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FIXED POINT THEOREMS ON S - MET R – I C SPACES

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Abstract . In this paper , we prove a general fixed point theorem in S - metric spaces which is a generalization of Theorem 3 . 1 from [S . Sedghi , N . Shobe , A . Aliouche , Mat . Vesnik 64 (20 1 2) ,
258 – 266] . As applications , we get many analogues of fixed point theorems from metric spaces to S - metric spaces .

1 . Introduction and preliminaries

In [1 3] , S . Sedghi , N . Shobe and A . Aliouche have introduced the notion of an $S-$ metric space as follows .

Definition 1 . 1 . [1 3 , Definition 2 . 1] Let X be a nonempty set . An S - *metric*
on X is a function $S : X^3 \to [0,\infty)$ that satisfies the following conditions for all

$$x, y, z, a \in X.$$

(S 1) $S(x,y,z) = 0$ if and only if $x = y = z$.

$$S(x,y,z) \leq S(x,x,a) + S(y,y,a) + S(z,z,a). \tag{S2}$$

The pair (X,S) is called an S - *metric space* .

This notion is a generalization of a $G-$ metric space [1 1] and a D^*- metric space [14] . For the fixed point problem in generalized metric spaces , many results have
been proved , see [1 , 7 , 9 , 1 0] , for example . In [1 3] , the authors proved some properties of S - metric spaces . Also , they proved some fixed point theorems for a self - map on an S - metric space .

In this paper , we prove a general fixed point theorem in S - metric spaces which is a generalization of [1 3 , Theorem 3 . 1] . As applications , we get many analogues of fixed point theorems in metric spaces for S - metric spaces .

Now we recall some notions and lemmas which will be useful later .

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DEFINITION 1 . 2 . [2] Let X be a nonempty set . A $B-$ *metric* on X is a function

$d : X^2 \to [0,\infty)$ if there exists a real number $b \geq 1$ such that the following

conditions hold for all $x,y,z \in X$. (B 1) $d(x,y)=0$ if and only if $x=y$.

$$d(x,y)=d(y,x). \tag{B2}$$

$$d(x,z)\leq b[d(x,y)+d(y,z)]. \tag{B3}$$

The pair (X,d) is called a $B-$ *metric space* .

DEFINITION 1 . 3 . [1 3] Let (X,S) be an S - metric space . For $r>0$ and $x\in X$,

we define the *open ball* $B_S(x,r)$ and the *clos ed ball* $B_S[x,r]$ with center x and radius r as follows

$$B_S(x,r)=\{y\in X : S(y,y,x)<r\},$$
$$B_S[x,r]=\{y\in X : S(y,y,x)\leq r\}.$$

The *topology induced by the S - metric* is the topology generated by the base of all open balls in X.

DEFINITION 1 . 4 . [1 3] Let (X,S) be an S - metric space .

(1) A sequence $\{x_n\} \subset X$ *converges* to $x\in X$ if $S(x_n,x_n,x)\to 0$ as $n\to\infty$.

That is , for each $\varepsilon>0$, there exists $n_0\in\mathbb{N}$ such that for all $n\geq n_0$ we have $S(x_n,x_n,x)<\varepsilon$. We write $x_n\to x$ for brevity .

(2) A sequence $\{x_n\}\subset X$ is a *Cauchy s equence* if $S(x_n,x_n,x_m)\to 0$ as $n,m\to\infty$. That is , for each $\varepsilon>0$, there exists $n_0\in\mathbb{N}$ such that for all $n,m\geq n_0$ we have

$$S(x_n,x_n,x_m)<\varepsilon.$$

(3) The S - metric space (X,S) is *complete* if every Cauchy sequence is a convergent

sequence .

LEMMA 1 . 5 . [1 3 , Lemma 2 . 5] *In an S - metric space , we have*

$$S(x,x,y)=S(y,y,x)$$

for al l $x,y\in X$.

LEMMA 1 . 6 . [1 3 , Lemma 2 . 1 2] *Let* (X,S) *be an S - metric space . If* $x_n\to x$

$$and y n\to y then S(x_n,x_n,yn)\to S(x,x,y).$$

As a special case of [1 3 , Examples in page 260] we have the following EXAMPLE 1 . 7 . Let $\mathbb{R}$ be the real line . Then

$$S(x,y,z)=\mid x-z\mid+\mid y-z\mid$$

for all $x,y,z \in \mathbb{R}$ is an S - metric on $\mathbb{R}$. This S - metric on $\mathbb{R}$ is called the *usual*

S - *metric* on $\mathbb{R}$.

2 . Main results

First , we prove some properties of S - metric spaces .

PROPOSITION 2 . 1 . *Let* (X, S) *be an* S *- metric space and le t*

$$d(x, y) = S(x, x, y)$$

for al l $x, y \in X$. *Then we have*

(1) *d is a B - metric on* X; (2) $x_n \to x$ *in* (X,S) *if and only if* $x_n \to x$ *in* (X,d); (3) $\{x_n\}$ *is a Cauchy s equence in* (X,S) *if and only if* $\{x_n\}$ *is a Cauchy s equence*

$in(X,d)$.

