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REMARKS ON COUPLED FIXED POINT THEOREMS IN CONE METRIC SPACES

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Abstract . In this paper , we first show that some coupled fixed point theorems in cone metric spaces are proper consequences of relevant fixed point theorems . Then we give and prove some corresponding coupled fixed point theorems in partially ordered cone metric spaces . Some examples are also given to illustrate our work .

1 . Introduction and preliminaries

The well - known Banach contraction principle is one of the pivotal results of analysis and has applications in a number of branches of mathematics . This prin - ciple has been extended and generalized in various directions for recent years by putting conditions on the mappings or on the spaces . Huang and Zhang in [1 6] introduced the notion of cone metric spaces , investigated the convergence in these spaces , introduced the notion of their completeness , and proved some fixed point theorems for contractive mappings on cone metric spaces . After that , many authors have fo cused on cone metric spaces and its topological properties , given and proved fixed point theorems in cone metric spaces (see [1 − 6 , 1 2 − 1 4 , 1 6 − 1 8 , 20 − 26 , 33 − 40 , 42 − 43] and references therein) .

Now we first recall some definitions and properties of cone metric spaces .

Definition 1 . [1 5] Let E be a real Banach space . A subset P of E is called a cone if and only if :

(a)P is closed , non - empty and $P \neq \{\theta\}$,

(b)$a, b \in \mathbb{R}, a, b \geq 0, x, y \in P$ imply that $ax + by \in P$,

(c)$P\cap \Rightarrow notdef$parenright − notdef$notdef{-}equal$ $\{$ θ $notdef{-}braceright^{\text{period}-\text{element}} notdef{-}notdef$

ven a c ne , d fine a p rtialo dering $\leq$ w th r spect t P b y $x \leq y$i − f an d ly G$_{i-f}$$y$ $-x \in P$. W es all w ite $\ll$ fo r $y - x \in ItP$ e , w e − h re ItP i − s t e n − i terior

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of P. Also we shall use $\prec$ to indicate that $x \leq y$ and $x \neq y$. The cone P in normed space E is called normal whenever there is a number $k > 0$ such that for all $x, y \in E, \theta \leq x \leq y$ implies $\| x \| \leq k \| y \|$. The least positive number k satisfying this norm inequality is called the normal constant of P. It is clear that $k \geq 1$. It is known that there exists ordered Banach space E with cone P which is not normal

$$\text{butwith} IntP \neq \varnothing.$$

Definition 2 . [1 6] Let X be a non - empty set . Suppose that the mapping

$$d : X \times X \to E \text{satisfies :}$$

(d 1)$\theta \leq d(x, y)$ for all $x, y \in X$ and $d(x, y) = \theta$ if and only if $x = y$, (d 2)$d(x, y) = d(y, x)$ for all $x, y \in X$, (d 3)$d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$. Then d is called a cone metric on X and (X, d) is called a cone metric space .

The concept of a cone metric space is more general than that of a metric space .

Definition 3 . [1 6] Let (X, d) be a cone metric space . We say that a sequence

$$\{x_n\} \text{in} X \text{is :}$$

(a) a Cauchy sequence if for every $c \in E$ with $0 \ll c$, there exists an N such that for all $n, m > N, d(x_n, x_m) \ll c$.

(b) a convergent sequence if for every $c \in E$ with $0 \ll c$, there exists an N such that for all $n > N, d(x_n, x) \ll c$ for some fixed $x \in X$.

A cone metric space X is said to be complete if every Cauchy sequence in X is convergent in X.

Let (X, d) be a cone metric space ; then we have the following properties

(p 1) If E is a real Banach space with a cone P and $a \leq ha$ where $a \in P$ and

$$h \in (0, 1) \text{then} a = \theta.$$

(p 2) if $\theta \leq u \leq c$ for each $\theta \leq c$ then $u = \theta$. (p 3) if $u \leq v$ and $v \ll w$ then $u \ll w$. (p 4) if $a \leq b + c$ for each $\theta \leq c$ then $a = b$. (p 5) if $c \in IntP, 0 \leq a_n$ and $a_n \to \theta$ then there exists a K such that for all

$$\ll c. \qquad n > K, \text{wehave} a_n$$

For the details about these properties see [2 1 , 24] .

It is known that the sequence $\{x_n\}$ converges to $x \in X$ if $d(x_n, x) \to \theta$ as $n \to \infty$ and $\{x_n\}$ is a Cauchy sequence if $d(x_n, x_m) \to \theta$ as $n, m \to \infty$. In the case when the cone is not necessarily normal , the fact that $d(x_n, yn) \to d(x, y)$ if $x_n \to x$ and $yn \to y$ is not applicable .

Definition 4 . [3] Let $f, g : X \to X$ be two self - mappings on X. An element $x \in X$ is called a coincidence point of f and g if $fx = gx$. f and g are said to be weakly compatible if they commute at their coincidence points , that is $gfx = fgx$

$$\text{if } fx = gx.$$

Using the concept of weakly compatible mappings , many authors have studied the existence and uniqueness of common fixed points of self - mappings in cone metric spaces (see , for example , [3 , 22 , 23] and references therein) . For our purpose , we now state the result of Jungck et . al . [22] .

