MATEMATIQKI VESNIK UDK 5 1 7 . 524
59 (2007) , 65 – 73 $original ninauqni$ research$_{\text{paper}}$ rad

SOME SUBCLASSES OF CLOSE - TO - CONVEX AND QUASI - CONVEX FUNCTIONS

Zhi - Gang Wang

Abstract . In the present paper , the author introduce two new subclasses $s_{\mathcal{S}}^{(k)}{}_c(\alpha,\beta,\gamma)$ of close - to - convex functions and $s_{\mathcal{C}}^{(k)}{}_c(\alpha,\beta,\gamma)$ of quasi - convex functions with respect to $2k-$ symmetric conjugate points . The coefficient inequalities and integral representations for functions belonging to these classes are provided , the inclusion relationships and convolution conditions for these classes are also provided .

1 . Introduction

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disk $\mathcal{U} = \{z \in \mathbf{C} : |z| < 1\}$. Let $\mathcal{S}$, $\mathcal{S}^*$, $\mathcal{K}, \mathcal{C}$ and $\mathcal{C}^*$ denote the familiar subclasses of $\mathcal{A}$ consisting of functions which are univalent , starlike , convex , close - to - convex and quasi - convex in $\mathcal{U}$, respectively (see , for details , [2 , 4 , 6 , 7 , 8] .

Al - Amiri , Coman and Mocanu [1] once introduced and investigated a class $s_{\mathcal{S}}^{(k)}{}_c$ of functions st arlike with respect to $2k-$ symmetric conjugate points , which satisfy the following inequality

$$\Re\left\{ \frac{zf'(z)}{f2k(z)} \right\} > 0 \quad (z \in \mathcal{U}),$$

where $k \geq 2$ is a fixed positive integer and $f2k(z)$ is defined by the following equality

$$f2k(z) = 2^1 k \sum_{k-1}^{\nu=0} [\varepsilon^{-\nu} f(\varepsilon^{\nu} z) + \varepsilon^{\nu} f(\varepsilon^{\nu} z)] \quad (\varepsilon = \exp(2\pi i/k); \quad z \in \mathcal{U}). \tag{1.2}$$

In the present paper , we introduce and investigate the following two more gen - eralized subclasses $s_{\mathcal{S}}^{(k)}{}_c(\alpha,\beta,\gamma)$ and $\mathcal{C}_{sc}^{(k)}(\alpha,\beta,\gamma)$ of $\mathcal{A}$ with respect to $2k-$ symmetric conjugate points , and obtain some interesting results .

AMS Subject Classification : 30 C 45 .

Keywords and phrases : Close - to - convex functions , quasi - convex functions , differential sub - ordination , Hadamard product , $2k-$ symmetric conjugate points . .

DEFINITION 1 . Let $s_{\mathcal{S}}^{(k)}{}_c(\alpha,\beta,\gamma)$ denote the class of functions $f(z)$ in $\mathcal{A}$ satis - fying the following inequality

$$\left| \begin{matrix} zf'f_{2k(z)}^{(z)} & -1 \\ \beta_{f2k(z)}^{zf'(z)} + (1-\gamma) & \end{matrix} \right| < 1-\alpha, \tag{1.3}$$

where $0 \leq \alpha < 1, 0 \leq \beta \leq 1, 0 \leq \gamma < 1$ and $f2k(z)$ is defined by equality (1 . 2) . And a function $f(z) \in \mathcal{A}$ is in the class $\mathcal{C}_{sc}^{(k)}(\alpha,\beta,\gamma)$ if and only if $zf'(z) \in s_{\mathcal{S}}^{(k)}{}_c(\alpha,\beta,\gamma)$.

Note that $s_{\mathcal{S}}^{(k)}{}_c(0,1,0) = s_{\mathcal{S}}^{(k)}{}_c$, so the class $s_{\mathcal{S}}^{(k)}{}_c(\alpha,\beta,\gamma)$ is a generalization of the class $s_{\mathcal{S}}^{(k)}{}_c$.

