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# Intersections of Closed Balls and Geometry of Banach *Spaces**

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## 1 . THE MAZUR INTERSECTION PROPERTY AND ITS RELATIVES

It was Mazur [ 39 ] who first drew attention t o the euclidean space property :
*every bounded c los ed convex s et can be represented as an intersection of c los ed balls .* He began the investigation t o determine those normed linear spaces
which posses this property , named after him the *Mazur intersection property*
or MIP . He proved Theorem 1 . 1 , whose proof i s so nice and clear that it
deserves to be the starting point for this survey . The following easy ( and useful ) fact will b e used extensively throughout the rest of the paper : a closed ,
convex and bounded set $C$ i s an intersection of balls if and only if for every $xelement - slashC$, there i s a closed ball containing the set but missing the point . Hence ,
the MIP can b e regarded as a separation property by balls which i s stronger than the classical separation property by hyperplanes . We denote by $B$ and
$S$ the unit ball and unit sphere of a Banach space . Analogously , $B^*$ and $S^*$
will stand for the corresp onding unit ball and unit sphere in the dual space .

THEOREM 1 . 1 . *If a norm* $\|\cdot\|$ *in a Banach space* $X$ *is* $F - r_{\acute{e}}$ *chet differentiable ,*
*then* $(X, \|\cdot\|)$ *satisfies the Mazur intersection property .*

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*Proof .* Consider a closed convex and bounded set $C$ and assume that $0element - slash\ C$. We will find $x \in X$ and $r > 0$ such that $C \subset x + rB$ but $0element - slash(x + rB)$. Since $0element - slashC$, there i s a norm one functional $f \in S^*$ such that inf $f(C) > 0$. Using Bishop - Phelps theorem , we can find a norm - attaining functional $g \in S^*$ close enough t o $f$ so that inf $g(C) > 0$. If we pick $x \in S$ satisfying $g(x) = 1$ then
$g = \| \cdot \|' (x)$. The idea now is considering a ball big enough so that it s boundary play the role of a separating hyperplane . To this end , put $\varepsilon = ($ inf $g(C))/2$ and , for $n \geq 2$, consider the ball $B_n = n\varepsilon x + (n-1)\varepsilon B$. Clearly , for every $n \geq 2$ we have $0element - slashB_n$. We will show that $C \subset B_n$ for some $n$. If this is not the case , for each $n \geq 2$ we can choose $x_n \in C \setminus B_n$. Then $\| x_n - n\varepsilon x \| > (n-1)\varepsilon$ and hence

$$\| x - (1/n\varepsilon)x_n \| > 1 - 1/n \tag{1}$$

Using that $\| \cdot \|$ i s $\mathrm{F} - \mathrm{r}_{\acute{e}}$ chet differentiable at $x$ and $g = \| \cdot \|' (x)$ we can write ,

$$\| x + h \| - \| x \| - g(h) = r(h), \quad \text{where} \quad \lim_{h \to 0} r(h)/ \| h \| = 0. \quad \text{forevery} h \in X, \tag{2}$$

Replacing now in the above equation $h$ by $-(1/n\varepsilon)x_n$, using ( 1 ) and the equality $\varepsilon = \inf\ g(C)/2$, we obtain

$$r(-(1/n\varepsilon)x_n) = \| x - (1/n\varepsilon)x_n \| - 1 + g((1/n\varepsilon)x_n) > 1/n.$$

Hence , for $n \geq 2$,

$$r\|(_- -^{(1/n\varepsilon)x_n}_{(} {}_{1/n\varepsilon)x_n}\| \geq \| (1/^{(1/n)}_{n)\varepsilon^{-1}x_n} \| \geq \sup_n \varepsilon\{ \| x_n \|\} \quad . \tag{3}$$

which contradicts ( 2 ) since $\{x_n\} \subset C, C$ i s bounded and $\lim_n \| (1/n\varepsilon)x_n \| = 0$.