Theorem 5 . [22] *Let* (X,d) *be a cone metric space* , P *a cone with non - empty in - t erior and mappings* $f,g:X\to X$. *Suppose that there exist non - negative constants* $a_i, i=1,2,...,5$ *satisfying* $\sum_{i=1}^{5} a_i<1$ *such that , for al l* $x,y\in X$,

$$d(fx,fy)\leq a_1d(gx,gy)+a_2d(gx,fx)+a_3d(gy,fy)+a_4d(gx,fy)+a_5d(gy,fx) \quad (1)$$

If $f(X)\subseteq g(X)$ *and* $f(X)$ *or* $g(X)$ *is a complete subspace of* X *th en* f *and* g *have a unique co in cidence po int in* X. *Moreover , if* f *and* g *are weakly compatible , then* f *and* g *have a unique common fixed point .*

Recently , existence of fixed points for contraction type mappings in partial - ly ordered metric spaces has been considered in [$7-1$ 1 , 1 9 , $27-32$, 41] and references therein , where some applications to matrix equations , ordinary differential equa - tions , and integral equations has been presented . Bhashkar and Lakshmikantham [1 0] introduced the concept of a coupled fixed point of a mapping $F:X\times X\to X$ (a non - empty set) and established some coupled fixed point theorems in partially ordered complete metric spaces which can be used to discuss the existence and uniqueness of solution for periodic boundary value problems . Later , Lakshmikan - tham and Ć iri ć[27] proved coupled coincidence and coupled common fixed point results for nonlinear mappings $F:X\times X\to X$ and $g:X\to X$ satisfying cer - tain contractive conditions in partially ordered complete metric spaces . Using the concepts of coupled fixed point and coupled coincidence point , some authors have proved coupled (coincidence , fixed) point theorems in cone metric spaces (see [1 , 1 4 , 25 , 40 , 42]) . Some of them are in non - ordered cone metric spaces .

Definition 6 . [1 0] Let $(X,\preceq)$ be a partially ordered set and $F:X\times X\to X$. The mapping F is said to have the mixed monotone property if F is monotone non - decreasing in x and F is monotone non - increasing in y, that is , for any $x,y\in X$,

$$\begin{aligned} x_1,x_2\in X, x_1\preceq x_2 \quad &\Rightarrow \quad F(x_1,y)\preceq F(x_2,y)\\ y1,y2\in X, y1\preceq y2 \quad &\Rightarrow \quad F(x,y1)\succeq F(x,y2). \end{aligned}$$

Definition 7 . [1 0] An element $(x,y)\in X\times X$ is called a coupled fixed point of the mapping $F:X\times X\to X$ if $x=f(x,y)$ and $y=f(y,x)$.

Definition 8 . [27] Let $(X,\preceq)$ be a partially ordered set and $F:X\times X\to X, g:X\to X$ be two mappings . The mapping F is said to have the mixed g - monotone property if F is monotone g - non - decreasing in its first argument and F is monotone g - non - increasing in it s second argument , that is , for any $x,y\in X$,

$$\begin{aligned} x_1,x_2\in X, gx1\preceq gx2 \quad &\Rightarrow \quad F(x_1,y)\preceq F(x_2,y)\\ y1,y2\in X, gy1\preceq gy2 \quad &\Rightarrow \quad F(x,y1)\succeq F(x,y2). \end{aligned}$$

 DEFINITION 9 . [27] An element $(x, y) \in X \times X$ is called

(1) a coupled coincidence point of the mapping $F : X \times X \to X$ and $g : X \to X$

$$\text{if} gx = F(x, y) \text{and} gy = F(y, x).$$

(2) a coupled common fixed point of the mapping $F : X \times X \to X$ and $g : X \to X$

$$\text{if} x = gx = F(x, y) \text{and} y = gy = F(y, x).$$

DEFINITION 1 0 . [27] The mappings F and g where $F : X \times X \to X, g : X \to X$
are said to commute if $F(gx, gy) = g(Fx, Fy)$ for all $x, y \in X$.

In [40] , Sabetghadam et al . proved the following coupled fixed point theorems .

THEOREM 1 1 . [40] *Let* (X, d) *be a cone metric space* , P *a cone with non - empty interio r . Suppose that the mapping* $F : X \times X \to X$ *satisfies the following contractive condition for al l* $x, y, u, v \in X$,

$$d(F(x, y), F(u, v)) \leq kd(x, u) + ld(y, v), \tag{2}$$

where k, l *are non - negative constants with* $k + l < 1$. *Then* F *has a unique coupled fixed point .*

THEOREM 1 2 . [40] *Let* (X, d) *be a cone metric space* , P *a cone with non - empty interio r . Suppose that the mapping* $F : X \times X \to X$ *satisfies the following contractive condition for al l* $x, y, u, v \in X$,

$$d(F(x, y), F(u, v)) \leq kd(F(x, y), x) + ld(F(u, v), u), \tag{3}$$

where k, l *are non - negative constants with* $k + l < 1$. *Then* F *has a unique coupled fixed point .*

THEOREM 1 3 . [40] *Let* (X, d) *be a cone metric space* , P *a cone with non - empty interio r . Suppose that the mapping* $F : X \times X \to X$ *satisfies the following contractive condition for al l* $x, y, u, v \in X$,

$$d(F(x, y), F(u, v)) \leq kd(F(x, y), u) + ld(F(u, v), x), \tag{4}$$

where k, l *are non - negative constants with* $k + l < 1$. *Then* F *has a unique coupled fixed point .*

Abbas et al . [1] introduced the concept of w - compatible mappings and proved some coupled coincidence point theorems which generalized the results of Sabet - ghadam et al . [40] .

