infty)$ a r − e nonc n − ov − e x

As mentioned before , the con ti nu i − t_y of h − t_e v a l − u e fun c t o − i n h a bein gs t − u die db ysevera authors and it should be stressed that , i − f h − t_e u n i − q ue nes of th minimize i observed Berge ' s Theorem immediately grants us the con t i − n ui y − t of th o pt i − m a selecto r A her an extension of the unbounded case of Berge quoteright − s The o e − r m i p resented thi continuit yca be straightforwardly estab il shed in important di s − c oun e − t d MDPs as , th slinear− quadrati
one or in the greatly non hyphen − l inear Lin d e − l y quoteright − s random wa lk us n − i g th result o [6]

4 . DISCOUNTED MARKOV DECISION PROCESSES

Decision Model . Let $(X, A, \{A(x) : x \in X \quad \}Q, c$ be th u s − u a discrete - tim Marko decision model (see [1 0]) , where both the st a t − e p − s a c − eX a d − n th contro spac A ar Borel spaces . For each $x \in X, A(x) \subset A$ i − s the me asu a − r bl s − u bse o d − a missibl action at a state x. The set $\mathbb{K} := \{(x, a) : \quad x \in X \quad a \in A(x$parenright − braceright i as u − s m e − d t b a Bore subse
of $X \times A$. Consider the transiti on proba b ili t − y a − l w $Q(B$ s $| x, a)$ w her
$B \in \mathbb{B}$parenleft − X ($\mathbb{B}(X$
denotes the Borel sigma - algebra of X) and $(xa) \in \mathbb{K}$ i a st o c − h asti e kerne o X give $\mathbb{K}($ i . e . $Q(\cdot \mid x, a)$ is a probabil^{i-t} y measure on X fo e ve y − r $(x, a \quad \in \mathbb{K}$ a − n d $Q(B \mid \cdot$ i a measurable function on $\mathbb{K}$ for every $B \in \mathbb{B}(X))$ F i − n ally th cos pe stag c i a nonnegative and measurable func t i − o n on $\mathbb{K}$

Remark 4 . 1 . In many important cases t − h_e r − t an sit i − o n a − l w Q i i − n duc d − e b a syst e − m equation of the form

$$x_{t+1} = F(\quad x_t, a_t, \xi t_) \tag{13}$$

with $t = 0, 1, ...,$ where $\{\xi t\}$ i − s a sequence of n − i d e − p en den a d − n identical l − y distribute (i . i . d .) random elements with values i − n some B ore p − s a c S a n − d w i h − t a c o − m mo densit $\Delta . F$ is a measurable function from $\mathbb{K} \times S$ t − oX and h − t_e t r − a nsiti n − o probabilit $n_{la-w}Q$ i
given by

$$Q(B \mid x, a) = \int I_B(F\text{parenleft} - \text{x}a, s)\Delta(s)\text{d}s \tag{14}$$

$B \in \mathbb{B}(X)$ and $(x, a) \in \mathbb{K}$, where $I[\cdot$ denot e − s the n − i d i − c ato u − f nc ti o − n o th subse $[\cdot]$

Policies . A *control po li cy* π is a (measura bl e po ssib l − y a − r ndo mized rul fo choosin actions , and at each time $t(t = 0, 1, ...$ the con , tro prescrib e − d by π may e depen do nth current state as well as on the h i − s tory of the p e − r v i − o u state a d − n a ction s Th se o al
policies will be denoted by Π. Given the initia sta t − e$x_0 =$ x a y − n p oli c − yπ d fi − e ne s auniqu probability distribution of the state - ac t i − o n proc es $\{(x_t, atbraceright - parenright$ (fo detail s se [10]) Thi distribution will be denoted by P_x^π, w h i l − eE_x^π st ands fo th corresp^{n-o} din expectatio
operator , and the sto chastic process $\{x_t\}$ wil be ca l l − e d M a k − ro − v decisi n − o proces parenleft − M D P) $\mathbb{F}$ denotes the set of measurable func t i − o ns $f \quad X \to A$ su h − c t − h a $f(x \quad \in A(x$ fo al $x \in X$. A policy $\pi \in \Pi$ is *s tationary* i − f the e − r exis t $f \in \mathbb{F}$ su c − h h − t a t un $)_{de}$ π th act^r io $f(x_t)$ is applied at each time t. The class of s tat^{s}i − o na y − r p ol icie i naturall identifie with $\mathbb{F}$.

Optimality Criterion . Given $\pi \in \Pi$ and initia stat $x_0 = x \in X$ le

$$V(\pi, x) = E_x^{\pi}[\sum_{t=0}^{\infty} \alpha^t c(x_t, at) \quad (15$$

be the *total expected dis counted cost* when u s i − n g h − t e po li c − y π giv n − e th initia stat x
The number $\alpha \in (0,1)$ is called the *dis count facto r*

A policy π^* is said to be *dis counted optimal* i − f$Vpi - parenleft^* x) = V^*(x$ fo a l $x \in X$ w her

$$V^*(x) = \quad \text{i} - \text{n}^{\text{f}}_{\pi} Vpi - parenleftx) \tag{16}$$

$x \in X.V^*$ defined in (1 6) is called the *optimal value func t i − o n*

An MDP with the total expected d i − s coun e − t d c os as th o p t i − m a li y − t criterio wil b referred to as *dis counted* MDP (and the p l − u r a of *th* t i tem w *il* be denot d − e b M DPs)

Assumption 4 . 1 .

(a) The one - stage cost c is lower sem i − c on t n − i uou s and inf −c − o mp ac n − o$\mathbb{K}$ (se Remar 2 . 2) .