Norm one functionals $f \in X^*$ satisfying that for every $\varepsilon > 0$ there exists a weak $*$ slice $\mathcal{S} = \{x^* \in B^* : x^*(x) \geq 1 - \delta\}$ ( where $x \in S$ and $\delta > 0$) such that diam $(f \cup \mathcal{S}) < \varepsilon$ were introduced in [ 7 ] under the name of *semi - denting* points . When , in addition , we ask that $f \in \mathcal{S}$, then we recover the classical definition of weak $*$ denting point . Semidenting points play an important role in questions related t o the MIP b ecause of the following key result , due t o Chen and Lin , whose proof can be found in [ 7 ] . It is the key to the subsequent characterization of MIP , probably the most useful between the several characterizations known of this property [ 1 8 ] .

PROPOSITION 1 . 2 . *A functional* $f \in S^*$ *is a semi - denting point of* $B^*$ *if*
*and only if for every closed convex and bounded set* $C$ *and every* $x \in X$, *if* $f$
*separates* $C$ *and* $x$ *then there is a ball* $D$ *in* $X$ *with* $C \subset D$ *and* $xelement - slashD$.

PROPOSITION 1 . 3 . *Given a Banach space , the following conditions are*

*equivalent* :

( i ) *The space has the Mazur intersection property .*

( ii ) *There is a dense set of semi - denting points in* $S^*$.

( iii ) *There is a dense set of weak* $*$ *denting points in* $S^*$.

*Proof .* To prove the equivalence between ( ii ) and ( iii ) , note that weak $*$ denting points are semi - denting points so we only need t o prove ( ii ) $\Longrightarrow$ ( iii ) . To this end , define $F_n$ as the set of those norm one functionals lying in the ( relative to $S^*$) interior of some $S^* \cap \Rightarrow notdef - w$ h e − notdef − negationslash r $notdef - eS - S$ i s a weak $*$ slice of diameter less than $1/n$. Then $F_n$ is open and , using ( ii ) , dense in $S^*$ *Therefore* $F = \cap \Rightarrow nnotdef$ a s od en s − e i n $S*($ a t − c u aly $, F$ i a $G\delta$ d en s − e s t ) . N o e − t, fi naly , t hat $F$ i t he t o fw eak $*$ d en t − i n gp on s − t o f $Sasteriskmath - period$

To prove that ( i ) implies ( ii ) , we will use Proposition 1 . 2 t o see that every norm one functional i s a semidenting point . Indeed , consider $f \in S^*$, $C$ a closed , convex and bounded set and , finally , $x \in X \setminus C$. Assume , for instance , that $f(x) > 0$ and sup $f(C) < 0$( otherwise we can consider a suitable translation $C - y$ and $x - y$). There i s $\lambda > 0$ satisfying $C \subset \lambda M_f$ where $M_f = \{z \in B : f(z) \leq 0\}$. Now , since $X$ has the MIP , $M_f$ is an intersection of balls , thus implying the existence of a ball $D$ containing $M_f$ but missing $x$. The same ball $D$ separates $C$ from $x$.

The arguments t o prove that ( ii ) implies ( i ) are quite similar . Let $C$ be convex , bounded and closed and let $xelement - slashC$. By using ( ii ) , we can find a semi - denting point $f \in S^*$ separating $C$ from $x$, say for instance that sup $f(C) < f(x)$. We may assume that sup $f(C) < 0$ and $f(x) > 0$. Clearly , for enough big $n \in \mathbb{N}$, $C \subset nB$ and $x \in nB$. Using that $f$ i s semi - denting , it i s not difficult t o prove that $M_f$ i s an intersection of balls , and so it i s $nM_f$. As a consequence , there i s a ball containing $M_f$ ( hence $C$) that miss the point $x$, thus implying that $C$ i s also an intersection of balls .

Clearly , the set of semi - denting points i s closed . Indeed , if $f \in S^*$ i s not semi - denting , there i s $\varepsilon > 0$ such that the set $B(f, \varepsilon) = \{x^* \in S^* : \| x^* - f \|_* < \varepsilon\}$ does not contain the intersection of $S^*$ with a weak $*$ slice and thus no point $g$ of $B(f, \varepsilon)$ i s semi - denting , either . As a consequence ,

condition ( ii ) of Proposition 1 . 3 easily implies that every norm - one functional
is a semi - denting point . A weak $*$ denting point is an extreme point . In a finite dimensional space , and extreme point i s always a weak $*$ denting point , so the classical Phelps ' result i s inmediate from the above proposition .