DEFINITION 1 4 . [1] The mappings F and g where $F : X \times X \to X, g : X \to X$
are said to be w - compatible if $gF(x, y) = F(gy, gx)$ whenever $gx = F(x, y)$ and

$$gy = F(y, x).$$

THEOREM 1 5 . [1] *Let* (X,d) *be a cone metric space with a cone* P *having non - empty interior* $, F : X \times X \to X$ *and* $g : X \to X$ *be mappings satisfying* .

$$d(F(x,y),F(u,v)) \leq a_1 d(gx,gu) + a_2 d(F(x,y),gx) + a_3 d(gy,gv) + a_4 d(F(u,v),gu) + a_5 d(F(x,y),gu) + a_6 d(F(u,v),gx) \tag{5}$$

for al l $x, y, u, v \in X$, *where* $a_i, i = 1, 2, ..., 6$ *are non - negative real numbers such that* $\sum_{i=1}^{6} a_i < 1$. *If* $F(X \times X) \subseteq g(X)$ *and* $g(X)$ *is a complete subspace of* X
then F *and* g *have a coupled co in cidence po int in* X. *Moreover ,* *if* F *and* g *are* w *- compatible , then* F *and* g *have a unique common coupled fixed point and common*
coupled fixed point of F *and* g *is of the form* (u, u) *for s ome* $u \in X$.

In this paper , we first show that Theorem 1 5 is a real consequence of Theorem 5 and so are Theorems 1 1 , 1 2 and 1 3 . Then we give and prove some coupled fixed point results in partially ordered cone metric spaces that are relevant to Theorem 1 5 . The results unify and extend some recent results .

2 . Main results

LEMMA 1 6 . *Let* $F : X \times X \to X$ *and* $g : X \to X$ *be* w *- compatible mappings .*
If the mapping $f : X \to X$ *is defined by* $fx = F(x, x)$ *for al l* $x \in X$, *then* f *and* g *are weakly compatible mappings .*

Proof . Suppose that x is a coincidence point of f and g, that is , $fx = gx$. By the definition of f, we have $F(x, x) = gx$. Since F and g are weakly compatible , we have $F(gx, gx) = gF(x, x)$. Therefore $fgx = gfx$, that is , f commute g at their coincidence point .

THEOREM 1 7 . *Theorem 1 5 is a consequence of Theorem 5 .*

Proof . Let $f : X \to X$ be the mapping defined by $fx = F(x, x)$ for all $x \in X$. In (5) , take $x = y, u = v$, we have

$$\begin{aligned} d(fx, fu) &= d(F(x, y), F(u, v)) \\ &\leq a_1 d(gx, gu) + a_2 d(F(x, x), gx) + a_3 d(gx, gu) \\ &+a_4 d(F(u, u), gu) + a_5 d(F(x, x), gu) + a_6 d(F(u, u), gx) \\ &= a_1 d(gx, gu) + a_2 d(fx, gx) + a_3 d(gx, gu) \\ &+a_4 d(fu, gu) + a_5 d(fx, gu) + a_6 d(fu, gx) \\ &= (a_1 + a_3) d(gx, gu) + a_2 d(fx, gx) \\ &+a_4 d(fu, gu) + a_5 d(fx, gu) + a_6 d(fu, gx). \end{aligned}$$

Moreover , we have $f(X) \subseteq F(X \times X) \subseteq g(X), g(X)$ is a complete subspace of X. Applying Theorem 5, f and g have a coincidence point $x \in X$, that is , $fx = gx$. This implies that $F(x, x) = gx$, that is , (x, x) is coupled coincidence point of F and g. Since f and g are weakly compatible , x is unique and $x = fx = gx$, that is $x = F(x, x) = gx$. Therefore F and g have unique common coupled fixed point of

$$\text{theform}(x, x).$$

The following example shows that Theorem 1 5 is a proper consequence of Theorem 5 .

EXAMPLE 1 8 . Let $X = \mathbb{R}$ with the cone metric $d(x, y) = | x - y |$, for all
$x, y \in X$. Let $F : X \times X \to X$ be given by

$$F(x, y) = \{x^{x/4,}_{+} y, \quad \text{if}^{\text{if}} x^{x} = \neq y^{y}_{,}$$

and $g : X \to X$ be given by $gx = x, \forall x \in X$. Then F and g do not satisfy the condition (5) for all $x, y, u, v \in X$. Indeed , suppose (5) holds for all $x, y, u, v \in X$, take $x = 2u \neq 0, y = v = 0$, we have

$$\begin{aligned} | u |=| x-u |= d(F(x,y),F(u,v)) \\ \leq a_1 d(gx,gu) + a_2 d(F(x,y),gx) + a_3 d(gy,gv) \\ +a_4 d(F(u,v),gu) + a_5 d(F(x,y),gu) + a_6 d(F(u,v),gx) \\ = a_1 | x-u | + a_3 | u | + a_5 | x-u | + a_6 | u-x | \\ = (a_1 + a_3 + a_5 + a_6) | u |, \end{aligned}$$

which is a contradiction .