In our proposed investigation of the classes $s_{\mathcal{S}}^{(k)}{}_c(\alpha,\beta,\gamma)$ and $\mathcal{C}_{sc}^{(k)}(\alpha,\beta,\gamma)$, we shall also make use of the following lemmas .

LEMMA 1 . [3] *Let* $H(z) = 1 + \sum_{n=1}^{\infty} h_n z^n$ *be analytic in* $\mathcal{U}, 0 \leq \alpha < 1,$ $0 \leq \beta \leq 1$ *and* $0 \leq \gamma < 1$, *then the inequality*

$$\left| \begin{matrix} H(z) - 1 \\ \beta H(z) + (1-\gamma) \end{matrix} \right| < 1-\alpha \quad (z \in \mathcal{U})$$

can be written as

$$H(z) \prec 1 + 1(_{-}1 - (_1\alpha_{-})(^{1-}_{\alpha)}{}_{\beta z}\gamma)z \quad (z \in \mathcal{U}),$$

w h − quotedblright$_{es-re-t}$ " n$_{bracketright-d-parenright}$ sT J − or − slash$_{Ft-one}$zero − h$_{e-suppress-nine}$nine − u − period s six − ua − suppress − Tl − f$_{s-suppress-three}$period − b − two o r − fivesuppress − zero − di − suppress − Tn − Dt − bracketleft − aparenleft − i$_{OE-on}$.

2. *L − e t* $0 \leq$ $\alpha <$ $,_1 0 \leq \beta \leq \frac{1}{a}$ *nd*$_0 \leq \gamma < 1$, *h n w e hav e*

$$\beta,\gamma) \subset \mathcal{C} \subset \quad .$$

Supp o ethat $fz) \in \mathcal{S}(c)$ α $\beta,\gamma)$, b y Le m ma 1, w − e n o wt h t − a t e n (1 . 3) c n^{s} b e wr i − te − t$_{n}$ as

$$f2k(z) \prec 1 + 1(_{-}1 - (_1\alpha_{-})(^{1-}_{\alpha)}{}_{\beta z}\gamma)z \quad (z \in \mathcal{U}). \tag{1.4}$$

Thus we have

$$\Re\left\{ \begin{matrix} zf'(z) \\ f2k(z) \end{matrix} \right\} > 0 \quad (z \in \mathcal{U}) \tag{1.5}$$

since

$$\Re\left\{ \begin{matrix} 1 + (1-\alpha)(1-\gamma)z \\ 1 - (1-\alpha)\beta z \end{matrix} \right\} > 0 \quad (z \in \mathcal{U}).$$

Now it suffices to show that $f2k(z) \in \mathcal{S}^* \subset \mathcal{S}$. Substituting z by $\varepsilon^{\mu} z$ $(\mu = 0,1,2,...,k-1)$ in (1 . 5) , then (1 . 5) is also true , that is ,

$$\Re\left\{ \begin{matrix} \varepsilon^{\mu} z f'(\varepsilon^{\mu} z) \\ f2k(\varepsilon^{\mu} z) \end{matrix} \right\} > 0 \quad (z \in \mathcal{U}). \tag{1.6}$$

 From inequality (1 . 6) , we have

$$\Re\left\{\begin{array}{c}\varepsilon^{\mu} z f'(\varepsilon^{\mu} z)\\ f2k(\varepsilon^{\mu} z)\end{array}\right\} > 0 \quad (z \in \mathcal{U}). \tag{1.7}$$

Note that $f2k(\varepsilon^{\mu} z) = \varepsilon^{\mu} f2k(z)$ and $f2k(\varepsilon^{\mu} z) = \varepsilon^{-\mu} f2k(z)$, then inequalities (1 . 6) and (1 . 7) can be written as

$$\Re\left\{\begin{array}{c} z f'(\varepsilon^{\mu} z)\\ f2k(z)\end{array}\right\} > 0 \quad (z \in \mathcal{U}), \tag{1.8}$$

and

$$\Re\left\{\begin{array}{c} z f'(\varepsilon^{\mu} z)\\ f2k(z)\end{array}\right\} > 0 \quad (z \in \mathcal{U}). \tag{1.9}$$