COROLLARY 1 . 4 . [ 47 ] *A finite dimensional normed linear space $X$ has the MIP if and only if the set of extreme points of $B^*$ is dense in $S^*$ .*

Since the weak $*$ denting points of $B^{**}$ must b e points of $X$, we get easily the following consequence of Proposition 1 . 3 . Besides , having in mind Pro - position 1 . 1 , note also that next corollary generalizes the well known result that $X$ is reflexive if the norm of $X^*$ i s F – r$_é$ chet differentiable .

COROLLARY 1 . 5 . *A Banach space whose dual $X^*$ satisfies the MIP is re - flexive .*

There exist some other characterizations of spaces with MIP , in t erms of the duality mapping , support mappings and points of $\varepsilon$– differentiability ( see
[ 1 8 ] ) , though probably the most useful i s the one given in Proposition 1 . 3 .

Among the several intersection properties that appeared as variations on the MIP , probably the most important is the *weak $*$ Mazur intersection prop - erty* or MIP $*$ introduced in [ 18 ] : a dual space satisfies the MIP $*$ if every
weak $*$ compact convex set i s an intersection of closed dual balls . In [ 18 ] it i s
shown that every result for MIP has an analogous formulation for MIP $*$. In particular , it is connected with convexity properties of the predual space :

PROPOSITION 1 . 6 . [ 1 8 ] *A dual space $X^*$ has the MIP $*$ if and only if the set of denting points of the predual unit ball is dense in its unit sphere .*

The nice piece of work contained in [ 1 8 ] was the culmination of previous
results obtained , among others , by Phelps [ 47 ] and Sullivan [ 56 ] . Since these pioneering works , the investigation on different intersection properties has been slow but st eady . Whitfield and Zizler studied in [ 60 ] the property that every compact convex set is an intersection of closed balls . F – u rther research on this property was carried out lat er by Sersouri in [ 52 ] and [ 53 ] and lat er by J . Vanderwerff [ 59 ] . The corresp onding intersection property for weakly
compact and convex sets was investigated by Zizler in [ 65 ] and J . Vander- werff
in [ 59 ] . Finally , an uniform version of the MIP was considered in [ 61 ] by Whitfield and Zizler . A unified approach t o different intersection properties is presented by Chen and Lin in [ 6 ] . Other authors have also contributed to

the study of MIP and MIP $*$ as Acosta and Galan in [ 1 ] , P . Bandyopadhyaya and A . Roy in [ 3 ] and finally , P . Georgiev and P . S . Kenderov , whose results will be mentioned in the next sections .

## 2 . Renorming Banach spaces with MIP or MIP $*$

Both MIP and MIP $*$ are metric properties and hence invariant under i so - metries but not under i somorphisms . The question of whether a Banach space can b e renormed with MIP or a dual space with MIP $*$ has not an easy answer . Indeed , one might well ask how , when provided with a norm , one can construct an equivalent norm such that every closed convex body i s an intersection of ( new ) closed balls . Zizler [ 65 ] realized that T – r oyansky renorming techniques for LUR norms ( [ 1 1 ] , Lemma 7 . 1 . 1 ) can b e applied to study intersection prop - erties . This fruitful idea turned out t o b e sp ecially successful when applied first t o MIP $*$ [41] and later t o MIP [ 3 1 ] . Recall that a biorthogonal system $\{x_i, x_i^*\} i \in I \subset X \times X^*$ i s *fundamental* provided $X = \text{span } (\{x_i\} i \in I)$. Through - out this section and also in Section 3 , all Banach spaces are assumed to b e infinite dimensional .