(b) The transition law Q is strongly con t i − n uo u s i . e

$$w(x,a) := \int u\text{parenleft} - \text{y})Q\text{d} - \text{parenleft}_y \mid xa)$$

is continuous and bounded on $\mathbb{K}$, f o − r ev r − e y me asu r − a bl b u − o n d e − d u − f nctio u o X

(c) There is a policy π' such that $V(\pi', x) < \infty$ f o e a c − h$x \in X$

Lemma 4 . 1 . (Hern $\acute{a}$ ndez - Lerma and La s − sr − er − e one − bracketleft0 The orm 4 . 2 . 3 Le Assumptio 4 . hold . Then the optimal value func t i − o n V^* defined n − i (16 i th)(poin t twise minima n solution of the *Optimality Equation* (OE) i . e f o − r a l $x \in X$

$$V^*(x) = \min_{\in_a A(x)} [c(x, a) + \alpha \int V * \text{parenleft} - \text{y})Q\text{d} - \text{parenleft}_y \mid xa)] \tag{17}$$

and , if u is another solution to the OE , then $u(\cdot \geq V^* \quad (\cdot)$ There is also $f^* \in \mathbb{F}$ such that :

$$V^*(x) = c(x, f^*(x) + \alpha \int V * \text{parenleft} - \text{y})Q\text{d} - \text{parenleft}_y \mid xf * (x)) \tag{18}$$

$x \in X$, and f^* is optimal .

The following assumption wil be va ld f o − r d s − ic − o unt d − e MD P fo w h i c − h Assumptio

$$4.1 \text{holds.Take} \gamma(x) = A(x), x \in X.$$

Assumption 4 . 2 .

(a) γ is closed - valued and continuous ; (b) f^* is unique ;

(c) $c(\cdot,\cdot)$ is a continuous func ti on , $\int V^*(y)Q($ d $ybar-x,a$ i fi nit fo ever $(x,a\in$ $\mathbb{K}$ an $\int V^*(y)Q($ d $y\mid\cdot\cdot)$ is a con ti nuous func t i − o n
(d) c satisfies the MC .

Remark 4 . 2 . (a) In [6] condit i − o ns w h c − i h ens u r − e th u n q − i uenes o f^* i (18 ar provided .

(b) In the next Section two examp l − e s f r − o w h i − c h A s u − s mp t i − o n 4 . a d − n 4. hol ar presented .

Theorem 4 . 1 . Consider a d i − s counted MDP f r − o w h i − c h As u − s mp tion 4. a − n d 4 . hol d Then V^* and f^* are continuous func t i − o n s

P r o o f . Fix a discounted MDP for wh i − c h Assump t o − i n 4 . a n − d 4.2 hol d Le $(X,A,\{A(x$ $x\in X\},Q,c)$ be the Markov decisi on model f o thi MD 1_{P} Le f^* be th optima lpolic whose existence is guaranteed in (18) l − e t V^* be h − $\mathrm{t_e}$ o p t i − m a v alu u − f ncti o − n define i (1 6) , and take $\gamma(x)=A(x),x\in X($ no $_{,\mathrm{ti-ce}}$ that $\mathbb{K}=Grgamma-parenleft))\mathrm{l_D}$e − fi n

$$G(x,a):=c(x,a)+\alpha\int V\quad *\mathrm{parenleft-y)}Q\mathrm{d-parenleft}_y\mid xa)\quad (19$$

$(x,a)\in\mathbb{K}.$ (Observe that the minim i − z a t o − i n prob l − e m i d fi − e n e − d v a − i th OE (17) . Now Assumptions in Corollary 3 . 1 for these G and f^* wil be verifie d Firs$^{\mathrm{e}}$ tly not tha from Assumptions 4 . 2 (a) , 4 . 2 (b) , and 4 2 (c) t − $\mathrm{h_e}$ mu lt$^{\mathrm{l}}$ i u − f ncti o − nγ i clos d − e− value an continuous , G is continuous , and the uniquen es of f^* fol l − o w s Sec n − o s dly observ d tha

$$A_r(x):=\{a\in A(x):G(x,a)\leq r\}\subseteq\{a\in A(x\quad c(xa\quad\leq\quad r\}$$

$x\in X$ and $r\in\mathbb{R}.$ Hence , Assump t i − o ns 4 . 1 (a) and 4two − period(c i − m p y − l t − h a G i inf - compac

on$Gr(\gamma)$.

Thirdly , let $\mathbb{K}_n$, $n=1,2,...$ be the compact s et i − n th MC fo th cos functio c (see Assumption 4 . 2 (d)) . Note that since ci − s nonn g − e a $\mathrm{s_{tive}}$ th e − nV^* a d − n$\int V\quad *(y)Q($ d $y\mid\cdot,\cdot$ are also nonnegative . Hence , since

$$(\inf_{a,_x)\notin\mathbb{K}_n}c(x,a)\quad\leq\quad(x\mathrm{n}-\mathrm{i}^{\mathrm{f}}_{comma-a)\notin\mathbb{K}_n}\mathrm{bracketleft-c}(xa)+\alpha\int V*(yQ-parenright\mathrm{parenleft-d}\quad y\mid x,a)$$

$$=\quad(x\mathrm{n}-\mathrm{i}^{\mathrm{f}}_{comma-a)element-negationslash\mathbb{K}_n}G(xa)\quad(20$$

letting $n\to\infty$ in (20) , it follows that G sa t i − sfi e h − $\mathrm{t_e}$ MC T herefor e V^* a − n d f^* ar continuous functions as a consequence of The o r − e m 3one − period and C orolla r − y 3.1 ☐

