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Renorming and Operators

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This communication presented in the summer course " Espacios de Banach y operadores " help in Laredo (Spain) , august 2003 , i s an annoucement of some results about MLUR renorming of Banach spaces . These results will appear in [6] .

Let us start by recalling some convexity properties of norms . Let $(X, \|\ \|)$ be a Banach space . We say that X (or the norm of X) is :

(1) *locally uniformly rotund* (LUR for short) if , for every x and every sequence $(x_n)_n$ in X such that $\| x_n + x \| \to 2 \| x \|$ and $\| x_n \| \to \| x \|$, we have

$$\| x_n - x \| \to 0;$$

(2) *midpoint locally uniformly rotund* (MLUR for short) if , for every x and every sequence $(x_n)_n$ in X such that $\| x_n + x \| \to \| x \|$ and $\| x_n - x \| \to \| x \|$,

$$\text{we have } \| x_n \| \to 0;$$

(3) *strictly convex or rotund* (R for short) if $x = y$ whenever x and y are points of X such that $\| x \| = \| y \| = \| \frac{x+y}{2} \|$, i . e . , if the unit sphere of X does not contain any nondegenerate segment .

It i s clear that LUR $\Rightarrow$ MLUR and that MLUR $\Rightarrow$ R . The converse implications are not true in general , even under renormings : as dual of a separable space , ℓ_∞ has an equivalent (dual) rotund norm , but it does not admit MLUR renorming [2] . In the paper [5] , Haydon showed the first example of MLUR space with no equivalent LUR norm .

Banach spaces with equivalent MLUR norms were characterized in [8] , in terms of countable decompositions of such spaces , involving the following

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DEFINITION 1 . Let A be a subset of a Banach space $(X, \| \|)$. A point
$x \in A$ i s said t o be a $\varepsilon-$ strongly extreme point of A if there i s $\delta > 0$ such that
$\| u - v \| < \varepsilon$ whenever u and v are points in A with $\|x - u{+}_2 v\| < \delta$.

It i s easy t o see that X is MLUR if and only if every point of the unit sphere is a $\varepsilon-$ strongly extreme point of the unit ball , for every $\varepsilon > 0$. The characterization of MLUR spaces mentioned above i s given by the following

THEOREM 1 . ([8] , THEOREM 1) *A Banach space X admits an equivalent MLUR norm if , and only if , for every $\varepsilon > 0$ we have a countable decomposition*

$$X = \bigcup_{n=1}^{\infty} X_{n,\varepsilon}$$

in such a way that every $x \in X_{n,\varepsilon}$ is a $\varepsilon-$ strongly extreme point of the convex

$$envelope \operatorname{co}(X_{n,\varepsilon}).$$

A similar result was proved for LUR renormability in [7] and [1 0] , where roughly sp eaking , $\varepsilon-$ strong extremality i s replaced by $\varepsilon-$ dentability .

THEOREM 2 . ([7] , MAIN THEOREM) *A Banach space X has an equival - ent LUR norm if , and only if , for every $\varepsilon > 0$ we have a countable decompos - ition*

$$X = \bigcup_{n=1}^{\infty} X_{n,\varepsilon}$$

in such a way that for every $n \in \mathbb{N}$ and every $x \in X_{n,\varepsilon}$ there is an open half space $H \subset X$ such that $x \in H$ and diam $(H \cap \Rightarrow_{notdef-parenright}\ notdef < epsilon - notdef notdef$ R $e - notdef - infinity - c_{notdef-alnotdef}$ t $notdef - ha - infinity$ t $notdef - notdef a - braceleft notdef - notdef^{o-notdef} negationslash - notdef - enotdef - notdef - nnotdef - notdef notdef$
lfsp ace o X i a s t o th e f–o rm $H = f \text{one} - \text{minus}(\alpha, \infty \text{parenright} - \text{comma} notdef_w$
th $f \in X * \setminus \{\}$ *a d*

$$\in \mathbb{R}.$$

This result has motivated the following notion , introduced and extensively studied by Molt ó, Orihuela , r – T oyanski and Valdivia in their recent memoir
[8] , where a non linear transfer method for LUR renormability is provided .

Definition 2 . Let X and Y b e Banach spaces , and let A be a subset of X. A map $\Psi : A \to Y$ i s said to be $\sigma-$ slicely continuous if for every $\varepsilon > 0$ we may write

$$A = \bigcup_{n} A_{n,\varepsilon}$$

in such a way that for every $x \in A_{n,\varepsilon}$ there exists an open half space H such that $x \in H$ and diam $\Psi(H \cap \Rightarrow_{notdef-parenright} notdef < notdef - epsilonnotdef$

We are going t o combine the covering characterization of Theorem 1 and some properties of $\sigma-$ slicely continuous maps to get some results about MLUR renormability on Banach spaces . Our first theorem contains , as a particular case , a version of the three space property for MLUR norms .

THEOREM 3 . *Let X be a Banach space . Suppose that there exist a closed MLUR renormable subspace Y of X and a $\sigma-$ slicely continuous map $\Phi : X \to X$ such that $x - \Phi x \in Y$ for all $x \in X$. Then X admits an equivalent MLUR norm .*

The basic idea t o prove this result i s to get $\varepsilon-$ MLUR decompositions on X from $\varepsilon-$ MLUR decompositions of Y via the operator $Id - \Phi$. The map $\Phi : X \to X$ given by $\Phi = g \circ Q$, where $Q : X \to X/Y$ i s the quotient map and X/Y i s LUR renormable , and $g : X/Y \to X$ is a continuous selector , i s $\sigma-$ slicely continuous . If moreover Y has an MLUR renorming , we obtain the following result Alexandrov [1] (see also [3 , p . 1 8 1]) .

COROLLARY 1 . *Let X be a Banach space . Suppose that there exists a closed subspace Y of X with an equivalent MLUR norm and such that the quotient X/Y is LUR renormable . Then X is MLUR renormable .*

Let us recall that MLUR i s not a three space property . In the paper [5] Haydon provided an example of Banach space X with a closed subspace Y sucht that Y and X/Y admit a LUR norm and a MLUR norm , resp ectively , while X does not have any equivalent rotund norm .

As another application of our t echnique we get a partial generalization of a result of Haydon ([5 , Proposition 5 . 3]) , which is the main tool for the construction of MLUR norms in $C(\Upsilon)$ spaces , Υ a tree .

THEOREM 4 . *Let K be a locally compact space . Suppose that there exist a $\sigma-$ slicely continuous map $\Psi : C_0(K) \to c_0(\Gamma)$ and a family $\{K_\gamma\}_\gamma \in \Gamma$ of closed and open subsets of K with the following properties :*

(1) *for each $\gamma \in \Gamma, C_0(K_\gamma)$ is MLUR renormable ;*

(2) *for each $x \in C_0(K), x \neq 0,$ supp $(x) \subset \bigcup\{K_\gamma : \Psi x(\gamma) \neq 0\}.$*

Then $C_0(K)$ admits an equivalent MLUR norm .

The idea now t o obtain the $\varepsilon-$ MLUR decompositions in $C_0(K)$ i s to use the $\sigma-$ slicely continuity of Ψ and condition (2) t o get a first decomposition where the functions x can b e approximated by its restriction on some K_γ, and t o transfer the MLUR decompositions of the spaces $C_0(K_\gamma)$.

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