Lemma 2 . 1 . *Let $X$ be a Banach space with a fundamental biorthogonal system $\{x_i, x_i^*\} i \in I \subset X \times X^*$. Then , the subspace $Y = \text{span } (\{x_i\} i \in I)$ admits a LUR norm .*

Theorem 2 . 2 . *Let $X$ be a Banach space with a fundamental biorthogonal system . Then $X^*$ admits an equivalent norm with the MIP $*$.*

The above theorem applies to a fairly wide class of Banach spaces includ - ing , for instance , the dual of $\ell_\infty(\Gamma)$. This fact will b e used lat er t o prove that almost every norm ( in the sense of Baire ) in this space i s F – r$_{é}$ chet differentiable on a dense set . We only know few Banach spaces which admits no fundamental biorthogonal system . This is the case , for instance , of Kunen , Shelah and the space $\ell_\infty^c(\Gamma)$ ( the subspace of all elements of $\ell_\infty(\Gamma)$ with countable support , card $\Gamma$ b eing strictly bigger than the cardinal of the continuum ) , spaces that will appear lat er in this survey . Before stating the analogous versions of these results for the MIP let us mention that , once we know that there i s an equi - valent norm with MIP ( or MIP $*$, if it i s dual ) in a Banach space , then there are many . In fact , Georgiev [ 1 6 ] proved that almost every norm ( again in the Baire sense , that will be precised latter ) satisfies this property provided there is one satisfying it .

PROPOSITION 2 . 3 . [ 1 6 ] *Given a Banach space* $X$, *the set of norms having the MIP is either empty or residual . Analogously , the set of dual norms having MIP* $*$ *is either empty or residual ( in the set of all dual norms ) .*

This result has many applications . For instance , it can b e used together with the following proposition t o show the density of norms which are $\mathrm{F} - \mathrm{r}_{\acute{e}}$ chet differentiable in open dense sets in spaces with MIP or MIP $*$. There exist even stronger results linking MIP , MIP $*$ and differentiability that will b e discussed lat er , in the section devoted t o almost Asplund spaces .

PROPOSITION 2 . 4 . [ 4 1 ] *If* $X^*$ *has MIP* $*$, *then the predual norm can be approximated by norms which are* $F - r_{\acute{e}}$ *chet differentiable on an open dense set . Also , if* $X$ *has MIP , then the dual norm can be approximated by ( dual ) norms which are* $F - r_{\acute{e}}$ *chet differentiable on an open dense set .*

It was for long t ime an open problem to determine whether spaces with the MIP are Asplund spaces . Also , it was unknown if every Asplund space admits a norm with the MIP or , in particular , a $\mathrm{F} - \mathrm{r}_{\acute{e}}$ chet differentiable norm . The latter was shown in the negative by Haydon [ 28 ] . First and second problems were also answered in the negative in [ 3 1 ] using , t ogether with Proposition 1 . 3 , the following results .

THEOREM 2 . 5 . *Let* $(X^*, \| \cdot \|^*)$ *be a dual Banach space with a biorthogonal system* $\{x_i, fi\} i \in I \subset X^* \times X$ *and* $X_0 = \mathrm{span}\ (\{x_i\} i \in I)$. *Then ,* $X^*$ *admits an equivalent dual norm* $| \cdot |^*$ *which is locally uniformly rotund at the points of* $X_0$. *Then , if* $X_0$ *is dense in* $X^*$, *the Banach space* $X$ *with the predual norm* $| \cdot |$ *has the Mazur Intersection property .*

*Outline of the proof .* We may assume that $\| fi \| = 1$, for every $i \in I$ and let us consider $\Delta = \{0\} \cup \mathbb{N} \cup I$. Define the map $T$ from $X^*$ into $\ell_\infty(\Delta)$ as

$$T(x)(\delta) = braceleftmid - braceleftbt \begin{array}{ll} \succ: & \\ \| x \|^* & \text{if } \delta = 0 \\ 2^{-n} G_n(x) & \text{if } \delta = n \in \mathbb{N} \\ fi(x) & \text{if } i \in I \end{array}$$

for every $x \in X^*$ and $\delta \in \Delta$, where

$$F_A(x) = \sum_{i \in A} | fi(x) |$$

$$E_A(x) = \mathrm{dist}\ (x, \mathrm{span}\ (\{x_i\} i \in A)) \quad A \subset I, \quad \mathrm{card}\ A < \infty$$

$$G_n(x) = \sup_{\text{card}A \leq n} \{E_A(x) + nF_A(x)\}.$$

Clearly $T(X^*) \subseteq \ell_\infty(\Delta)$ and $T(X_0) \subseteq c_0(\Delta)$. On the other hand , since

$2^{-n}(1+n^2) \leq 2$ for every $n \in \mathbb{N}$, we have $\| x \|^* \leq \| T(x) \|_\infty \leq 2 \| x \|$ .