5 . EXAMPLES ON THE CONTROLLED LINDLEY quoteright − S RANDOM W L − A K **Example 5 . 1 .** Let $X=A=A(x)=[0,\infty)$ f o − r a l $x\in X$ T h d n − y a m i o th syst e − m is given by

$$x_{t+1}=[\quad x+a_t-\quad\xi t^+\quad(21$$

$t=0,1,....$ Here $z^{+}=\max\{0,z\}$, and $\xi 0$ $\xi 1$ a r i . i . d r − a n d m − o variable takin values in $S=[0,\infty)$ and with a common de nsi t − yΔ e Beside s i i as u − s m e − d tha Δ i a bounded continuous func ti on . Let H be t − h e distr b − i u t i − o n u − f nct t i o − n o ξ w her ξ i a generic element of the sequence $\{\xi\}$(no t − e h − t@H i a c ont i − no − u u f − u nction) T e h cos function is given by :

$$c(x,a)=x+(a-1)^{2} \quad (22$$
$$x,a\in[0,\infty).$$

Lemma 5 . 1 . Assumptions 4 . 1 and 4 . 2 ho l − d f r − o Examp l 5 . 1

With the results developed so far a theor et i − c al but n evertheles interestin situatio can arise :

Example 5 . 2 . Let $X=A=[0,\infty)$, and $A(x)=\quad[x\infty)x\in X$ T h d ynamic o th system is given by

$$x_{t+1}=[\quad x+g(\quad at-\quad \xi t^{+} \qquad (23$$

$t=0,1,....$ Here $\xi 0,\xi 1,...$ are i . i d . random v ar i − a ble a − t k i − n g v alue n − i$S=$ $[0\infty$ and with a common density Δ. Besi· des , i − ts − i a s s − u med h − t a Δ i a b o − u n s d d − e continuou function . The cost function is given by

$$c(x,a)=x^{2}+a^{2} \quad (24$$
$$(x,a)\in\mathbb{K}.$$

Assumption 5.1. $g:[0,\infty)\to\mathbb{R}$ i − s po sit v − i e co nt i − no − u u s c − o nv x − e a − n d decreasing

Lemma 5 . 2 . Under Assump t i − o n 5 . 1 , Examp e − l5period − two satisfie As u − s mp tion 4 . an 4 . 2

6 . CONCLUSIONS It would seem that theorems like Berge quoteright − s h − t@ a r − e w d − i e l − y k o − n wn t th mathematica economists , should be better known to a l r e − s e a c − r her interest e − d n − i MD P s An th efac that such a Theorem can s t i − l l be a source of new fin d n − i g w h e − n th opt· ima polic i unique , suggests that further research on t hi a r − e a i stil p o − r mis i − n g

APPENDIX A : PROOFS RELATED TO EXAMPLES 3 . AND 3 . P r o o f o f L e mm a 3 . 1 . Let x be a fixed e l − e me n of $\mathbb{R}$ Comp u tin th firs an th second derivatives of P with respect to w, deno t − e d by P_{w} a d − n$Pw-w$ respectively d i i obtained that

$$P_{w}(x,w)=w^{2k+1}+2\varphi(\quad xw+\varphi(x) \qquad (25$$

and

$$P_{ww}(x,w)=(2k+1)w\text{two}-\text{k}+2\varphi(x) \qquad (26$$

$P_{w}(x,\cdot)$ has odd degree , hence i − t has at e − l a s o n − e rea o − r o t A $Pw-w(x,\cdot$ i positiv e by the well - known Rolle ’ s Theorem , there on l − y e xist o n rea roo fo $P_{w}(x,\cdot$ denote by $h^*(x)$. Furthermore , again , the po sit i − v ene s of $P_{ww}(x \cdot$ m − i p lie t − h a $h*(x$ i th unique minimum for (8) . Final ly $h^*(x)$ i − s nega t v − i e b e c − a us $P_w(x, w$ > 0 fo $w \geq$ 0 and , obviously , $P(x, h^*(x)) < P(x, 0) = \varphi(x)$ S i − n ce x i a rbitrary L m − e ma 3 . follow s period − square

P r o o f o f L e mma 3 . 2 . Fix $x \in \mathbb{R}$. If $a >$ 0 t − h e n trivially $G(x, a >$ 0 Suppos that $a \leq 0$, then

$$G(x,a) = \frac{1}{2(k+1)} a^{2(k+1)} + \varphi(x \text{parenright} - \mathrm{a}^2 + \varphi(xa - parenright + \varphi(x$$

$$= \frac{1}{2(k+1)} a^{2(k+1)} + \varphi(x) \text{parenleft} - \mathrm{a}^2 + a + \quad 1$$

$$= \frac{1}{2(k+1)} a^{2(k+1)} + \varphi(x)[\text{parenleft} - \mathrm{a} + \quad slash - one \quad 2^2 + \quad 3/4 > \quad 0$$

Consequently , as x is arbitrar y G i − s nonnega t v − i e (i − n fac G i p ositive)

Clearly , G is continuous (observe that , $G(x0) =$ $\mathrm{lm}_{a \to 0} + G(xa) =$ $\lim a \to^+$ $(a1/+$

$$\varphi(x)) = \varphi(x), \text{foreach} x \in \mathbb{R}).$$

Let $A_r(x) := \{a \in \mathbb{R} : \quad G(x,a) \leq r\}, x \in \mathbb{R} \quad r \in \mathbb{R}$ O bserv t − h a fo ea h − c$x \in \mathbb{R}$ an $r \in \mathbb{R}, A_r(x)$ is closed in $\mathbb{R}$ as a consequence of t − h$_\mathrm{e}$ co nt n − i ui t − y of G Sinc G i positiv e it follows that $A_r(x) = \varnothing($ and hence $A_r(x)$ i − s compa c t i $r \leq$ 0 Not tha t fo $x \in$ $\mathbb{R}$

$$\lim_{\to_a + \infty} G(x,a) = \mathrm{lm}^{(a}_{\to_a + \infty} 1/ + \varphi(x) = +\infty \tag{27}$$

and

$$\lim_{\to_a - \infty} G(x,a) = \mathrm{il}_{\to_a} \mathrm{m}_{-\infty} P(x,a)$$

$$= \mathrm{il}_{\to_a} \mathrm{m}_{-\infty} a^{2(k+1)} [\frac{1}{2(k+1)} + \varphi(x) \frac{1}{a2} + \frac{1}{a2k+} + a21_{(k} + 1 \quad)$$

$$= +\infty. \tag{28}$$

Therefore , if for some $x \in \mathbb{R}$ and $r > 0, A_r(x$ i − s unbo n − u de d usin (27 an (28 i is possible to choose $a' \in A_r(x)$ such that $G(xa\prime$ $>$ r w h i c − h i a *contradict* d ion)
I conclusion G is inf - compact on $Gr(\gamma)$, and t − h$_\mathrm{e}$e − r e xist a selecto $f^* \in \mathbb{F}$ suc tha fo each $x \in \mathbb{R}, G(x, \cdot)$ attains its minimum i − n$f^*(x$ (e − s e Rem a r − k 2.2)