Proof . For the statement (1) , conditions (B 1) and (B 2) are easy to check . It follows from (S 2) and Lemma 1 . 5 that

$$\begin{aligned} d(x,z) = S(x,x,z) &\leq S(x,x,y) + S(x,x,y) + S(z,z,y) \\ &= 2S(x,x,y) + S(y,y,z) = 2d(x,y) + d(y,z) \\ d(x,z) = S(z,z,x) &\leq S(z,z,y) + S(z,z,y) + S(x,x,y) \\ &= 2S(z,z,y) + S(x,x,y) = 2d(y,z) + d(x,y). \end{aligned}$$

It follows that $d(x,z) \leq 3/2[d(x,y) + d(y,z)]$. Then d is a $B-$ metric with $b = 3/2$. Statements (2) and (3) are easy to check .

The following property is trivial and we omit the proof .

PROPOSITION 2 . 2 . *Let* (X,S) *be an S - metric space . Then we have* (1) X *is first - countable ;* (2) X *is regular .*

REMARK 2 . 3 . By Propositions 2 . 1 and 2 . 2 we have that every S - metric space is topologically equivalent to a B - metric space .

COROLLARY 2 . 4 . *Let* $f : X \to Y$ *be a map from an S - metric space* X *to an S - metric space* Y. *Then* f *is continuous at* $x \in X$ *if and only if* $f(x_n) \to f(x)$

$$whenever x_n \to x.$$

Now , we introduce an implicit relation to investigate some fixed point theorems on S - metric spaces . Let $\mathcal{M}$ be the family of all continuous functions of five variables $M : \mathbb{R}_+^5 \to \mathbb{R}_+$. For some $k \in [0,1)$, we consider the following conditions . (C 1) For all $x,y,z \in \mathbb{R}_+$, if $y \leq M(x,x,0,z,y)$ with $z \leq 2x + y$, then $y \leq kx$.

(C 2) For all $y \in \mathbb{R}_+$, if $y \leq M(y,0,y,y,0)$, then $y = 0$.

(C 3) If $x_i \leq yi + z_i$ for all $x_i, yi, z_i \in \mathbb{R}_+, i \leq 5$, then

$$M(x_1,...,x_5) \leq M(y1,...,y5) + M(z_1,...,z_5).$$

Moreover , for all $y \in X, M(0,0,0,y,2y) \leq ky$.

REMARK 2 . 5 . Note that the coefficient k in conditions (C 1) and (C 3) may be different , for example , k_1 and k_3 respectively . But we may assume that they are equal by putting $k = \max\ \{k_1, k_3\}$.

A general fixed point theorem for S - metric spaces is as follows .

THEOREM 2 . 6 . *Let* T *be a s e lf - map on a complete S - metric space* (X,S) *and*

$$\begin{aligned} S(Tx,Tx,Ty) \leq M(&S(x,x,y), S(Tx,Tx,x), S(Tx,Tx,y), \\ &S(Ty,Ty,x), S(Ty,Ty,y)) \end{aligned} \tag{2.1}$$

for al l $x,y,z \in X$ *and s ome* $M \in \mathcal{M}$. *Then we have*

 (1) *If M satisfies the condition (C 1) , then T has a fixed point . Moreover , for any $x_0 \in X$ and the fixed point x, we have*

$$S(Tx_n, Tx_n, x) \leq 1^{2k^n}_{-} k^{S(x_0,} x_0, Tx_0).$$

(2) *If M satisfies the condition (C 2) and T has a fixed point , th en the fixed po int is unique .*

(3) *If M satisfies the condition (C 3) and T has a fixed point x, th en T is continuous at x.*

Proof . (1) For each $x_0 \in X$ and $n \in \mathbb{N}$, put $x_{n+1} = Tx_n$. It follows from (2 . 1) and Lemma 1 . 5 that

$$\begin{aligned} S(x_{n+1}, x_{n+1}, x_{n+2}) &= S(Tx_n, Tx_n, Tx_{n+1}) \\ \leq M(S(x_n, x_n, x_{n+1}), &S(x_{n+1}, x_{n+1}, x_n), S(x_{n+1}, x_{n+1}, x_{n+1}), \\ &S(x_{n+2}, x_{n+2}, x_n), S(x_{n+2}, x_{n+2}, x_{n+1})) \\ &= M(S(x_n, x_n, x_{n+1}), S(x_n, x_n, x_{n+1}), 0, \\ &S(x_n, x_n, x_{n+2}), S(x_{n+1}, x_{n+1}, x_{n+2})). \end{aligned}$$