For every $\delta \in \Delta$, consider the map $T_\delta(x) = T(x)(\delta), \quad x \in X^*$. Obviously , if $\delta \in I \cup \{0\}$ the map $T_\delta$ is weak $*-$ l . s . c . . Moreover , the maps $F_A$ and , the maps $E_A$ are weak $*-$ l . s . c . , so $T_\delta$ i s weak $*-$ l . s . c . for every $\delta \in \Delta$.

Let $p$ b e the Day norm [ 1 1 , p . 69 ] in $\ell_\infty(\Delta)$, and consider in $X^*$ the map $n(x) = p(T(x)), \quad x \in X^*$. It can be easily proved that $n(\cdot)$ i s an equivalent norm in $X^*$. The norm $n(\cdot)$ has the following expression :

$$n(x)^2 = \sup\{\sum_{n}^{i=1} | T^{(x)}_{\delta i 4\, i} |^2 \quad : \quad (\delta_1, \delta_2, ..., \delta_n) \subset \Delta, \quad \delta_i \neq \delta_j, \quad n \in \mathbb{N}\}$$

so $n(\cdot)$ is weak $*-$ l . s . c . , that i s , $n$ is a dual norm $| \cdot |^*$ . The norm $p$ defined in $\ell_\infty(\Delta)$ i s lo cally uniformly rotund at the points of $c_0(\Delta)$. It can b e checked that the norm $| \cdot |^*$ i s locally uniformly rotund at the points of $X_0$[31]. Now , it i s straightforward to verify that the points of $X_0 \cap arrowdblright - asteriskmathnotdef$ a e $-$ notdef$notdef$ w $notdef -$ $ak -$ infinity$* \in notdef - infinitynotdef - n_{\text{t}-\text{notdef}}\text{i} - \text{n}notdef$ g $\infty notdef - notdef$ ints o $B*$. F naly , if th e s bspace $X$ is d nsein $Xasteriskmath - comma$ b yt e P opos t $-$ i io n 3 , t e s ace $X$ e dowed w th t e p edual n r mo $| \ * \ |$ h s t e M a $-$ $\text{z}_{\text{ur}}$ tersection p operty .

COROLLARY 2 . 6 . *Let $X, Y$ be Banach spaces such that* dens $X^* \leq$ dens $Y^*$. *Suppose that $Y^*$ has a fundamental biorthogonal system* $\{yi, fi\}i \in I \subset Y^* \times Y$. *Then , the Banach space $X \oplus Y$ admits an equivalent norm with the*
*MIP .*

*Proof .* Let us consider $Z = X \oplus Y$ with the norm $\| (x, y) \|_Z = \| x \|_X + \| y \|_Y$. By Theorem 2 . 5 we need only t o show that $Z^* \approx X^* \oplus Y^*$ has also a fundamental biorthogonal system in $Z^* \times Z$. An element $x^* + y^*$ of $X^* \oplus Y^*$ i s considered an element of $Z^*$ in the usual way : $(x^*+y^*)(x+y) = x^*(x)+y^*(y)$ for every $x \in X, y \in Y$. Relabel the fundamental biorthogonal system given in $Y^*$ as $\{y_i^n, f_i^n\}i \in I, n \in x4e$ . We may assume that $\| y_i^n \|_Y \leq 1/n$ for every
$i \in I, n \in \mathbb{N}$. Let us take a dense set $\{x_i\}i \in I$ of $X^*$. Then , the system

$$S = \{x_i + y_i^n, \quad f_i^n\}i \in I, n \in x4e \subset Z^* \times Z$$

is a fundamental biorthogonal system in $Z^*$ and we conclude the proof .