Now , note that from Lemma 3 . 1

$$G(x, h^*(x)) = P(x, h^*(x) < G(xa)$$

$x \in \mathbb{R}, a \leq 0, \quad a \neq h^*(x)$; and a l − s o by Lemma 3 1 f o $x \in \mathbb{R}$ $a >$ 0

$$G(x, h^*(x)) = P(xh*(x)$$

$$< P(x0) = \varphi(x$$

$$\leq a^1 slash - three + \varphi(x = G(x,a)$$

Then , $h^*(x)$ is a minimum of $G(x, \cdot)$ for a l $x \in \mathbb{R}$ and evident l − y$h*(x = f$ $*(x)$ fo al $x \in \mathbb{R}$.
Consequently , the uniqueness of f^* f ol l − o w s

Now , it is direct to verify that G sa t i − s fies the MC To p o − r v thi s t − a k $\mathbb{K} = [-n, n\times$

$$[-n, n], n = 1, 2, ...$$

Obviously , for each $n, \mathbb{K}_n$ is compact and $\mathbb{K}_n \uparrow \mathbb{K} = Gr(\gamma = \mathbb{R}^2$ Fi a positiv integer n and take (x, a) $element{-}negationslash\mathbb{K}_n.$ Consider the f ol l − o w n − i g f − o u case s $n{-}minus \leq x \leq n$ an $a > n; -n \leq x \leq n$ and $a < -n; | x |> n$ and $a > 0$ or $bar - x$ $> n$ a d − na $\leq$ 0
In the first case , it follows from (9) that

$$G(x, a) = a^{1/3} + \varphi(x) \geq a1/ \geq n1/_3 \qquad (29$$

If $-n \leq x \leq n$ and $a < -n,$ then , from (1 0) i f ol l − o ws h − t a

$$\begin{aligned} G(x, a) &= \frac{1}{2(k+1)} a^2 \text{parenleft} - \text{k}_{+1}) + \varphi(xa - parenright^2 + \varphi(x\text{parenright} - \text{a} + \varphi(x \\ &= \frac{1}{2(k+1)} a^2 \text{parenleft} - \text{k}_{+1}) + \varphi(x)[a - parenleft + \quad \text{one} - \text{slash} \quad 2) + 3/4 \\ &\geq \frac{1}{2(k+1)} a^2 \text{parenleft} - \text{k}_{+1}) \geq \frac{1}{2k - parenleft + 1)} n2(k+1_) \qquad (30 \end{aligned}$$

Similarly , it is possible to obtain tha t f o − r $x - bar$ $> n$ and $a >$ 0

$$G(xa) \geq n1^{slash-three} \qquad (31$$

and that , for $| x |> n$ and $a \leq 0,$

$$G(x, a) \geq (3/4)n1/_3 \qquad (32$$

Hence , (30) − (32) imply that for every $(x, a) element - negationslash\mathbb{K}_n$

$$G(x, a) \geq \min\{(3/4)n^{1/3} \quad (1/\text{parenleft} - \text{two}(k+1))n - parenright2(k+1)_\}$$

Since n is arbitrary , it results that

$$(\inf_{a,_x)\notin\mathbb{K}_n} G(x, a) \geq \min\{(3/4)n1^{slash-three} \quad (1/\text{two} - \text{parenleftparenleft} - \text{k} + \quad 1))n - parenright2(k+1)_\} \qquad (33$$

for every $n = 1, 2, ...$ Then , let tng $n \to +\infty$ i − n three − parenleft3) i result t − h a $G(\cdot, \cdot$ satisfie th MC .

Finally , $G(\cdot, \cdot)$ is nonconvex as a con s − e quence of th no nc n − o vexi t − y o $G(-1, a)$

$a1^{\in\mathbb{R},}_{/\,(2(k+\text{which}_{1}))a^{2}(}\text{is}^{\text{given}}_{k+\quad 1),\,a\leq}\text{by}_0 G(^{(-1,\quad a)}_{\text{observe}} = a^1\text{that}G^{slash-three}(+^1_{x0)}a_= > \varphi(x^0_)x^{\text{a}}\text{d} - \text{n}_\in\mathbb{R}^{G(-}_{\text{i}}\text{a1}a\text{ls} \quad = \quad 1{+}a{+}\text{noncor}$

P r o o f o f L e mma 3 . 3 . Simiar to the p r − o of of Lemma 3.2 ☐ APPENDIX B : PROOFS RELATED TO EXAMPLES 5 . AND 5 .