By (S 2) and Lemma 1 . 5 we have

$$\begin{aligned} S(x_n, x_n, x_{n+2}) &\leq 2S(x_n, x_n, x_{n+1}) + S(x_{n+2}, x_{n+2}, x_{n+1}) \\ &= 2S(x_n, x_n, x_{n+1}) + S(x_{n+1}, x_{n+1}, x_{n+2}). \end{aligned}$$

Since M satisfies the condition (C 1) , there exists $k \in [0, 1)$ such that

$$S(x_{n+1}, x_{n+1}, x_{n+2}) \leq kS(x_n, x_n, x_{n+1}) \leq k^{n+1} S(x_0, x_0, x_1). \tag{2.2}$$

Thus for all $n < m$, by using (S 2) , Lemma 1 . 5 and (2 . 2) , we have

$$\begin{aligned} S(x_n, x_n, x_m) &\leq 2S(x_n, x_n, x_{n+1}) + S(x_m, x_m, x_{n+1}) \\ &= 2S(x_n, x_n, x_{n+1}) + S(x_{n+1}, x_{n+1}, x_m) \\ &\quad ... \\ &\leq 2[k^n + \cdots + k^{m-1}]S(x_0, x_0, x_1) \\ &\leq 1^{2k^n}_{-} k^{S(x_0,} x_0, x_1). \end{aligned}$$

Taking the limit as $n, m \to \infty$ we get $S(x_n, x_n, x_m) \to 0$. This proves that $\{x_n\}$ is a Cauchy sequence in the complete S - metric space (X, S). Then $x_n \to x \in X$. Moreover , taking the limit as $m \to \infty$ we get

$$S(x_n, x_n, x) \leq 2_1 k^{n+1}_{-\,k} S(x_0, x_0, x_1).$$

It implies that

$$S(Tx_n, Tx_n, x) \leq 1^{2k^n}_{-} k^{S(x_0,} x_0, Tx_0).$$

 Now we prove that x is a fixed point of T. By using (2 . 1) again we get

$$\begin{aligned} S(x_{n+1}, x_{n+1}, Tx) &= S(Tx_n, Tx_n, Tx) \\ &\leq M(S(x_n, x_n, x), S(Tx_n, Tx_n, x), S(Tx_n, Tx_n, x_n), \\ &\quad S(Tx, Tx, x_n), S(Tx, Tx, x)) \\ &= M(S(x_n, x_n, x), S(x_{n+1}, x_{n+1}, x), S(x_{n+1}, x_{n+1}, x_n), \\ &\quad S(Tx, Tx, x_n), S(Tx, Tx, x)). \end{aligned}$$

Note that $M \in \mathcal{M}$, then using Lemma 1 . 6 and taking the limit as $n \to \infty$ we obtain

$$S(x, x, Tx) \leq M(0, 0, 0, S(Tx, Tx, x), S(Tx, Tx, x)).$$

Then , from Lemma 1 . 5 , we obtain

$$S(x, x, Tx) \leq M(0, 0, 0, S(x, x, Tx), S(x, x, Tx,)).$$

Since M satisfies the condition (C 1) , then $S(x, x, Tx) \leq k \cdot 0 = 0$. This proves that

$$x = Tx.$$

(2) Let x, y be fixed points of T. We shall prove that $x = y$. It follows from (2 . 1) and Lemma 1 . 5 that

$$\begin{aligned} S(x, x, y) &= S(Tx, Tx, Ty) \\ &\leq M(S(x, x, y), S(Tx, Tx, x), S(Tx, Tx, y), S(Ty, Ty, x), S(Ty, Ty, y)) \\ &= M(S(x, x, y), 0, S(x, x, y), S(y, y, x), 0) \\ &= M(S(x, x, y), 0, S(x, x, y), S(x, x, y), 0). \end{aligned}$$

Since M satisfies the condition ((C 2) , then $S(x, x, y) = 0$. This proves that $x = y$. (3) Let x be the fixed point of T and $yn \to x \in X$. By Corollary 2 . 4 , we need to prove that $Ty_n \to Tx$. It follows from (2 . 1) that

$$\begin{aligned} S(x, x, Ty_n) &= S(Tx, Tx, Ty_n) \\ &\leq M(S(x, x, yn), S(Tx, Tx, x), S(Tx, Tx, yn), \\ &\quad S(Ty_n, Ty_n, x), S(Ty_n, Ty_n, yn)) \\ &= M(S(x, x, yn), 0, S(x, x, yn), S(Ty_n, Ty_n, x), S(Ty_n, Ty_n, yn)). \end{aligned}$$

Since M satisfies the condition (C 3) and by (S 2)

$$S(Ty_n, Ty_n, yn) \leq 2S(Ty_n, Ty_n, x) + S(yn, yn, x)$$

then we have

$$\begin{aligned} S(x, x, Ty_n) &\leq M(S(x, x, yn), 0, S(x, x, yn), 0, S(x, x, yn)) \\ &\quad + M(0, 0, 0, S(Ty_n, Ty_n, x), 2.S(Ty_n, Ty_n, x)) \\ &\leq M(S(x, x, yn), 0, S(x, x, yn), 0, S(x, x, yn)) + kS(Ty_n, Ty_n, x). \end{aligned}$$