However , if we define $f : X \to X$ by $fx = F(x, x)$ for all $x \in X$ then f and g satisfy all the conditions of Theorem 5 . Applying Theorem 5 , we conclude that f and g have the unique common fixed point 0 . Therefore , F and g have the common coupled fixed point (0 , 0) .

We next give and prove some coupled fixed point results in partially ordered cone metric space for compatible mappings .

DEFINITION 1 9 . Let (X, d) be a cone metric space . The mappings F and g where $F : X \times X \to X, g : X \to X$ are said to be compatible if

$$\lim_{n\to\infty} d(gF(x_n, yn), F(gxn, gyn)) = \theta \text{and} \lim_{n\to\infty} d(gF(yn, x_n), F(gyn, gxn)) = \theta,$$

where $\{x_n\}$ and $\{yn\}$ are sequences in X such that

$$\lim_{n\to\infty} F(x_n, yn) = \lim_{n\to\infty} gxn = x \text{and} \lim_{n\to\infty} F(yn, x_n) = \lim_{n\to\infty} gyn = y$$

for all $x, y \in X$ are satisfied .

It is easy to see that if F and g commute then they are compatible .

THEOREM 2 0 . *Let $(X, \preceq)$ be a partially ordered s e t and suppose there is a metric d such that (X, d) is a complete cone metric space . Let $F : X \times X \to X$ and $g : X \to X$ be such F has the mixed $g-$ monotone property and the re exist non - negative constants α, β, γ and λ satisfying $\alpha + \beta + 2\gamma + 2\lambda < 1$ such that*

$$\begin{aligned} d(F(x,y),F(u,v)) \leq \alpha d(gx,gu) + \beta d(gy,gv) + \gamma[d(F(x,y),gx) + d(F(u,v),gu)] \\ +\lambda[d(F(x,y),gu) + d(F(u,v),gx)] \quad (6) \end{aligned}$$

for al l $x, y, u, v \in X$ with $gx \preceq gu$ and $gy \succeq gv$. Further suppose that $F(X \times X) \subseteq g(X)$, g is continuous and g and F are compatible . Suppose e ither

(a) F is continuous o r

(b) X has the following property

(i) If $\{x_n\}$ is a non - decreasing s equence and $\lim_{n\to\infty} x_n = x$ then $gxn \preceq gx$ for al l n,

(ii) If $\{yn\}$ *is a non - increasing s equence and* $\lim_{n\to\infty} yn = y$ *th en* $gy \preceq gyn$ *for al l* n.

If the re exist $x_0, y0 \in X$ *such that* $gx0 \preceq F(x_0, y0)$ *and* $gy0 \succeq F(y0, x_0)$ *then* F *and* g *have a coupled coincidence point .*

Proof . Let $x_0, y0 \in X$ be such that $gx0 \preceq F(x_0, y0)$ and $gy0 \succeq F(y0, x_0)$.
Since $F(X \times X) \subseteq g(X)$, we construct sequences $\{x_n\}$ and $\{yn\}$ in X as follows

$$gxn + 1 = F(x_n, yn) \quad \text{and } gyn + 1 = F(yn, x_n), \quad \text{for all } n \geq 0 \tag{7}$$

We shall show that

$$gxn \preceq gxn + 1, \text{forall} n \geq 0 \tag{8}$$

and

$$gyn \succeq gyn + 1, \text{forall} n \geq 0 \tag{9}$$

Since $gx0 \preceq F(x_0, y0)$ and $gy0 \succeq F(y0, x_0)$ and as $gx1 = F(x_0, y0)$ and $gy1 = F(y0, x_0)$, we have $gx0 \preceq gx1$ and $gy0 \succeq gy1$. Thus (8) and (9) hold for $n = 0$.

Suppose that (8) and (9) hold for some $n \geq 0$. Then , since $gxn \preceq gxn+1$ and $gyn \succeq gyn + 1$, and by the $g-$ mixed monotone property of F, we have

$$gxn + 2 = F(x_{n+1}, yn + 1) \succeq F(x_n, yn + 1) \succeq F(x_n, yn) = gxn + 1 \tag{10}$$

and

$$gyn + 2 = F(yn + 1, x_{n+1}) \preceq F(yn, x_{n+1}) \preceq F(yn, x_n) = gyn + 1. \tag{11}$$

Now from (1 0) and (1 1) , we obtain

$$gxn + 1 \preceq gxn + 2 \quad \text{and} gyn + 1 \succeq gyn + 2$$

Thus by the mathematical induction we conclude that (8) and (9) hold for all $n \geq 0$.
Since $gxn - 1 \preceq gxn$ and $gyn - 1 \succeq gyn$, from (6) and (7) , we have

$$\begin{aligned} d(gxn, gxn + 1) &= d(F(x_{n-1}, yn - 1), F(x_n, yn)) \\ &\leq \alpha d(gxn - 1, gxn) + \beta d(gyn - 1, gyn) \\ &+\gamma[d(F(x_{n-1}, yn - 1), gxn - 1) + d(F(x_n, yn), gxn)] \\ &+\lambda[d(F(x_{n-1}, yn - 1), gxn) + d(F(x_n, yn), gxn - 1)] \\ \leq \alpha d(gxn - 1, gxn) + \beta d(gyn - 1, gyn) &+ \gamma[d(gxn, gxn - 1) + d(gxn + 1, gxn)] \\ &+\lambda d(gxn + 1, gxn - 1) \\ \leq \alpha d(gxn - 1, gxn) + \beta d(gyn - 1, gyn) &+ \gamma[d(gxn, gxn - 1) + d(gxn + 1, gxn)] \\ &+\lambda[d(gxn + 1, gxn) + d(gxn, gxn - 1)] \end{aligned} \tag{12}$$