P r o o f o f L e mm a 5 . 1 . The cost func t i − o n cs − i n o − n n g − e ativ e cont i − n uou s an observ

$\text{that}_{x\leq}\text{for}_{r\leq}\text{each}^{x}_{x+1,} \in [0\text{and}^{\infty),\,Ar(x)}_{Ar(x)}\,{}^{,}_{=[0,} = 1\varnothing^{,}_{+(r}\text{i} - \text{f}^r_- <)_x 1x/_{2]} A^r_{\text{f}-\text{i}}(_x x^{=}_{+}1\text{bracketleft} - \text{one}_< -r \quad (-\text{th}_{\text{n}-\text{e}}x)1\text{th}^{/2\;\;1+}_{\;\;\inf-\text{compa}}$

c follows , concluding that Assump t i − o n 4 . 1 (a) ho ld s

The proof of the strong con ti nu i − t$_y$ of h − t$_e$ ta nsit i − o n a − l w Q i − n duc d − e b − y (21 i a follows : if $u : X \to \mathbb{R}$ is a measurab l − e and bounded u − f nct i − o n th n − e fr o − m (21 an s th well - known Change of Variable Theorem i i − s o bta n − i ed t − h a

$$\int u(y)Q(\mathrm{d}y \mid x, a) = u(0)[1 - H(xa) + \int I[0^{comma-x} + a](z)u(z\Delta x - parenleft + a - z)\mathrm{d}z$$

$(x, a) \in \mathbb{K}$, where $I_{[}\cdot$ denotes the ind i − c a o − t r func t i − o n of th se $[\cdot$ a d − n$H(x, a = P[\leq x + a]$. The continui$^]$ ty of H imp l i − e s the con t n − i ui t − y of u e $(0)[-H(x, a)$ o $)_{\mathbb{K}}$ A u is bounded and Δ is a bounded con tnuous func t i − o n i result fr m − o th n$_{\text{Domina}}$ te Convergence Theorem that

$$\int I_{[0}comma - x + a(z)u(z)\Delta(x + a - z)\mathrm{d}z$$

is a continuous function on $\mathbb{K}$. In conc l − u s o − i n Q i st r − o n g y − l cont i − n uou s Let $f \in \mathbb{F}$, given by $f(x) = 1$, for a l $x \in X$ Th e − n fo a − ec − h$x \in X$

$$E_x^f[c(x_0, a_0) = c(xf(x) = \quad x \tag{34}$$

Now , for each $x \in X$,

$$\begin{aligned} E_x^f[c(x_1, a_1)] &= \int c(y, f(y)Q(dybar - xf) = \int yQ\mathrm{d} - \text{parenleft}_y \mid xf) \\ &= \int I_{[0,\infty)}(s)[x + 1 - s +_\Delta (s\mathrm{d}s \\ &= \int I_{[0,x+1}(s)(x + 1 - s)\Delta(s\mathrm{d}s \\ &= (x + 1)P[\xi \leq x + 1 \quad - \int I[0, x - plus1](s)\Delta(s)\mathrm{d} \leq x + \quad 1 \end{aligned} \tag{35}$$

and by a direct induction argument , i − t f ol o − l ws t − h$_@$

$$E_x^f[c(xta_t) \leq x + \quad t \quad (36$$
$$t = 0, 1, ...$$

Now , for each $x \in X$, using (34) and (36) ,

$$V(f, x) = \sum_{t=0}^{\infty} \alpha^t E_x^f[c(x_t, a_t) \leq x/(1 - \alpha) + \alpha \quad /(-\alpha 2 \tag{37}$$

Therefore , Assumption 4 . 1 holds for Examp l − e5.1

On the other hand , clearly $\gamma(x) = A(x) = [0\infty)x \in X$ i clos e − d− value an continuous (in fact γ is constant) .

Observe that it is trivial to prove that $c(\cdot\cdot$ i strict l − y o − c n ve x $F(x, a, s =$ $[+a- s]^+$, $x, a \in \mathbb{R}$ and $s \in S$, is convex in (x, a) f o each $s \in S$ a n − d increasin n − i x fo eac $a \in \mathbb{R}$ and $s \in S$, and the mult i − f$_{\text{unc}}$ t o − i$_{\text{n}}$$x \to A($ $x =$ $[0\infty)$ i c o − n ve x tha i s i i vali that $(1-\lambda)a+\lambda a' \in A((1-\lambda)x+\lambda x')$ f o − r a l x x' aa $\in$ $[0\infty)$ a d − n$\lambda \in$ $[0,1]$ an A and $A(x)$ are convex for each $x \in X$. Mo r − e o ve r X i o − c n v e − x a w el l S thi exampl satisfies Condition 1 in [6] , and the uniquene s − s of h − t$_{\mathrm{e}}$ o pt i − m al poli y − c follow s

Now the finiteness and the continu i − t$_{\mathrm{y}}$ of $\int V^*(yQ$d − parenleft$_y$ | ·· w il be verifie d

From (37) ,

$$0 \le V^*(x) \le x/(1-\alpha)+\alpha/(1-\alpha)^2 \quad (38$$

$$x \in X.$$

Then , from (38) and a computa t i − o n smi a − l r t − o h − t$_{\mathrm{e}}$ o n − e n − i parenleft − three 5) i follow tha tfo reac

$$(x,a) \in \mathbb{K},$$

$$\begin{aligned} \int V^*(y)Q(\mathrm{d}y \mid x,a) &= \int I_{\text{bracketleft−zero},\infty)}(s)V^*(x - bracketleft + a - s +)_{\Delta}\,(s\mathrm{d}s \\ &\le [1/(1-\alpha)\int I_{[0\infty)}(s)\text{bracketleft} - \mathrm{x} + a - \quad s^{+}\Delta(s)\mathrm{d}+\alpha \quad /(-\alpha) \\ &\le \frac{x+a}{1-\alpha}+\frac{\alpha}{(1-\alpha)^2} < +\infty \end{aligned}$$

In [6] it has been proved that , i − f condit o − i n C 1 h old s t − hn − e th opt i − m a v alu functio V is an increasing function on $X = [0,+\infty)$(s − e e Lemma 6one − period i − n [6]) Henc i i obta$^{\mathrm{n}}$ ine that V^* is continuous almost everywhere (a period − e.n − i $[0+\infty)$ (se T heor e − me$_4$.3. s i [2 an the paragraph j ust next to the end of the p r − o of of thi theorm) Le $(x_k, ak \in \mathrm{n}_{\mathbb{K}}k = 1$, 2 , . . . , such that $(x_k,a_k) \to (x,a) \in \mathbb{K}$. Let $T>0$ s − uh − c t − h a e$_{\mathrm{fo}}$ ea h − c$k =$ 1, 2, ...