Therefore

$$S(x, x, Ty_n) \leq 11_{-}k^{M(S(x,} x, yn), 0, S(x, x, yn), 0, S(x, x, yn)).$$

 Note that $M \in \mathcal{M}$, hence taking the limit as $n \to \infty$ we get $S(x, x, Ty_n) \to 0$. This

$$\text{provesthat} Ty_n \to x = Tx.$$

Next , we give some analogues of fixed point theorems in metric spaces for S - metric spaces by combining Theorem 2 , 6 with examples of $M \in \mathcal{M}$ and M satisfies conditions (C 1) , (C 2) and (C 3) . The following corollary is an analogue of Banach ' s contraction principle .

Corollary 2 . 7 . [1 3 , Theorem 3 . 1] *Let T be a s e lf - map on a complete S - metric*

$$space (X, S) and$$

$$S(Tx, Tx, Ty) \leq LS(x, x, y)$$

for s ome $L \in [0, 1)$ *and al l* $x, y \in X$. *Then* T *has a unique fixed point in* X.
Moreover ,T *is continuous at th e fixed point .*

Proof . The assertion follows using Theorem 2 . 6 with $M(x, y, z, s, t) = Lx$ for some $L \in [0, 1)$ and all $x, y, z, s, t \in \mathbb{R}_+$.

The following corollary is an analogue of R . Kannan ' s result in [8] .

Corollary 2 . 8 . *Let T be a s e lf - map on a complete S - metric space* (X, S) *and*

$$S(Tx, Tx, Ty) \leq a(S(Tx, Tx, x) + S(Ty, Ty, y))$$

for s ome $a \in [0, 1/2)$ *and al l* $x, y \in X$. *Then* T *has a unique fixed po int in* X. *Moreover* ,T *is continuous at th e fixed point .*

Proof . The assertion follows using Theorem 2 . 6 with $M(x, y, z, s, t) = a(y + t)$ for some $a \in [0, 1/2)$ and all $x, y, z, s, t \in \mathbb{R}_+$. Indeed , M is continuous . First , we have $M(x, x, 0, z, y) = a(x + y)$. So , if $y \leq M(x, x, 0, z, y)$ with $z \leq 2x + y$, then $y \leq a/(1 - a)', x$ with $a/(1 - a) < 1$. Therefore ,T satisfies the condition (C 1) .

Next , if $y \leq M(y, 0, y, y, 0) = 0$, then $y = 0$. Therefore ,T satisfies the condi - tion (C 2) .

Finally , if $x_i \leq yi + z_i$ for $i \leq 5$, then

$$M(x_1, ..., x_5) = a(x_2 + x_5) \leq a[(y2 + z_2) + (y5 + z_5)]$$
$$= a(y2 + z_2) + a.(y5 + z_5) = M(y1, ..., y5) + M(z_1, ..., z_5).$$

Moreover

$$M(0, 0, 0, y, 2y) = a(0 + 2y) = 2ay$$

where $2a < 1$. Therefore ,T satisfies the condition (C 3) .

Example 2 . 9 . Let $\mathbb{R}$ be the usual S - metric space as in Example 1 . 7 and let

$$Tx = \{1^{1/2}_{\ /4} \quad \text{if}^{\text{if}} x^x \in_= [^{0,}_{1} .1)$$

Then T is a self - map on a complete S - metric space $[0, 1] \subset \mathbb{R}$. For all $x \in (3/4, 1)$ we have

$$S(Tx,Tx,T1) = S(1/2,1/2,1/4) = \mid 1/2 - 1/4 \mid + \mid 1/2 - 1/4 \mid = 1/2$$
$$S(x,x,1) = \mid x - 1 \mid + \mid x - 1 \mid = 2 \mid x - 1 \mid < 1/2.$$

 Then T does not satisfy the condition of Corollary 2 . 7 . We also have

$$S(Tx,Tx,x) = \{3^{2|1/2}_{/2} - x \mid \quad \text{if}^{\text{if}} x^{x} \in_{=} [^{0,}_{1}.1)$$

It implies that

$$5/12((S(Tx,Tx,x)+S(Ty,Ty,y))$$
$$= \{5^{5/6(|1/2}_{/12|1/2} -_{-} x^{|}_{x|} +_{+} |^{1/2}_{5} \ {}_{/8} - y \mid) \quad \text{if}^{\text{if}} x^{x},y_{\in} \in [_{0,}[0^{;}_{1)},1_{y}) = 1.$$

Then we get $S(Tx,Tx,Ty) \leq 5/12((S(Tx,Tx,x)+S(Ty,Ty,y))$. Therefore , T satisfies the condition of Corollary 2 . 8 . It is clear that $x = 1/2$ is the unique fixed point of T.