Therefore ,

$$d(gxn, gxn + 1) \leq \alpha 1 +_{-} \gamma^{\gamma} +_{-} \lambda^{\lambda_d}(gxn - 1, gxn) + 1 - \beta_{\gamma} - \lambda^{d}(gyn - 1, gyn). \tag{13}$$

 Similarly $, gyn \preceq gyn-1$ and $gxn \succeq gxn-1$, from (6) and (7) , and we have

$$\begin{aligned} d(gyn+1, gyn) &= d(F(yn, x_n), F(yn-1, x_{n-1})) \\ &\leq \alpha d(gyn, gyn-1) + \beta d(gxn, gxn-1) \\ &+\gamma[d(F(yn, x_n), gyn) + d(F(yn-1, x_{n-1}), gyn-1)] \\ &+\lambda[d(F(yn, x_n), gyn-1) + d(F(yn-1, x_{n-1}), gyn)] \\ \leq \alpha d(gyn, gyn-1) + \beta d(gxn, gxn-1) &+ \gamma[d(gyn+1, gyn) + d(gyn, gyn-1)] \\ &+\lambda d(gyn+1, gyn-1) \\ \leq \alpha d(gyn, gyn-1) + \beta d(gxn, gxn-1) &+ \gamma[d(gyn+1, gyn) + d(gyn, gyn-1)] \\ &+\lambda[d(gyn+1, gyn) + d(gyn, gyn-1)] \end{aligned} \quad (14)$$

Therefore ,

$$d(gyn, gyn+1) \leq \alpha 1 +_{-} \gamma^{\gamma} +_{-} \lambda^{\lambda_d}(gyn-1, gyn) + 1 - \beta_{\gamma} - \lambda^{d}(gxn-1, gxn). \quad (15)$$

From (1 3) and (1 5) , we have

$$d(gxn, gxn+1) + d(gyn, gyn+1) \leq \alpha + 1_{-}\beta\gamma^{+}\gamma_{-} + \lambda\lambda[d(gxn-1, gxn) + d(gyn-1, gyn)]. \quad (16)$$

for all n. Set $k = \alpha + {}^{\beta+\gamma+\lambda}_{1-\gamma-}\lambda < 1$; from (1 6) , we have

$$\begin{aligned} d(gxn, gxn+1) + d(gyn, gyn+1) &\leq k[d(gxn-1, gxn) + d(gyn-1, gyn)] \\ &\leq k^2[d(gxn-2, gxn-1) + d(gyn-2, gyn-1)] \end{aligned}$$

. . .

$$\leq k^n[d(gx0, gx1) + d(gy0, gy1)]$$

This implies

$$d(gxn, gxn+1) \leq k^n[d(gx0, gx1) + d(gy0, gy1)],$$

and

$$d(gyn, gyn+1) \leq k^n[d(gx0, gx1) + d(gy0, gy1)].$$

We shall show that $\{gxn\}$ and $\{gyn\}$ are Cauchy sequences . For $m > n$, we have

$$\begin{aligned} d(gxn, gxm) &\leq d(gxn, gxn+1) + \cdots + d(gxm-1, gxm) \\ \leq k^n[d(gx0, gx1) + d(gy0, gy1)] &+ \cdots + k^{m-1}[d(gx0, gx1) + d(gy0, gy1)] \\ &\leq 1k^n_{-k}[d(gx0, gx1) + d(gy0, gy1)]. \end{aligned}$$

Let $\theta \ll c$ be given . Then there is a neighborhood of θ

$$N_\delta(\theta) = \{y \in E : \| y \| \leq \delta\},$$

 where $\delta > 0$, such that $c+N_\delta(\theta) \subseteq IntP$. Since $k < 1$, choose $N \in \mathbb{N}$ such that

$$\| - 1k^n_{-k}[d(gx0, gx1) + d(gy0, gy1)]\| \quad < \delta.$$

Then

$$-1^{kn}_{-}k[d(gx0, gx1) + d(gy0, gy1)] \in N_\delta(\theta)$$

for all $n > N$. Hence

$$c - 1^{kn}_{-}k[d(gx0, gx1) + d(gy0, gy1)] \in c + N_\delta(\theta) \subseteq IntP.$$