Summing inequalities (1 . 8) and (1 . 9) , we can get

$$\Re braceleftmid - braceleftbtz(f'(\varepsilon \mu z) f^{+}_{2k(z)} f'(\varepsilon^{\mu} z)) bracerightbt - bracerightmid > 0 \quad (z \in \mathcal{U}). \tag{1.10}$$

Letting $\mu = 0, 1, 2, ..., k-1$ in (1 . 1 0) , respectively , and summing them we can get

$$\Re braceex - braceex - braceex - braceex - braceleftmid - braceex - braceex - braceex - braceex - braceleftbtz[^{1}_{2k} \sum_{k-1}^{\mu=0} (f$$

or equivalently ,

$$\Re\left\{\begin{array}{c} z f'_{2k}(z)\\ f2k(z)\end{array}\right\} > 0 \quad (z \in \mathcal{U}),$$

that is $f2k(z) \in \mathcal{S}^{*} \subset \mathcal{S}$. This means that $s^{(k)}_{\mathcal{S}}{}_{c}(\alpha, \beta, \gamma) \subset \mathcal{C} \subset \mathcal{S}$, hence the proof of Lemma 2 is complete .

Similarly , for the class $\mathcal{C}^{(k)}_{sc}(\alpha, \beta, \gamma)$, we have

Lemma 3 . *Let* $0 \leq \alpha < 1, \quad 0 \leq \beta \leq 1$ *and* $0 \leq \gamma < 1$, *then we have*

$$\mathcal{C}^{(k)}_{sc}(\alpha, \beta, \gamma) \subset \mathcal{C}^{*} \subset \mathcal{C}.$$

Lemma 4 . [5] *Let* $-1 \leq B_2 \leq B_1 < A_1 \leq A_2 \leq 1$, *then we have*

$$1^{1} +^{+} A^{1^{z}}_{B_1 z} \prec 1^{1} +^{+} A^{2^{z}}_{B_2 z}.$$

In the present paper , we shall provide the coefficient inequalities and integral representations for functions belonging to the classes $s^{(k)}_{\mathcal{S}}{}_{c}(\alpha, \beta, \gamma)$ and $\mathcal{C}^{(k)}_{sc}(\alpha, \beta, \gamma)$, we shall also provide the inclusion relationships and convolution conditions for these classes .

2 . Inclusion relationships

We first give some inclusion relationships for the classes $s_{\mathcal{S}}^{(k)}{}_c(\alpha,\beta,\gamma)$ and

$$\mathcal{C}_{sc}^{(k)}(\alpha,\beta,\gamma).$$

THEOREM 1 . *Let* $0 \le \beta 2 \le \beta 1 \le 1,\quad 0 \le \alpha_1 \le \alpha_2 < 1$ *and* $0 \le \gamma 1 \le \gamma 2 < 1,$ *then we have*

$$s_{\mathcal{S}}^{(k)}{}_c(\alpha_2,\beta 2,\gamma 2) \subset s_{\mathcal{S}}^{(k)}{}_c(\alpha_1,\beta 1,\gamma 1).$$

Proof . Suppose that $f(z) \in s_{\mathcal{S}}^{(k)}{}_c(\alpha_2,\beta 2,\gamma 2)$, by (1 . 4) , we have

$$f^{zf\prime(z)}{}_{2k(z)} \prec 1 + 1(_{-}1 - ({}_1\alpha_{-}^{2})_{\alpha_2)}^{(1\ -\gamma 2)z}{}_{\beta 2}\ z.$$

Since $0 \le \alpha_1 \le \alpha_2 < 1,\quad 0 \le \beta 2 \le \beta 1 \le 1$ and $0 \le \gamma 1 \le \gamma 2 < 1$, then we have

$$-1 \le -(1-\alpha_1)\beta 1 \le -(1-\alpha_2)\beta 2 < (1-\alpha_2)(1-\gamma 2) \le (1-\alpha_1)(1-\gamma 1) \le 1.$$