$$0 \le x_k \le T \quad \text{and} \quad 0 \le a_k \le T \qquad (39$$

From (38) , for each $k = 1,2,...$, and $s \in S$,

$$0 \le V^*([x_k + a_k - s +)_{\Delta}\,(s \le h_k(s)$$

where $h_k(s) = ([x_k + a_k - s]^+/(1-\alpha)+\alpha/(1-\alpha) \quad 2\Delta(s)s \in S$ O bserv tha tfr o − m (39 and a computation similar to the one in (35) f o ea c − h$k =$ 1, 2 ...

$$\begin{aligned} 0 &\le \int h_k(s)\mathrm{d}s \\ &= \int I_{[0,\infty)}(s)(\frac{[x_k + a_k - s+}{1-\alpha}+\frac{\alpha}{(1-\alpha)^2}{)}\Delta(s)\mathrm{d}s \\ &\le \frac{x_k+a_k}{1-\alpha}+\frac{\alpha}{(1-\alpha)^2} \le \frac{2T}{1-\alpha}+\frac{\alpha}{\text{parenleft − one} - \alpha)^2} < +\infty \end{aligned} \qquad (40$$

Moreover , it is direct to ver i − f$_{\mathrm{y}}$ that $\{h_k\}$ conv r − e ge p ointwise l − y t th u − f nctio $h(s =$

$([x+a-s]^+/(1-\alpha)+\alpha/(1-\alpha)^2)\Delta(s)s \in S$ and t − ha $\mid hk(s) \le (T - two \ /(-\alpha +$ $\alpha/(1-\alpha)^2)\Delta(s), s \in S, k = 1,2,....$ Now us i − n g h − t$_{\mathrm{e}}$ s t − a d − n a r − d Dom inat d − e Convergenc

Theorem it follows that $\int h_k(s)ds \to \int h(s)$ d s

On the other hand , due to the con t i − n u t − i$_{\mathrm{y}}$ a . e of V^* i i o b tain e − d a wel tha

$$V^*([x_k + a_k - s+)_{\Delta(s)} \to V^*(\text{bracketleft} - \mathrm{x} + a - \quad s] +^{\text{Delta−parenright}}\,(s)$$

 when $k \to \infty, s-$ a . e . So , applying Theorem 1 7 p period − nine2n − i bracketleft − two9] i result tha

$$\lim_{k\to\infty} \int V^*(y)Q(\mathrm{d}y \mid x_k, a_k) = k\mathrm{lm}_{\to\infty} \int I_{[0\infty)}(sV^* \quad ([x+ \quad a- \quad s] +^{\mathrm{parenright-Delta}} (s)\mathrm{d}$$
$$= \int I_{[0\infty)}(sV^* \quad (\mathrm{bracketleft-x}+a- \quad s^+)\Delta(s)\mathrm{d}$$
$$= \int V^*(y)Q\mathrm{parenleft}-\mathrm{d}_y x - bara)$$

i . e . $\int V^*(y)Q($ d $y \mid \cdot,\cdot)$ is a con ti nuous func t i − o n

Therefore , Assumption 4 . 2 (c) holds .

Finally , let $\mathbb{K}_n = [0,n] \times [0,n], n = 1,2$. Ev i − d ently fo ea h − c $n\mathbb{K}_n$ i compac tan $\mathbb{K}_n \uparrow \mathbb{K} = Gr(\gamma) = [0,\infty) \times [0,\infty)$. Now , fix a po sitiv intege n a n − d a − t k $(x, a element - negationslash$ $\mathbb{K}n$ There are two cases : $0 \le x \le n$ and $a > n$ or $x > n$ and $a \ge$ 0 I − n th firs cas e i follows from (22) that

$$c(x,a) \ge (n-1)^2 \tag{41}$$

If $x > n$ and $a \ge 0$, then again , from (22) i − t r esult h − t@

$$c(xa) \ge n \tag{42}$$

(4 1) and (42) imply that for every $(x,a) element - negationslash \mathbb{K}_n$

$$c(x,a) \ge \mathrm{mi-n}\{(n-1)^2 n\} \tag{43}$$

and since n is arbitrary , it follows that

$$\inf_{(a,x) \notin \mathbb{K}_n} c(x,a) \ge \mathrm{mi-n}\{n - parenleft - 1)^2 n \quad \} \tag{44}$$

for every $n = 1,2,\ldots.$ Now , let ti ng $n \to +\infty$ i − n four − parenleft4 i result t − h a $c(\cdot,\cdot$ satisfie th MC . □

P r o o f o f L e mm a 5 . 2 . Clear ly X and A a r − e c n − o v e − x set s a d − nc i nonnegative inf - compact , continuous , and strict l − y conv x − e on $\mathbb{K}$ b eside s $x \to A(x$ i clos e − d− value d

It is direct to verify that $F(x,a,s) =$ bracketleft − x + $g(a) - s +$ $xa \in \mathbb{R}$ a d − ns $\in S$ i conve

in (x,a) for each $s \in$ S and increasing i − nx f r − o e a − c h $a \in$ $A(x$ a d − ns $\in$ S an th multifunction $x \to A(x)$ is convex , and $A(x)$ i − s con e − v x fo ea c − h$x \in X$

Similar to Example 5 . 1 (see the proof of Lemma 5 . 1) i i possibl t prov tha t

- Q induced by (23) is strongly con t i − n uou s
- For $f \in \mathbb{F}$, given by $f(x) = x, x \in X$

$$V(f,x) \le \eta x^2 + \beta x + \theta \quad x \in X \tag{45}$$

where$\eta = 2/(1-\alpha), \beta = (4g(0)\alpha)/(1-\alpha)^2$ and$\theta = 2\mathrm{parenleft-g(0))2}[(\alpha(+\alpha)$
$(1-\alpha)^3) + (\alpha(1+4\alpha+\alpha^2)/(1-\alpha)^4)]$

 Hence , Assumption 4 . 1 and 4 . 2 (b) hold . In pa rt i − c ula r Con diti n − o C 1 i − n [6 hold s

Again , as in Example 5 . 1 (see the proof of Lemma 5 . 1) i i a ls possibl et oestablis

that :

- Foreach$(x, a) \in \mathbb{K}$,

$$\int V^*(y)Q(\mathrm{d}y \mid x, a) \leq \eta(x+g(0)) + \beta\text{parenleft} - \mathrm{x} + g(0) + \theta < +\infty \quad (46$$

(To verify (46) it is necessary to use (45) . • V^* is continuous almost everywhere i − n zero − bracketleft+ $\infty)$ (thi fol l − o w fr m − o C o − n ditio C 1 i

[6]).