The following corollary is an analogue of R . M . T . Bianchini ' s result in [3] .

Corollary 2 . 1 0 . *Let T be a s e lf - map on a complete S - metric space (X,S) and*

$$S(Tx,Tx,Ty) \leq h\max\{S(Tx,Tx,x),S(Ty,Ty,y)\}$$

for s ome $h \in [0,1)$ and all $x,y \in X$. Then T has a unique fixed point in X. Moreover , if $h \in [0,1/2)$, then T is continuous at th e fixed point .

Proof . The assertion follows using Theorem 2 . 6 with $M(x,y,z,s,t) = h$ max $\{y,t\}$ for some $h \in [0,1)$ and all $x,y,z,s,t \in \mathbb{R}_+$. Indeed , M is continu - ous . First , we have $M(x,x,0,z,y) = h$ max $\{x,y\}$. So , if $y \leq M(x,x,0,z,y)$ with $z \leq 2x+y$, then $y \leq hx$ or $y \leq hy$. Therefore , $y \leq hx$. Therefore , T satisfies the condition (C 1) .

Next , if $y \leq M(y,0,y,y,0) = h$ max $\{y,0\} = hy$, then $y = 0$ since $h < 1/2$. Therefore , T satisfies the condition (C 2) .

Finally , if $x_i \leq yi + z_i$ for $i \leq 5$, then

$$M(x_1,...,x_5) = h\max\{x_2,x_5\} \leq h\max\{y2+z_2,y5+z_5\}$$
$$\leq h\max\{y2,y5\} + h\max\{z_2,z_5\} = M(y1,...,y5) + M(z_1,...,z_5).$$

Moreover , if $h \in [0,1/2)$, then $2h < 1$ and $M(0,0,0,y,2y) = h$ max $\{0,2y\} = 2hy$ where $2h < 1$. Therefore , T satisfies the condition (C 3) .

Example 2 . 1 1 . Let $\mathbb{R}$ be the usual S - metric space as in Example 1 . 7 and let $Tx = x/3$ for all $x \in [0,1]$. We have

$$S(Tx,Tx,Ty) = S(x/3,x/3,y/3) =\mid x/3-y/3 \mid + \mid x/3-y/3 \mid= 2/3 \mid x-y \mid$$
$$S(Tx,Tx,x) = S(x/3,x/3,x) =\mid x/3-x \mid + \mid x/3-x \mid= 4/3 \mid x \mid$$
$$S(Ty,Ty,y) = S(y/3,y/3,y) =\mid y/3-y \mid + \mid y/3-y \mid= 4/3 \mid y \mid$$
$$S(Tx,Tx,x)+S(Ty,Ty,y) = 4/3(\mid x \mid + \mid y \mid)$$
$$\max\{S(Tx,Tx,x),S(Ty,Ty,y)\} = 4/3\max\{\mid x \mid, \mid y \mid\}.$$

It implies that $S(T1,T1,T0) = 2/3, S(T1,T1,1)+S(T0,T0,0) = 4/3$. This proves that T does not satisfy the condition of Corollary 2 . 8 . We also have that T satisfies the condition of Corollary 2 . 1 0 with $h = 3/4$ and T has a unique fixed point $x = 0$.

The following corollary is an analogue of S . Reich ' s result in [1 2] .

Corollary 2 . 1 2 . *Let* T *be a s e lf - map on a complete* S *- metric space* (X,S) *and*

$$S(Tx,Tx,Ty) \leq aS(x,x,y) + bS(Tx,Tx,x) + cS(Ty,Ty,y)$$

for s ome $a,b,c \geq 0, a+b+c < 1,$ *and al l* $x,y \in X.$ *Then* T *has a unique fixed point in* $X.$ *Moreover , if* $c < 1/2,$ *then* T *is continuous at the fixed po int .*

Proof . The assertion follows using Theorem 2 . 6 with $M(x,y,z,s,t) = ax + by + ct$ for some $a,b,c \geq 0, a+b+c < 1$ and all $x,y,z,s,t \in \mathbb{R}_+$. Indeed , M is continuous . First , we have $M(x,x,0,z,y) = ax+bx+cy$. So , if $y \leq M(x,x,0,z,y)$ with $z \leq 2x+y$, then $y \leq (a+b)/(1-c)x$ with $(a+b)/(1-c) < 1$. Therefore , T satisfies the condition (C 1) .

Next , if $y \leq M(y,0,y,y,0) = ay$, then $y = 0$ since $a < 1$. Therefore , T satisfies the condition (C 2) .