Thus , by Lemma 4 , we have

$$f^{zf\prime(z)}{}_{2k(z)} \prec 1 + 1(_{-}1 - ({}_1\alpha_{-}^{2})_{\alpha_2)}^{(1\ -\gamma 2)z}{}_{\beta 2}\ z \prec 1 + 1(_{-}1 - ({}_1\alpha_{-}^{1})_{\alpha\ 1)\beta 1}^{(1-\gamma 1)z}\ z,$$

that is $f(z) \in s_{\mathcal{S}}^{(k)}{}_c(\alpha_1,\beta 1,\gamma 1)$. This means that $s_{\mathcal{S}}^{(k)}{}_c(\alpha_2,\beta 2,\gamma 2) \subset s_{\mathcal{S}}^{(k)}{}_c(\alpha_1,\beta 1,\gamma 1)$.

Similarly , for the class $\mathcal{C}_{sc}^{(k)}(\alpha,\beta,\gamma)$, we have

COROLLARY 1 . *Let* $0 \le \beta 2 \le \beta 1 \le 1,\quad 0 \le \alpha_1 \le \alpha_2 < 1$ *and* $0 \le \gamma 1 \le \gamma 2 < 1,$ *then we have*

$$\mathcal{C}_{sc}^{(k)}(\alpha_2,\beta 2,\gamma 2) \subset \mathcal{C}_{sc}^{(k)}(\alpha_1,\beta 1,\gamma 1).$$

3 . Coefficient inequalities

In this section , we give some coefficient inequalities for functions belonging to the classes $s_{\mathcal{S}}^{(k)}{}_c(\alpha,\beta,\gamma)$ and $\mathcal{C}_{sc}^{(k)}(\alpha,\beta,\gamma)$.

THEOREM 2 . *Let* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ *be analytic in* $\mathcal{U}$, *if for* $0 \le \alpha <$
$1,\quad 0 \le \beta \le 1$ *and* $0 \le \gamma < 1,$ *we have*

$$\sum_{n=2}^{\infty} n[1+(1-\alpha)\beta] \mid a_n \mid + \sum_{l=1}^{\infty}[(1-\alpha)(1-\gamma)+1] \mid \Re(a_{lk+1}) \mid \le (1-\alpha)(1+\beta-\gamma), \tag{3.1}$$

then $f(z) \in s_{\mathcal{S}}^{(k)}{}_c(\alpha,\beta,\gamma).$

Proof . Suppose that $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, and $f2k(z)$ is defined by equality (1 . 2) . We now let M be denoted by

$$M := \mid zf'(z) - f2k(z) \mid -(1-\alpha) \mid \beta z f'(z) + (1-\gamma) f2k(z) \mid$$

$$= \left| \sum_{n=2}^{\infty} n a_n z^n - \sum_{n=2}^{\infty} \Re(a_n) c_n z^n \right|$$

$$-(1-\alpha)|\beta \left(z + \sum_{n=2}^{\infty} n a_n z^n \right) + (1-\gamma) \left(z + \sum_{n=2}^{\infty} \Re(a_n) c_n z^n \right)|,$$

 where

$$c_n = k^1 \sum_{k-1}^{\nu=0} \varepsilon^{(n-1)\nu} = \{0^1_, \quad n^n = \neq lk^{lk} +^+ 1^1_. , (\varepsilon = \exp(2\pi i/k); \quad l \in \mathbf{N} = \{1, 2, ...\}). \tag{3.2}$$