Therefore ,

$$1k^n_{-k}[d(gx0, gx1) + d(gy0, gy1)] \quad \ll c,$$

for all $n > N$. This means ,

$$d(gxn, gxm) \quad \ll c, \quad \text{forall} m > n > N.$$

Hence we conclude that $\{gxn\}$ is a Cauchy sequence . Similarly , one can show that $\{gyn\}$ is also a Cauchy sequence . Since X is a complete cone metric space , there exist $x, y \in X$ such that

$$\lim_{n\to\infty} gxn = x\text{and} \lim_{n\to\infty} gyn = y. \tag{17}$$

Thus

$$\lim_{n\to\infty} F(x_n, yn) = \lim_{n\to\infty} gxn = x\text{and} \lim_{n\to\infty} F(yn, x_n) = \lim_{n\to\infty} gyn = y. \tag{18}$$

Since F and g are compatible , from (1 8) we have

$$\lim_{n\to\infty} d(gF(x_n, yn), F(gxn, gyn)) = \theta \tag{19}$$

and

$$\lim_{n\to\infty} d(gF(yn, x_n), F(gyn, gxn)) = \theta. \tag{20}$$

Now , suppose that assumption (a) holds . Since F, g is continuous , by (1 8) , $gF(x_n, yn) \quad \to gx$ and $F(gxn, gyn) \quad \to F(x, y)$ as $n \to \infty$. Let $\theta \quad \ll c$ be given ; there exists $k \in \mathbb{N}$, such that , for all $n > k$,

$$d(gx, gF(x_n, yn)) \ll \quad c3, \quad d(F(gxn, gyn), F(x, y)) \quad \ll c3$$
$$\text{and} \quad d(gF(x_n, yn), F(gxn, gyn)) \quad \ll c3$$

Therefore ,

$$d(gx, F(x, y)) \le d(gx, gF(x_n, yn)) + d(gF(x_n, yn), F(gxn, gyn))$$
$$+d(F(gxn, gyn), F(x, y)) \quad \ll c$$

 1 3 1 for all $n > k$. Since c is arbitrary , we get

$$d(gx, F(x, y)) \ll m^c, \forall m \in \mathbb{N}$$

Notice that $m^c \to \theta$ as $m \to \infty$, and we conclude that $m^c - d(gx, F(x, y)) \to$
$-d(gx, F(x, y))$ as $m \to \infty$. Since P is closed , we get $-d(gx, F(x, y)) \in P$. Thus $d(gx, F(x, y)) \in P\cap \Rightarrow)notdef{-}periodnotdef$ H n − notdef$_{element-cenotdef-notdefd-notdef}g-notdefnegationslash-notdef-notdef-xcomma-braceleft^{notdef-notdef-F}notdef-xcomma-notdefy_{infinity-parenright})notdef-notdef$equal − notdef − notdef$\theta notdefnotdef$
T h − e refore , $gx = Fx - parenleft, y)$. mi l − a r y − l, w ec n s o wt at g $y = F$parenleft − y, $x)$.
naly , s ppose t at ( ) h lds . nce { xn} i a n n - decreasing s quence d $gn \to x$ a d a $\{yn\}$ i − s a n n −n − ii $g$creasing s quence a d $gn \to y$ w eh ve
$xn \preceq g$ x a d $gyn \succeq g$ y f r a l n nce F a d g a e c m p − atible' a d g i ntinuous , f r om (7) , (1 9) a d (0) w eh ve

$$\mathrm{m}_{\to\infty g} gxn^{=} xg = \mathrm{lm}_{\to\infty} gF^{\text{parenleft}-\text{x}} n, y_n) = \mathrm{lm}_{\to\infty} Fg - parenleftxn'gyn) \quad 1)$$

d

$$\mathrm{m}_{\to\infty g} gyn = yg = \mathrm{lm}_{\to\infty} gF^{\text{parenleft}-\text{y}} n, x_n) = \mathrm{lm}_{\to\infty} Fg - parenleftyn, gxn^{)}. \quad 2)$$

eh ve

$$\begin{aligned}
gx, F\text{parenleft} - \text{x}, y)) &\leq dgx, ggxn + 1) + \quad dggxn + 1, F \quad x - parenleft, y)) \\
&dgx, ggxn + 1) + \quad dgF \quad \text{parenleft} - \text{x}n, y_n), F \quad x - parenleft, y)) \\
&dgx, ggxn + 1) + \quad dF \quad \text{parenleft} - \text{g}xn'gyn), F \quad x - parenleft, y)) \\
&dgx, ggxn + 1) + \quad \alpha(ggxn'gx) + \quad \beta(ggyn, gy) \\
&\gamma d(F \quad \text{parenleft} - \text{g}xn'gyn), ggxn^{)} + \quad dF \quad x - parenleft, y), gx)] \\
&\lambda d(F \quad \text{parenleft} - \text{g}xn'gyn), gx) + \quad dF \quad x - parenleft, y), ggxn^{)}] \\
&dgx, ggxn + 1) + \quad \alpha(ggxn'gx) + \quad \beta(ggyn, gy) \\
&\gamma d(F \quad \text{parenleft} - \text{g}xn'gyn), ggxn^{)} + \quad dF \quad x - parenleft, y), gx)] \\
\lambda d(F \quad \text{parenleft} - \text{g}xn'gyn), gx) &+ \quad dF \quad x - parenleft, y), gx) + \quad dgx, ggxn^{)}].
\end{aligned}$$

h − i sim l − p e s

$$\begin{aligned}
gx, F\text{parenleft} - \text{x}, y)) \leq \quad & 11_{-\gamma-\lambda}(d(gx, ggxn + 1) + \alpha d(ggxn, gx) + \beta d(ggyn, gy) \\
& +\gamma d(F(gxn, gyn), ggxn) + \lambda[d(F(gxn, gyn), gx) + d(gx, ggxn)])
\end{aligned} \tag{23}$$