As a corollary , we get that every Banach space $X$ can be embedded into a Banach space with the MIP : just consider $X \oplus \ell_2(\Gamma)$ with card $\Gamma =$ dens $X^*$ Thus , for instance , the non - Asplund space $\ell_1 \oplus \ell_2(c)$ admits an equivalent norm with the MIP . We also obtain as an application of the above corollary the following result of Deville [ 8 ] .

COROLLARY 2 . 7 . [ 8 ] *For every ordinal $\eta$, the long James space $J(\eta)$, i ts predual $M(\eta)$ and every finite dual of $J(\eta)$ admit an equivalent norm with the Mazur intersection property .*

*Proof .* First , we need t o observe that $\ell_2(\eta)$ can be complementably em - bedded into $J(\eta)$. Indeed , consider the subset

$$A = \{\alpha \in [0, \eta] : \alpha = 2n \text{ or } \alpha = \gamma + 2n, \text{ with } \gamma \text{ ordinal limit and } n \geq 1\}$$

and the subspace $H(\eta) = \{f \in J(\eta) : f(\alpha) = 0 \text{ if } \alpha element - slash A\}$. The subspace $H(\eta)$ i s i somorphic t o $\ell_2(A)$ and card $A =$ card $\eta$. On the other hand , the proj ection $f \in J(\eta) \to p(f) \in H(\eta)$ defined as

$$p(f)(\alpha) = \begin{cases} f(\alpha) - f(\alpha - 1) & \text{if } \alpha \in A \\ 0 \text{ if } & \alpha element - slash A \end{cases}$$

is continuous and , therefore $, H(\eta)$ i s complemented in $J(\eta)$. Thus , we have that $J(\eta) \approx \ell_2(\eta) \oplus Y$ for a Banach space $Y$ ( which can b e easily identified

$$J(\eta)) \text{and} J(\eta)^* \approx \ell_2(\eta) \oplus Y^*. \quad \text{with}$$

On the other hand , $M(\eta)$, $J(\eta)$ and every finite dual of $J(\eta)$ are As - plund spaces [ 1 2 ] . Consequently dens $\ell_2(\eta) =$ card $\eta \geq$ dens $Y =$ dens $Y^* =$ dens $Y^{**}$ and , applying Corollary 2 . 6 , we obtain that $J(\eta)$ and $J(\eta)^*$ admit a norm with the Mazur intersection property . The assertion for $M(\eta)$ and the dual spaces of $J(\eta)^*$ follows from the fact that $M(\eta)$ i s isometric t o $J(\eta)^*$ ( cf . [ 1 2 ] ) .

Consider the James ' tree space $JT$. It i s shown in [ 37 ] that $JT^{**}$ i s i so - morphic to $JT \oplus \ell_2(\mathbb{R})$. Then , as a consequence of Corollary 2 . 6 , we obtain that $JT^{**}$ and finite even duals of $JT^{**}$ admit an equivalent norm with the Mazur intersection property . On the other hand , notice that the space $JT^*$ and finite odd duals of $JT$ admit a $\mathrm{F} - \mathrm{r}_{\acute{e}}$ chet differentiable norm since their duals are WCG . We finish this section with the following consequences , the first one already mentioned .

COROLLARY 2 . 8 . ( i ) *Every Banach space* $X$ *can be almost isometrically complementably embedded into a Banach space with the Mazur intersection property .*

( ii ) *Every Banach space* $X$ *may be isometrically embedded into a Banach space* $Z$ *with the Mazur intersection property .*

*Proof .* ( i ) Let us consider the Banach space $Z = X \oplus \ell_2(\Gamma)$ with card $\Gamma =$ dens $X^*$. By Corollary 2.6, $Z$ can b e renormed with the MIP and a useful result of Georgiev [ 1 6 ] ensures that the set of equivalent norms with the Mazur intersection property in a Banach space i s either empty or residual . In this case the set is residual and implies the assertion . Notice that dens $Z =$ dens $X^*$. Clearly , this i s sharp in the sense that , necessarily , if a Banach space $Z$ has the MIP , dens $Z =$ dens $Z^*$. In addition , if $X$ i s a subspace of $Z$, dens $Z \geq$