Thus , for $| z |= r < 1$, we have

$$M \leq \sum_{n=2}^{\infty} (n \mid a_n \mid + \mid \Re(a_n) \mid c_n) r^n$$

$$-(1-\alpha)\left[(1+\beta-\gamma)r - \sum_{n=2}^{\infty}[n\beta \mid a_n \mid +(1-\gamma) \mid \Re(a_n) \mid c_n] r^n\right]$$

$$< \left(\sum_{n=2}^{\infty}\{n[1+(1-\alpha)\beta] \mid a_n \mid +[(1-\alpha)(1-\gamma)+1] \mid \Re(a_n) \mid c_n\} -(1-\alpha)(1+\beta-\gamma)\right) r$$

$$< \sum_{n=2}^{\infty}\{n[1+(1-\alpha)\beta] \mid a_n \mid +[(1-\alpha)(1-\gamma)+1] \mid \Re(a_n) \mid c_n\} - (1-\alpha)(1+\beta-\gamma)$$

$$= \sum_{n=2}^{\infty} n[1+(1-\alpha)\beta] \mid a_n \mid + \sum_{l=1}^{\infty}[(1-\alpha)(1-\gamma)+1] \mid \Re(a_{lk+1}) \mid -(1-\alpha)(1+\beta-\gamma).$$

From inequality (3 . 1) , we know that $M < 0$, thus we can get inequality (1 . 3) , that is $f(z) \in s_{\mathcal{S}}^{(k)}{}_c(\alpha, \beta, \gamma)$. This completes the proof of Theorem 2 .

Similarly , for the class $\mathcal{C}_{sc}^{(k)}(\alpha, \beta, \gamma)$, we have

Corollary 2 . *Let* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ *be analytic in* $\mathcal{U}$, *if for* $0 \leq \alpha < 1$, $0 \leq \beta \leq 1$ *and* $0 \leq \gamma < 1$, *we have*

$$\sum_{n=2}^{\infty} n^2[1+(1-\alpha)\beta] \mid a_n \mid + \sum_{l=1}^{\infty}[(1-\alpha)(1-\gamma)+1](lk+1) \mid \Re(a_{lk+1}) \mid \leq (1-\alpha)(1+\beta-\gamma),$$

then $f(z) \in \mathcal{C}_{sc}^{(k)}(\alpha, \beta, \gamma)$.

4 . Integral representations

In this section , we provide the integral representations for functions belonging to the classes $s_{\mathcal{S}}^{(k)}{}_c(\alpha, \beta, \gamma)$ and $\mathcal{C}_{sc}^{(k)}(\alpha, \beta, \gamma)$.

THEOREM 3 . *Let* $f(z) \in s_{\mathcal{S}}^{(k)}{}_{c}(\alpha,\beta,\gamma)$, *then we have*

$$f2k(z) = z \cdot \exp\{2^1 k \sum_{k-1}^{\mu=0} \int_0^z (1-\alpha)(1+\zeta\beta-\gamma)\times$$

$$\times \quad [_1-(1\omega_-(\varepsilon_{\alpha)}^{\mu\zeta)}{}_{\beta\omega(\varepsilon^\mu\zeta)} + 1 - (1\omega_-(\varepsilon_{\alpha)}^{\mu\zeta)}{}_{\beta\omega(\varepsilon^\mu\zeta)}]d\zeta\}, \quad (4.1)$$

where $f2k(z)$ *is defined by equality* (1.2), $\omega(z)$ *is analytic in* $\mathcal{U}$ *and* $\omega(0) = 0$,

$$\mid \omega(z) \mid < 1.$$

Proof . Suppose that $f(z) \in s_{\mathcal{S}}^{(k)}{}_c(\alpha,\beta,\gamma)$, by (1 . 4) , we have

$$f^{zf\prime(z)}{}_{2k(z)} = 1 + 1(_-1 - (_1\alpha_-)(^{1-\gamma)\omega(z)}_{\alpha)\ \beta\omega}{}_{(z)}, \tag{4.2}$$

where $\omega(z)$ is analytic in $\mathcal{U}$ and $\omega(0) = 0, |\ \omega(z)\ | < 1$. Substituting z by $\varepsilon^{\mu} z(\mu = 0, 1, 2, ..., k-1)$ in (4 . 2) , we have

$$\varepsilon\mu z f' f^{(\varepsilon^{\mu} z)}_{2k(\varepsilon^{\mu} z)} = 1 + 1(_-1 - (_1\alpha_-)(^{1-\gamma)\omega(\varepsilon^{\mu} z)}_{\alpha)\ \beta\omega}{}_{(\varepsilon^{\mu} z)}. \tag{4.3}$$