Finally , if $x_i \leq yi + z_i$ for $i \leq 5$, then

$$\begin{aligned} M(x_1,...,x_5) &= ax_1 + bx_2 + cx_5 \\ &\leq a(y1+z_1) + b(y2+z_2) + c(y5+z_5) \\ &= (ay1+by2+cy5) + (az_1+bz_2+cz_5) \\ &= M(y1,...,y5) + M(z_1,...,z_5). \end{aligned}$$

Moreover $M(0,0,0,y,2y) = 2cy$ where $2c < 1$. Therefore , T satisfies the condi - tion (C 3) .

Example 2 . 1 3 . Let $\mathbb{R}$ be the usual S - metric space as in Example 1 . 7 and let $Tx = x/2$ for all $x \in [0,1]$. We have

$$\begin{aligned} S(Tx,Tx,Ty) &= \mid x/2 - y/2 \mid + \mid x/2 - y/2 \mid = \mid x - y \mid \\ S(x,x,y) &= \mid x-y \mid + \mid x-y \mid = 2 \mid x-y \mid \\ S(Tx,Tx,x) &= \mid x/2 - x \mid + \mid x/2 - x \mid = \mid x \mid . \end{aligned}$$

Then $S(Tx,Tx,T0) = \mid x \mid$, max $\{S(Tx,Tx,x), S(T0,T0,0)\} = \mid x \mid$. This proves that T does not satisfy the condition of Corollary 2 . 1 0 . We also have

$$S(Tx,Tx,Ty) \leq 1/2S(x,x,y) + 1/3S(Tx,Tx,x) + 1/3S(Ty,Ty,y).$$

Then T satisfy the condition of Corollary 2 . 1 2 . It is clear that T has a unique fixed point $x = 0$.

The following corollary is an analogue of S . K . Chatterjee ' s result in [4] .

Corollary 2 . 1 4 . *Let* T *be a s e lf - map on a complete* S *- metric space* (X,S) *and*

$$S(Tx,Tx,Ty) \leq h \max\{S(Tx,Tx,y), S(Ty,Ty,x)\}$$

for s ome $h \in [0,1/3)$ *and al l* $x,y \in X.$ *Then* T *has a unique fixed po int in* $X.$ *Moreover* , T *is continuous at th e fixed point .*

Proof . The assertion follows using Theorem 2 . 6 with $M(x,y,z,s,t) = h$ max $\{z,s\}$ for some $h \in [0,1/3)$ and all $x,y,z,s,t \in \mathbb{R}_+$. Indeed , M is con - tinuous . First , we have $M(x,x,0,z,y) = h$ max $\{0,z\}$. So , if $y \leq M(x,x,0,z,y)$ with $z \leq 2x+y$, then $y \leq 2hx+hy$. So $y \leq 2h/(1-h)x$ with $2h/(1-h) < 1$. Therefore , T satisfies the condition (C 1) .

Next , if $y \leq M(y,0,y,y,0) = hy$, then $y = 0$ since $h < 1/3$. Therefore , T satisfies the condition (C 2) .

Finally , if $x_i \leq yi + z_i$ for $i \leq 5$, then

$$M(x_1,...,x_5) = h\max\{x_3,x_4\} \leq h\max\{y3+z_3, y4+z_4\}$$
$$\leq h\max\{y3,y4\} + h\max\{z_3,z_4\} = M(y1,...,y5) + M(z_1,...,z_5).$$

Moreover

$$M(0,0,0,y,2y) = h\max\{0,y\} = hy$$

where $h < 1$. Therefore , T satisfies the condition (C 3) .

Corollary 2 . 1 5 . *Let T be a s e lf - map on a complete S - metric space (X,S) and*

$$S(Tx,Tx,Ty) \leq a.(S(Tx,Tx,y) + S(Ty,Ty,x))$$

for s ome $a \in [0,1/3)$ and al l $x,y \in X$. Then T has a unique fixed po int in X. Moreover , T is continuous at th e fixed point .

Proof . The assertion follows using Theorem 2 . 6 with $M(x,y,z,s,t) = a(z+s)$ for some $a \in [0,1/3)$ and all $x,y,z,s,t \in \mathbb{R}_+$. Indeed , M is continuous . First , we have $M(x,x,0,z,y) = a(0+z) = az$. So , if $y \leq M(x,x,0,z,y)$ with $z \leq 2x+y$, then $y \leq 2ax+ay$. So $y \leq 2a/(1-a)x$ with $2a/(1-a) < 1$. Therefore , T satisfies the condition (C 1) .

Next , if $y \leq M(y,0,y,y,0) = a(y+y) = 2ay$ then $y = 0$ since $2a < 2/3$. Therefore , T satisfies the condition (C 2) .

Finally , if $x_i \leq yi + z_i$ for $i \leq 5$, then

$$M(x_1,...,x_5) = a(x_3+x_4) \leq a(y3+z_3+y4+z_4)$$
$$= a(y3+y4) + a(z_3+z_4) = M(y1,...,y5) + M(z_1,...,z_5).$$

Moreover $M(0,0,0,y,2y) = a(0+y) = ay$ where $a < 1$. Therefore , T satisfies the condition (C 3) .