- $\int V^*(y)Q($ d $y \mid \cdot\cdot)$ is a con ti nuous func t i − o n

Consequently , Assumption 4 . 2 (c) holds .

Now the continuity of $x \rightarrow A(x)$ wil be p r − oe − v d Firstly i w il b p rov e − d tha $x \rightarrow A(x)$ is l . s . c , and later that $x \rightarrow A(x)$ i − s u s − periodperiod − c Let $x_n \rightarrow x$ n − iX a n − d$a \in A(x =$ $[x, \infty)$ If $x = a$, then take $a_n = x_n \in A(x_n), n = 12$. and $a_n \rightarrow$ a I $a \neq$ x i . e $x <$ a then take $a_n = x_n + (a-x), n = 1, 2, ..$. and o .bserv t − h a $a_n \in A(xn_) n =$ .1, 2, ... and $a_n \rightarrow a$; hence , Remark 2 . 1 imp l i − e s that $x \rightarrow A(x$ i l . s . c Le $F \subset A$ b a close set , and let $x_n \in$ $\{x \in X :$ $[x, \infty) \cap F \neq \varnothing\}, n =$ 1, 2 . and s − u p pos tha $x \rightarrow y \in X$ For each $n = 1, 2, ...,$ let $b_n \in [x_n, \infty) \cap F.$ I − f h − t e r − e e xis' t a p osi tiv intege m suc htha $b_m > y$, then $b_m \in [y, \infty) \cap F$, i . e . $y \in$ $\{x \in X$ $[x\infty) \cap F \neq \mathrm{e}_{\varnothing\}}$ I $x_n \leq$ $b \leq$ y for all $n = 1, 2, ...,$ then , since $x_n \rightarrow y$ i − t f ol o − l ws t − h a l m − $\mathrm{i}_{n\rightarrow\infty}$ $b =$ y As b $\in F$ for all n, and F is closed , it resul t − s that $y \in F$ i . e $y \in$ $\{x \in X$ $[x, \infty \cap F \neq$ $\varnothing\}$ hence ,$\{x \in X :$ $[x, \infty) \cap F \neq \varnothing\}$ i − s c o − l sed i − nX T . herefor e D fi − e niti o − n 2.1(b implie that $x \rightarrow A(x)$ is u . s . c .

Finally , for each $n = 1, 2, ...,$ let $\mathbb{K}_n = \{(xa)$ $x \in$ $[0n]a \in$ $[xn]\}$ I i direc t verify that for each $n, \mathbb{K}_n$ is compact , and a l − s o h − t$_{@}\mathbb{K}_n \uparrow \mathbb{K}$ Le n be a fixe posit$^{\mathrm{t}}$ iv integer , and take $(x, a) \in \mathbb{K} \setminus \mathbb{K}_n$. Then $a > n$ w h i − c h mp lie h − t a $c(xa$ $= x^2 + a$ $\geq$

$$a^2 > n^2.\mathrm{So}$$

$$(x\mathrm{n} - \mathrm{i}^{\mathrm{f}}_{comma-a)\notin\mathbb{K}_n} c(xa) > n^2 \quad (47$$

Since n is arbitrary , it follows that (47) holds f o e a − c h $n =$ 1, 2, ... Henc e lettin $n \rightarrow \infty$ in (47) , it results that c sa t i − s fies the MC

Therefore , Assumptions 4 . 2 (a) and 4period − two(d) ho l d □ (Receive Marc h 4 , 2 0 1 1

RE F E RE N C E S

[1] C . D . Aliprantis and K . C . Borde r Infini t − e Dme ns o − i n a An alysi s . Thi r − d E ditio n . Springer Verlag , Berlin 2006 .

[2] R . B . Ash : Real Variables w i − t h Ba s c − i Me tr i − c S a − pc − e T p − o ol o − g y E − I E − E Pres s N w − e Yor 1 993 .

[3] J . P . Aubin and I . Ekeland : App l i − e d Non l n − i e a An alysi s J o − $\mathrm{h_n}$ W iley N w − e Yor 1984

[4] L . M . Ausubel and R . J . Denecker e A g e − n eraliz d − e theor e − m o th m aximum Econom Theory *3* (1 993) , 99 – 1 7 .

[5] C . Berge : Topological Space s O l i − vr − e and Boy d E d i − n bu g − r h a n − d Londo 196 (reprinte by Dover Publications , Inc . M i − n eo la New Y o k − r one − nine97)

[6] D . Cruz - Su *á* rez , R . Montes - de - Oca , and F Sa l − e m - S i v − l a C o − n dition fo th uniquenes of optimal policies of d i − s counted Ma k − ro − v d ecis o − i n p rocesses Ma t h s Method Oper . Re s *60* (2004) , 41 5 – 436 .