From equality (4 . 3) , we have

$$\varepsilon\mu z f^{f'(\varepsilon^{\mu} z)}_{2k}{}_{(\varepsilon^{\mu} z)} = 1 + 1(_-1 - (_1\alpha_-)(^{1-\gamma)\omega(\varepsilon^{\mu} z)}_{\alpha)\ \beta\omega}{}_{(\varepsilon^{\mu} z)}. \tag{4.4}$$

Summing equalities (4 . 3) and (4 . 4) , and making use of the same method as in Lemma 2 , we have

$$z f'_{f2^2{}_k^k(z)}{}^{(z)} = 2^1 k \sum_{k-1}^{\mu=0} [^1 + 1(_-1 - (_1\alpha_-)(^{1-\gamma)\omega(\varepsilon^{\mu} z)}_{\alpha)\ \beta\omega}{}_{(\varepsilon^{\mu} z)} + 1 + 1(_-1 - (_1\alpha_-)(^{1-\gamma)\omega(\varepsilon^{\mu} z)}_{\alpha)\ \beta\omega}{}_{(\varepsilon^{\mu} z)}], \tag{4.5}$$

from equality (4 . 5) , we can get

$$\begin{aligned} 2k^{2k}_{ff'}{}^{(z)}_{(z)} - 1_z = 2^1 k \sum_{k-1}^{\mu=0} 1_z\{[1 + 1(_-1 - (_1\alpha_-)(^{1-\gamma)\omega(\varepsilon^{\mu} z)}_{\alpha)\ \beta\omega}{}_{(\varepsilon^{\mu} z)} + \\ + \quad 1 + 1(_-1 - (_1\alpha_-)(^{1-\gamma)\omega(\varepsilon^{\mu} z)}_{\alpha)\ \beta\omega}{}_{(\varepsilon^{\mu} z)}] \quad - 2\}. \end{aligned} \tag{4.6}$$

Integrating equality (4 . 6) , we have

$$\begin{aligned} \log \left\{ \begin{matrix} f2k(z) \\ z \end{matrix} \right\} = 2^1 k \sum_{k-1}^{\mu=0} \int_0^z (1-\alpha)(1+\zeta\beta-\gamma) \times \\ \times \quad [_1 - (1\omega_-(\varepsilon^{\mu\zeta}_{\alpha)}{}_{\beta\omega(\varepsilon^{\mu}\zeta)} + 1 - (1\omega_-(\varepsilon^{\mu\zeta}_{\alpha)}{}_{\beta\omega(\varepsilon^{\mu}\zeta)}]d\zeta. \end{aligned} \tag{4.7}$$

From equality (4 . 7) , we can get equality (4 . 1) easily . This completes the proof of Theorem 3 .

Theorem 4 . *Let* $f(z) \in s_{\mathcal{S}}^{(k)}{}_c(\alpha,\beta,\gamma)$, *then we have*

$$\begin{aligned} f(z) = \int_0^z \exp\{2^1 k \sum_{k-1}^{\mu=0} \int_0^{\xi} (1-\alpha)(1+\zeta\beta-\gamma) \left[\begin{matrix} \omega(\varepsilon^{\mu}\zeta) \\ 1-(1-\alpha)\beta\omega(\varepsilon^{\mu}\zeta) \end{matrix} \right. \\ +1 - (1\omega_-(\varepsilon^{\mu\zeta}_{\alpha)}{}_{\beta\omega(\varepsilon^{\mu}\zeta)}]d\zeta\} \cdot 1 + 1(_-1 - (_1\alpha_-)(^{1-\gamma)\omega(\xi)}_{\alpha)\ \beta\omega}{}_{(\xi)} d\xi, \end{aligned} \tag{4.8}$$

where $\omega(z)$ *is analytic in* $\mathcal{U}$ *and* $\omega(0) = 0, |\ \omega(z)\ | < 1$.