Let $\theta \ll c$. By (2 1) , (22) , there exist $n_0 \in \mathbb{N}$ such that

$$d(gx, ggxn) \ll \quad 1 +^{c} (^{1}_{\alpha} - + \gamma\beta - + \lambda^{)}_{\gamma+} 2\lambda, \quad d(ggyn, gy) \quad \ll 1 +^{c} (^{1}_{\alpha} - + \gamma\beta - + \lambda^{)}_{\gamma+} 2\lambda,$$

$$d(F(gxn, gyn), ggxn) \quad \ll 1 +^{c} (^{1}_{\alpha} - + \gamma\beta - + \lambda^{)}_{\gamma+} 2\lambda$$

$$\text{and} \quad d(F(gxn, gyn), gx) \quad \ll 1 +^{c} (^{1}_{\alpha} - + \gamma\beta - + \lambda^{)}_{\gamma+} 2\lambda$$

 for all $n > n_0$. Thus , from (23) , we have $d(gx, F(x,y)) \ll c$ for all $n > n_0$.

$$\text{Therefore}, gx = F(x,y).$$

Similarly , one can show that $gy = F(y,x)$. Thus we have proved that F and g have a coupled coincidence point .

Corollary 2 1 . *Let $(X, \preceq)$ be a partially ordered s e t and suppose there is a metric d such that (X,d) is a complete cone metric space . Let $F : X \times X \to X$ is such F has the mixed monotone property and there exist non - negative constants α, β, γ and λ satisfying $\alpha + \beta + 2\gamma + 2\lambda < 1$ such that*

$$d(F(x,y), F(u,v)) \leq \alpha d(x,u) + \beta d(y,v) + \gamma[d(F(x,y),x) + d(F(u,v),u)] + \lambda[d(F(x,y),u) + d(F(u,v),x)] \quad (24)$$

for al l $x, y, u, v \in X$ with $x \preceq u$ and $y \succeq v$. Suppose e ither

(a)F is continuous or

(b)X has th e fo l lowing property :

(i) if $\{x_n\}$ is a non - decreasing s equence and $\lim_{n\to\infty} x_n = x$ then $x_n \preceq x$ for al l n,

(ii) if $\{yn\}$ is a non - in creasing s equence and $\lim_{n\to\infty} yn = y$ th en $y \preceq yn$ for al l n.

If there exist $x_0, y0 \in X$ such that $x_0 \preceq F(x_0, y0)$ and $y0 \succeq F(y0, x_0)$ then F has a coupled fixed po int .

Remark 2 2 . Theorems 2 . 2 and 2 . 3 in [1 4] , Theorems 2 . 1 and 2 . 2 in [1 0] are special cases of Corollary 22 .

Theorem 23 . *In addition to the hypotheses of Theorem 20 , suppose that for every (x,y), $(z,t) \in X \times X$ th ere exists a $(u,v) \in X \times X$ such that (gu, gv) is comparable to $(gx, gy))$ and (gz, gt). Then F and g have a unique coupled common fixed point .*

Proof . Suppose (x,y) and (z,t) are coupled coincidence points of F and g, that is , $gx = F(x,y), gy = F(y,x), gz = F(z,t)$ and $gt = F(t,z)$. We shall show that $gx = gz$ and $gy = gt$. By the assumption , there exists $(u,v) \in X \times X$ that (gu, gv) is comparable to (gx, gy) and (gz, gt).

Since $F(X \times X) \subseteq g(X)$, we define sequences $\{u_n\}, \{v_n\}$ as follows

$$u_0 = u, v_0 = v, gun + 1 = F(u_n, v_n) \quad \text{and} gvn + 1 = F(v_n, u_n),$$

for all n. Since (gu, gv) is comparable with (gx, gy), we may assume that $(gx, gy) \preceq$

$$(gu, gv) = (gu0, gv0).$$

By using the mathematical induction , it is easy to prove that

$$(gx, gy) \preceq (gun, gvn), \quad \text{forall} n. \quad (25)$$

From (6) and (25) , we have

$$d(gx, gun + 1) = d(F(x, y), F(u_n, v_n))$$

$$\leq \alpha d(gx, gun) + \beta d(gy, gvn) + \gamma[d(F(x, y), gx) + d(F(u_n, v_n), gun)]$$
$$+\lambda[d(F(x, y), gun) + d(F(u_n, v_n), gx)]$$
$$+\lambda[d(gx, gun) + d(gun + 1, gx)]$$
$$\leq \alpha d(gx, gun) + \beta d(gy, gvn) + \gamma[d(gun + 1, gx) + d(gx, gun)]$$
$$+\lambda[d(gx, gun) + d(gun + 1, gx)].$$

This implies

$$d(gx, gun + 1) \leq \alpha 1 +_- \gamma^\gamma +_- \lambda^{\lambda_d}(gx, gun) + 1 - \beta_\gamma - \lambda^d(gy, gvn). \tag{26}$$