[7] J . Dugundji : Topolog y A l l − y n and Baco n n − I c . B os t − o n 1966

[8] P . K . Dutta , M . K . Majumdar , and R K Sunda r − a m P a a − r metri continuit i dynami programming problems . J . Econom Dynam Co ntro *1 8* (1 994) c 1069 – 109 2

[9] P . K . Dutta and T . M i − $\mathrm{t_{ra:}}$ Maximum h − te − oe − r ms fo c − o n v x − e structure wit a applicatio to the theory of optimal intertemporal a l l − o ca t o − i n s J Ma t h Econom *one − eight* h (1989) 77 – 8 6

[1 0] O . Hern *á* ndez - Lerma and J . B . La s − sr − e r e D i − s cre e − t− T i − m e Ma k − ro − v Contro lProcesse s Basi Optimality Criteria . Springer - V erag , New Y o r − k nine − one96

[1 1] O . Hern *á* ndez - Lerma and W . J . Rungga ldie r Mon o t − o n a pproximation fo conve stochastic control problems . J . Ma t − h Sy s e − t ms Est i − m Co ntro *4* (1994) 99 – 140

[1 2] K . Hinderer : Lipsch i − $\mathrm{t_z}$ con t n − i u i − $\mathrm{t_y}$ of v a u − l e fun ct i − o n n − i Ma rkovia decisio Processes Math . Methods Oper . Res . *60* (2005) 3 – 2 2

[1 3] K . Hinderer and M . Stieg l i − $\mathrm{t_{z:}}$ Inc e − r a s i − n g a d − n Lipschit continuou m inimizer i one dimensional linear - convex sys t − e ms wi h − $\mathrm{t_{out}}$ c onstraints z th continuou an th $\mathrm{s_{discret}}$ case . Math . Methods Oper . Re s 44(1996)189endash − two04

[14] A . Horsley , A . J . Wrobe l and T Van Zan d t Be r − g e ' ma x m − i um theor e − m wit htw otopolo gies on the action set . Econom Le t t 61(1998)2eight − five – 2 9 1

[1 5] A . Ja *ś* kiewicz and A . S . Nowak : D s − i coun t − e d dyna mi $\mathrm{cp^{o-r}}$ g r − a m n − i g wit hunbounde dreturns Application to economic mode l − s. J Ma t h Ana l Ap p l *378* (2 0 1 1) 450 – 46 2

[1 6] J . S . Jordan : The continu i − $\mathrm{t_y}$ of optimal dyna m c − i decis i − o n rule s Econometric *five − four* (1977) 1 365 – 1 376 .

[1 7] T . Kamihigashi : Stochastic optimal g r − o w h − t w i t − h bo n − u d e − d or un b u − o nde dutilit yan dwit bounded or unbounded shock s J . Ma t − h Econom 43 two − parenleftzero − zero7) seven − four7 – 50 0

[1 8] T . Kamihigashi and S . Roy : A nonsmoo t h n on c − o n v x − e mode o optima growt h J Econom . Theory *1 32* (2007) , 435 – 460

[1 9] R . B . King : Beyond Quar t i − c Equa t o − i n Bi r − k ha use r B os o − t n 1996

[20] M . Kitayev : Semi - Markov and j − u mp Ma k − ro − v c ontro m odel s a verag cos criterion Theory Probab . Appl . *30* (1 985) 272 – 2 8 8

[2 1] A . Leizarowitz : Optimal contro l − s f r − o d i − $\mathrm{ff_u}$ s o − i ns i − n$\mathbb{R}^n$ J Ma t h Anal Appl $1^{nine-four}$ (1990) 1 80 – 209 .

[22] D . V . Lindley : The theory of queues wi t – h a s n – i g l serve r Proc Ca mbridg Philo s So c
4 8 (1 952) , 277 – 289 .
[23] M . Majumdar and R . *Radne* r : *Sta* t i – o na y – r op t i – m al po licie swi h – t discountin gi n astochasti
activity analysis model . Econome tr i – c a *51* (1 98 3) 1 8 21 – eight – one37
[24] S . P . Meyn : Ergodic Theorems f o – r di c – s r e t – e tme sto h – c asti system usin a stochasti
Lyapunov functions . SIAM J Con t – r o Op tm 27 (nine – one89) 1409 – 143 9

[25] E . A . Ok : Real Analys i − s w t − i h Econom i − c App l c − i a t i − o n s Princet n − o U niversit yPress , Prince
ton 2007 .
[26] A . L . Peressini , F . E . Su lli van , and J J Uh l The Ma thma tic so fNonlinea rProgr a − m ming
Springer - Verlag , New York 1 988 .
[27] M . L . Puterman : Markov Decis i − o n Proc esse s Discre e − t Sto c − h asti cDynami cProgr m − a ming
John Wiley , New York 1 994 .
[28] U . Rieder : Measurable selec t i − o n theo e − r ms f o o ptimizat i − o n p roblems Manuscript aMath
24 (1 978) , 1 1 5 – 1 31 .
[29] H . L . Royden : Real Analy s i − s. T h r − i d E dit i − o n Mac m il a − l n N e − w Yo k − r 1988
[30] R . H . Stockbridge : Time - average con t − r o of m art n − i gal problems a linea progr^{a-m} min
formulation . Ann . Probab . *1 8* (1 990) 29 1 – 31 4
[31] R . Sundaram : A First Course in Op tm i − z at o − i n T heory C a − m bridg Universit Pres s
Cambridge 1 996 .
[32] G . Tian and J . Zhou : The maximum t − ho − er − e m a d − n th existenc o Nas equilibri u − m
of (generalized) games w i − t hout o − l wer sem i − c on t n − i uitie s J Ma t h A nal Appl *one − six*6 (1992)
35 1 – 364 .
[33] G . Tian and J . Zhou : Transfer con t i − n uit e − i s ge neralizat i − o n of th Weierstras an dmaxi
mum theorem : a full characte r i − z a t o − i n J Ma t h E c − oo − n m *24* (1995) 281 – 30 3
[34] M . Walker : A general i − z a ti on of the maximum theo e − r m Inte r − n a t . Econom . Re v . *2 0* (1979)
267 – 272 .
[35] A . Yushkevich : Blackwe l optima li t − yn − i B orel i − a n o − c nt i − n uous −n − i− actio Marko decisio
processes . SIAM J . Control Op ti m 35(1997)2one − five7 – 2 18 2

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