 Proof . Suppose that $f(z) \in s_{\mathcal{S}}^{(k)}{}_{c}(\alpha,\beta,\gamma)$, from equalities (4 . 1) and (4 . 2) , we can get

$$f'(z) = f2k_{z}^{(z)} \cdot 1 + 1(_{-}1 - (_{1}\alpha_{-})(^{1-\gamma)\omega(z)}_{\alpha)\ \beta\omega}{}_{(z)}$$

$$= \exp\{2^{1}k \sum_{k-1}^{\mu=0} \int_{0}^{z} (1-\alpha)(1+\zeta\beta-\gamma) \left[\begin{array}{l} \omega(\varepsilon^{\mu}\zeta) \\ 1-(1-\alpha)\beta\omega(\varepsilon^{\mu}\zeta) \end{array} \right.$$

$$+1 - (1\omega_{-}(\varepsilon^{\mu\zeta)}_{\alpha)}{}_{\beta\omega(\varepsilon^{\mu}\zeta)}]d\zeta \left.\right\} \begin{array}{l} 1+(1-\alpha)(1-\gamma)\omega(z) \\ \cdot \quad 1-(1-\alpha)\beta\omega(z) \end{array} .$$

Integrating the above equality , we can get equality (4 . 8) easily . Hence the proof of Theorem 4 is complete .

Similarly , for the class $\mathcal{C}_{sc}^{(k)}(\alpha,\beta,\gamma)$, we have

COROLLARY 3 . *Let* $f(z) \in \mathcal{C}_{sc}^{(k)}(\alpha,\beta,\gamma)$, *then we have*

$$f2k(z) = \int_{0}^{z} \exp\{2^{1}k \sum_{k-1}^{\mu=0} \int_{0}^{\xi} (1-\alpha)(1+\zeta\beta-\gamma) \times$$

$$\times \quad [_{1} - (1\omega_{-}(\varepsilon^{\mu\zeta)}_{\alpha)}{}_{\beta\omega(\varepsilon^{\mu}\zeta)} + 1 - (1\omega_{-}(\varepsilon^{\mu\zeta)}_{\alpha)}{}_{\beta\omega(\varepsilon^{\mu}\zeta)}]d\zeta\}d\xi,$$

where $f2k(z)$ *is defined by equality* (1.2), $\omega(z)$ *is analytic in* $\mathcal{U}$ *and* $\omega(0) = 0,$

$$|\ \omega(z)\ | < 1.$$

COROLLARY 4 . *Let* $f(z) \in \mathcal{C}_{sc}^{(k)}(\alpha,\beta,\gamma)$, *then we have*

$$f(z) = \int_{0}^{z} 1_{t} \int_{0}^{t} \exp\{2^{1}k \sum_{k-1}^{\mu=0} \int_{0}^{\xi} (1-\alpha)(1+\zeta\beta-\gamma) \left[\begin{array}{l} \omega(\varepsilon^{\mu}\zeta) \\ 1-(1-\alpha)\beta\omega(\varepsilon^{\mu}\zeta) \end{array} \right.$$

$$+1 - (1\omega_{-}(\varepsilon^{\mu\zeta)}_{\alpha)}{}_{\beta\omega(\varepsilon^{\mu}\zeta)}]d\zeta\} \cdot 1 + 1(_{-}1 - (_{1}\alpha_{-})(^{1-\gamma)\omega(\xi)}_{\alpha)\ \beta\omega}{}_{(\xi)}d\xi dt,$$

where $\omega(z)$ *is analytic in* $\mathcal{U}$ *and* $\omega(0) = 0, |\ \omega(z)\ | < 1.$