Similarly , from (6) and (25) , we also have

$$d(gy, gvn + 1) \leq \alpha 1 +_- \gamma^\gamma +_- \lambda^{\lambda_d}(gy, gvn) + 1 - \beta_\gamma - \lambda^d(gx, gun). \tag{27}$$

Summing up (26) and (27) , we obtain

$$d(gx, gun + 1) + d(gy, gvn + 1) \leq \alpha + 1_- \beta\gamma^+\gamma_- + \lambda\lambda[d(gx, gun) + d(gy, gvn)]$$
$$\leq k^2[d(gx, gun - 2) + d(gy, gvn - 2)]$$

. . .

$$\leq k^{n+1}[d(gx, gu0) + d(gy, gv0)]$$

where $k = \alpha + {}^{\beta+\gamma+\lambda}_{1-\gamma-}\lambda < 1$. This implies

$$d(gx, gun + 1) \leq k^{n+1}[d(gx, gu0) + d(gy, gv0)],$$

for all n. Let $\theta \ll c$ be given . Then there is a neighborhood of θ

$$N_\delta(\theta) = \{y \in E :\| y \| \leq \delta\},$$

where $\delta > 0$, such that $c + N_\delta(\theta) \subseteq IntP$. Since $k < 1$, there is an $N_1 \in \mathbb{N}$ such that

$$\| - k^{n+1}[d(gx, gu0) + d(gy, gv0)]\| < \delta.$$

Then

$$-k^{n+1}[d(gx, gu0) + d(gy, gv0)] \in N_\delta(\theta).$$

for all $n > N_1$. Hence

$$c - k^{n+1}[d(gx, gu0) + d(gy, gv0)] \in c + N_\delta(\theta) \subseteq IntP.$$

Therefore ,

$$k^{n+1}[d(gx, gu0) + d(gy, gv0)] \ll c,$$

for all $n > N_1$. That means $d(gx, gun + 1) \ll c$, for all $n > N_1$. Thus , $gun \to gx$ as $n \to \infty$. Similarly , one can show that $gvn \to gy, gun \to gz$ and $gvn \to gt$ as $n \to \infty$. By the uniqueness of limits , we have $gx = gz$ and $gy = gt$.

Since $gx = F(x, y)$ and $gy = F(y, x)$, by the compatibility of F and g , it is easy to find that

$$ggx = gF(x, y) = F(gx, gy) \quad \text{and} ggy = gF(y, x) = F(gy, gx).$$

Denote $gx = p$ and $gy = q$, then $gp = F(p, q), gq = F(q, p)$. Thus (p, q) is a coupled coincidence of F and g. Hence $gx = gp$ and $gy = gq$. Therefore ,

$$p = gp = F(p, q) \quad \text{and} q = gq = F(q, p).$$

This means that (p, q) is a coupled common fixed point of F and g .

Suppose (a, b) is another coupled common fixed point of F and g. Then from the previous argument , $p = gp = ga = a$ and $q = gq = gb = b$.

We end the paper with a simple example which can be applied to Theorem 20 but not to Theorem 1 5 .

Example 2 0 . Let $X = \mathbb{R}, E = C^1_{\mathbb{R}}[0, 1]$ and $P = \{\phi \in E : \phi \geq 0\}$. Define $d : X \times X \to E$ by $d(x, y) =| x - y | \phi$ for all $x, y \in X$, where $\phi : [0, 1] \to \mathbb{R}$ such that $\phi(t) = e^t$. It is clear that (X, d) is a complete cone metric space . On the set X, we consider the following order relation

$$x, y \in X, \quad x \preceq y \quad \Leftrightarrow \quad x = y \text{or} \quad (x, y) = (0, 1).$$

Let $F : X \times X \to X$ be given by

$$F(x, y) = \begin{cases} 1, & \text{if} x, y \text{arecomparable}, \\ 0, & \text{otherwise}. \end{cases}$$

and $g : X \to X$ be given by $gx = x$, for all $x \in X$ It is easy to see that all the conditions of Theorem 2 . 5 are satisfied with $\alpha, \beta, \gamma, \delta \geq 0$ and $\alpha + \beta + 2\gamma + 2\delta < 1$. Moreover , (1 , 1) is a coupled coincidence point of F and g.

However , the condition (5) in Theorem 1 5 is not satisfied . In fact , suppose (5) holds . Take $x = 1, y = 0, u = 1/2$ and $v = 0$; we have

$$\begin{aligned} \phi &= d(F(1, 0), F(1/2, 0)) \\ &= d(F(x, y), F(u, v)) \\ &\leq a_1 d(gx, gu) + a_2 d(F(x, y), gx) + a_3 d(gy, gv) \\ &+ a_4 d(F(u, v), gu) + a_5 d(F(x, y), gu) + a_6 d(F(u, v), gx) \\ &= a_1 d(g1, g1/2) + a_2 d(F(1, 0), g1) + a_3 d(g0, g0) \\ &+ a_4 d(F(1/2, 0), g1/2) + a_5 d(F(1, 0), g1/2) + a_6 d(F(1/2, 0), g1) \\ &= 1/2 a_1 \phi + 1/2 a_4 \phi + 1/2 a_5 \phi + a_6 \phi \\ &< \phi, \end{aligned}$$

which is a contradiction . Thus we cannot apply Theorem 1 5 to this example .

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