Example 2 . 1 6 . Let $\mathbb{R}$ be the usual S - metric space as in Example 1 . 7 and let $Tx = x/3$ for all $x \in [0,1]$. Then we have $S(Tx,Tx,Ty) = 2 \mid x/3 - y/3 \mid = 2/3 \mid x-y \mid, S(Tx,Tx,y) = 2 \mid x/3 - y \mid, S(Ty,Ty,x) = 2 \mid y/3 - x \mid$. It implies that $S(T1,T1,T0) = 2/3, S(T1,T1,0) = 2/3, S(T0,T0,1) = 2$. This proves that T does not satisfy the condition of Corollary 2 . 1 4 . We also have

$$S(Tx,Tx,y) + S(Ty,Ty,x) = 2 \mid x/3 - y \mid + 2 \mid y/3 - x \mid \geq 8/3 \mid x - y \mid .$$

Therefore , T satisfies the condition of Corollary 2 . 1 5 . It is clear that T has a unique fixed point $x = 0$.

Corollary 2 . 1 7 . *Let T be a s e lf - map on a complete S - metric space (X, S) and*

$$S(Tx, Tx, Ty) \leq aS(x, x, y) + bS(Tx, Tx, y) + cS(Ty, Ty, x)$$

for s ome $a, b, c \geq 0, a + b + c < 1, a + 3c < 1$ and al l $x, y \in X$. Then T has a unique fixed po int in X. Moreover ,T is continuous at the fixed point .

Proof . The assertion follows using Theorem 2 . 6 with $M(x, y, z, s, t) = ax + bz + cs$ for some $a, b, c \geq 0, a + b + c < 1, a + 3c < 1$ and all $x, y, z, s, t \in \mathbb{R}_+$. Indeed ,$M$ is continuous . First , we have $M(x, x, 0, z, y) = ax + cz$. So , if $y \leq M(x, x, 0, z, y)$ with $z \leq 2x + y$, then $y \leq ax + 2cx + cy$. So $y \leq (a + 2c)/(1 - c)x$ with $(a + 2c)/(1 - c) < 1$. Therefore ,T satisfies the condition (C 1) .

Next , if $y \leq M(y, 0, y, y, 0) = ay + by + cy = (a + b + c)y$ then $y = 0$ since $a + b + c < 1$. Therefore ,T satisfies the condition (C 2) .

Finally , if $x_i \leq yi + z_i$ for $i \leq 5$, then

$$\begin{aligned} M(x_1, ..., x_5) = ax_1 + bx_3 + cx_4 &\leq a(y1 + z_1) + b(y3 + z_3) + c(y4 + z_4) \\ &= (ay1 + by3 + cy4) + (az_1 + bz_3 + cz_4) \\ &= M(y1, ..., y5) + M(z_1, ..., z_5). \end{aligned}$$

Moreover $M(0, 0, 0, y, 2y) = cy$ where $c < 1$. Therefore , T satisfies the condi - tion (C 3) .

Example 2 . 1 8 . Let $\mathbb{R}$ be the usual S - metric space as in Example 1 . 7 and let $Tx = 3/4(1-x)$ for all $x \in [0, 1]$. Then we have $S(Tx, Tx, Ty) = 3/2 \mid x - y \mid, S(Tx, Tx, y) = 2 \mid 3/4(1 - x) - y \mid$. It implies that $S(T1, T1, T0) = 3/2$, max $\{S(T1, T1, 0), S(T0, T0, 1)\} =$ max $\{0, 1/2\} = 1/2$. This proves that T does not satisfy the condition of Corollary 2 . 1 4 . We also have

$$4/5S(x, x, y) + 0 \cdot S(Tx, Tx, y) + 0 \cdot S(Ty, Ty, x) = (8/5) \mid x - y \mid \geq S(Tx, Tx, Ty).$$

Therefore ,T satisfies the condition of Corollary 2 . 1 7 . It is clear that T has a unique fixed point $x = 3/7$.

The following corollary is an analogue of G . E . Hardy and T . D . Rogers ' result in [6] .

Corollary 2 . 1 9 . *Let T be a s e lf - map on a complete S - metric space (X, S) and*

$$\begin{aligned} S(Tx, Tx, Ty) \leq a_1S(x, x, y) + a_2S(Tx, Tx, x) + a_3S(Tx, Tx, y) \\ + a_4S(Ty, Ty, x) + a_5S(Ty, Ty, y) \end{aligned}$$

for s ome $a_1, ..., a_5 \geq 0$ such that $\max\{a_1 + a_2 + 3a_4 + a_5, a_1 + a_3 + a_4, a_4 + 2a_5\} < 1$ *and all $x, y \in X$. Then T has a unique fixed point in X. Moreover ,T is continuous at the fixed point .*