5 . Convolution conditions

Finally , we provide the convolution conditions for the classes $s_{\mathcal{S}}^{(k)}{}_{c}(\alpha,\beta,\gamma)$ and $\mathcal{C}_{sc}^{(k)}(\alpha,\beta,\gamma)$. Let $f, \quad g \in \mathcal{A}$, where $f(z)$ is given by (1 . 1) and $g(z)$ is defined by

$$g(z) = z + \sum_{n=2}^{\infty} b_{n}z^{n},$$

then the Hadamard product (or convolution) $f * g$ is defined (as usual) by

$$(f*g)(z) = z + \sum_{n=2}^{\infty} a_{n}b_{n}z^{n} = (g*f)(z).$$

THEOREM 5 . *A function* $f(z) \in s_{\mathcal{S}}^{(k)}{}_c(\alpha,\beta,\gamma)$ *if and only if*

$$1_z\{f * \{(1z_{-}z)^2[1-(1-\alpha)\beta e^{i\theta}] - 1 + (1-\alpha)2(1-\gamma)e^{i\theta}h\}(z)$$
$$-1 + (1-\alpha)2(1-\gamma)e^{i\theta} \cdot \quad (f*h)(z)\} \neq 0 \tag{5.1}$$

for al l $z \in \mathcal{U}$ *and* $0 \le \theta < 2\pi$, *where* $h(z)$ *is given by (5 . 6) .*

Proof . Suppose that $f(z) \in s_{\mathcal{S}}^{(k)}{}_c(\alpha,\beta,\gamma)$, we know that the condition (1 . 3) can be written as (1 . 4) , since (1 . 4) is equivalent to

$$zf'_{f2k(z)}{}^{(z)} \neq 1 + 1(_{-}1 - (_1\alpha_{-})(^{1-\gamma)e^{i\theta}}_{\alpha)\ \beta e}\ {}^{i\theta} \tag{5.2}$$

for all $z \in \mathcal{U}$ and $0 \le \theta < 2\pi$. It is easy to know that the condition (5 . 2) can be written as

$$1_z\{[1-(1-\alpha)\beta e^{i\theta}]zf'(z) - [1+(1-\alpha)(1-\gamma)e^{i\theta}]f2k(z)\} \neq 0. \tag{5.3}$$

On the other hand , it is well known that

$$zf'(z) = f(z) * (1z_{-}z)^2. \tag{5.4}$$

And from the definition of $f2k(z)$, we know that

$$f2k(z) = 2^1[(f*h)(z) + (f*h)(z)], \tag{5.5}$$

where

$$h(z) = k^1 \sum_{k-1}^{v=0} 1z_{-\varepsilon^v} z. \tag{5.6}$$

Substituting (5 . 4) and (5 . 5) into (5 . 3) , we can get (5 . 1) easily . This completes the
proof of Theorem 5 .

Similarly , for the class $\mathcal{C}_{sc}^{(k)}(\alpha,\beta,\gamma)$, we have

COROLLARY 5 . *A function* $f(z) \in \mathcal{C}_{sc}^{(k)}(\alpha,\beta,\gamma)$ *if and only if*

$$1_z\{f * \{z\{(1z_{-}z)^2 \quad [1-(1-\alpha)\beta e^{i\theta}] - 1 + (1-\alpha)2(1-\gamma)e^{i\theta}h\}'\}(z)$$
$$-1 + (1-\alpha)2(1-\gamma)e^{i\theta} \cdot \quad [f*(zh')](z)\} \neq 0$$

for al l $z \in \mathcal{U}$ *and* $0 \le \theta < 2\pi$, *where* $h(z)$ *is given by (5 . 6) .*

ACKNOWLEDGEMENTS . This work was supported by the Scientific Research Fund of Hunan Provincial Education Department and the Hunan Provincial Natu - ral Science Foundation (No . 5 JJ 300 1 3) of People ' s Republic of China . The author would like to thank Prof . Chun - Yi Gao and Prof . Yue - Ping Jiang for their support and encouragement .

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(received 6 . 1 2 . 2006)

School of Mathematics and Computing Science , Changsha University of Science and Technology , Changsha , 410076 Hunan , People ' s Republic of China .

E - mail : zhigw^{a-n} g @163. com