Proof . The assertion follows using Theorem 2 . 6 with $M(x, y, z, s, t) = a_1x + a_2y + a_3z + a_4s + a_5t$ for some $a_1, ..., a_5 \geq 0$ such that max $\{a_1 + a_2 + 3a_4 + a_5, a_1 + a_3 + a_4, a_4 + 2a_5\} < 1$ and all $x, y, z, s, t \in \mathbb{R}_+$. Indeed ,$M$ is continuous . First , we have $M(x,x,0,z,y) = a_1x + a_2x + a_4z + a_5y$. So , if $y \leq M(x,x,0,z,y)$ with

$$z \leq 2x + y, \text{then}$$
$$y \leq a_1x + a_2x + a_4z + a_5y \leq a_1x + a_2x + a_4(2x+y) + a_5y.$$
$$\text{Then} y \leq (a_1 + a_2 + 2a_4)/(1 - a_4 - a_5)x \text{with} (a_1 + a_2 + 2a_4)/(1 - a_4 - a_5) < 1.$$

Therefore , T satisfies the condition (C 1) .

Next , if $y \leq M(y,0,y,y,0) = a_1y + a_3y + a_4y = (a_1 + a_3 + a_4)y$ then $y = 0$ since $a_1 + a_3 + a_4 < 1$. Therefore , T satisfies the condition (C 2) .

Finally , if $x_i \leq yi + z_i$ for $i \leq 5$, then

$$\begin{aligned} M(x_1, ..., x_5) &= a_1x_1 + \cdots + a_5x_5 \\ &\leq a_1(y1 + z_1) + \cdots + a_5(y5 + z_5) \\ &= (a_1y1 + \cdots + a_5y5) + (a_1z_1 + \cdots + a_5z_5) \\ &= M(y1, ..., y5) + M(z_1, ..., z_5). \end{aligned}$$

Moreover $M(0,0,0,y,2y) = a_4y + 2a_5y = (a_4 + 2a_5)y$ where $a_4 + 2a_5 < 1$. Therefore , T satisfies the condition (C 3) .

Example 2 . 2 0 . Let T be the map in Example 2 . 1 6 . Then we have

$$S(T1, T1, T1/2) = 1,$$
$$aS(1,1,1/2) + bS(T1,T1,1/2) + cS(T1/2,T1/2,1) = a + 2c.$$

This proves that T does not satisfy the condition of Corollary 2 . 1 7 . We also have

$$\begin{aligned} &0 \cdot S(x,x,y) + (3/4)S(Tx,Tx,x) + (3/4)S(Tx,Tx,y) + 0 \cdot S(Ty,Ty,x) + 0 \cdot S(Ty,Ty,y) \\ &\quad = (3/4)S(Tx,Tx,x) + (3/4)S(Tx,Tx,y) \geq S(Tx,Tx,Ty). \end{aligned}$$

Therefore , T satisfies the condition of Corollary 2 . 1 9 . It is clear that T has a unique fixed point $x = 0$.

The following corollary is an analogue of L . B .Ć iri ć ' s result in [5] .

Corollary 2 . 2 1 . *Let T be a s e lf - map on a complete S - metric space (X,S) and*

$$\begin{aligned} S(Tx,Tx,Ty) \leq h \max\{&S(x,x,y), S(Tx,Tx,x), S(Tx,Tx,y), \\ &S(Ty,Ty,x), S(Ty,Ty,y)\} \end{aligned}$$

for s ome $h \in [0,1/3)$ and al l $x,y \in X$. Then T has a unique fixed po int in X. Moreover , T is continuous at th e fixed point .

Proof . The assertion follows using Theorem 2 . 6 with $M(x,y,z,s,t) = h \max\{x,y,z,s,t\}$ for some $h \in [0,1/3)$ and all $x,y,z,s,t \in \mathbb{R}_+$. Indeed , M is continuous . First , we have $M(x,x,0,z,y) = h \max\{x,x,0,z,y\}$. So , if $y \leq M(x,x,0,z,y)$ with $z \leq 2x + y$, then $y \leq hx$ or $y \leq hz \leq h(2x+y)$. Then $y \leq kx$ with $k = \max\{h, 2h/(1-h)\} < 1$. Therefore , T satisfies the condition (C 1) .

Next , if $y \leq M(y,0,y,y,0) = h.y$, then $y = 0$ since $h < 1/3$. Therefore , T satisfies the condition (C 2) .

 Finally , if $x_i \leq yi + z_i$ for $i \leq 5$, then

$$M(x_1,...,x_5) = h\max\{x_1,...,x_5\} \leq h\max\{y1+z_1,...,y5+z_5\} \\ \leq h\max\{y1,...,y5\} + h\max\{z_1,...,z_5\} \\ = M(y1,...,y5) + M(z_1,...,z_5).$$

Moreover $M(0,0,0,y,2y) = 2hy$ where $2h < 1$. Therefore , T satisfies the condi - tion (C 3) .

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