$$\text{dens} X \quad {}^{*}$$
$$.$$

( ii ) We denote by $\alpha =$ dens $X, \beta = \alpha^+ (=$ min $\{\gamma$ ordinal number : card $\gamma > \alpha\})$, and the Banach space

$$m_\alpha(\beta) = \{x \in \ell_\infty(\beta) : \text{ supp } x \text{ has cardinality at most } \alpha\},$$

with the supremum norm $\| x \| = \sup_{\gamma<\beta} | x_\gamma |$ . Obviously , $X$ may b e i somet - rically embedded into $(m_\alpha(\beta), \| \cdot \|)$. On the other hand , by Corollary 2 . 8 , $m_\alpha(\beta)$ embeds into a Banach space $(Z, | \cdot |)$ with the Mazur intersection prop - erty and , by a result of Partington [46], $(m_\alpha(\beta), \| \cdot \|$ ) embeds *is ometrically* into $(m_\alpha(\beta), | \cdot |)$. Therefore , $X$ embeds isometrically into $(Z, | \cdot |)$. Note that , with this argument , we have dens $X^* <$ dens $Z^*$

We are concern now with the three - space problem for the MIP . The fol - lowing result states that being isomorphic to a Banach space with the MIP is a three space property [ 5 1 ] . An application of this result states that every space of continuous functions over a tree can b e renormed with the MIP [ 3 1 ] .

PROPOSITION 2 . 9 . *Let* $X$ *be a Banach space and* $Y$ *be a closed subspace of* $X$ *such that* $Y$ *admits a norm with the Mazur Intersection Property and* $X/Y$ *admits a norm with the Mazur intersection property . Then* $X$ *admits a norm with the Mazur intersection property .*

*Sketch of the Proof .* M . Raja [ 5 1 ] proved the following renorming the - orem : *Consider the set* $D$ *of all weak* $*-$ *denting points of the ( dual ) unit ball of a dual Banach space* $X^*$. *Then* $, X^*$ *admits an equivalent* dual *norm which is locally uniformly rotund at every point of* $D$. Thus , we may assume that both

$Y^*$ and $(X/Y)^*$ admit equivalent dual norms with a $(G_\delta)$ dense set of LUR points . The existence of an equivalent dual norm in $X^*$ with a $(G_\delta)$ dense
set of LUR points follows by imitating the proof of the three - space property for locally uniform rotund renormings given in [ 2 1 ] : We consider , under the
standard identifications $,(X/Y)^*$ to b e the annihilator subspace $Y^\perp$ with the weak $*$ t opology in $(X/Y)^*$ being the same as the induced weak $*$ t opology
which $Y^\perp$ inherites as a subspace of $X^*$. Then , we may assume that there is a norm on $Y^\perp$ which is $\sigma(Y^\perp, X)-$ l . s . c . and has a $G_\delta$ dense set of lo cally
uniformly rotund points . The subspace $Y^\perp$ i s weak $*$ closed so this norm can be extended to an equivalent dual norm $\|\cdot\|^*$ on $X^*$. Let $|\cdot|^*$ be an equi - valent dual norm on $Y^*$ which i s lo cally uniformly rotund at a $G_\delta$ dense set . Consider the restriction map $Q : X^* \to Y^*$, which is weak $*-$ weak $*$ continuous
and the Bartle - Graves continuous selection mapping $B : Y^* \to X^*$, which i s bounded on bounded set s , $B(y^*) = |y^*|^* B(y^*/|y^*|^*)$ and $B(0) = 0$. For every $y^* \in S_{|\cdot|^*} = \{y^* \in Y^* : |y^*|^* = 1\}$, take $y \in Y$ such that $y^*(y) = 1$ and
$|y| \leq 2$. Define $P_{y^*}(x^*) = x^*(y)B(y^*)$, for $x^* \in X^*$, which i s weak $*-$ weak $*$ continuous . The following family of weak $*$ l . s . c . convex functions defined

$$\text{on} X^*$$
$$\phi_y * (x^*) = |Q(x^*) + y^*|^*,$$
$$\psi_y * (x^*) = ||x^* - P_{y^*}(x^*)||^*, \quad y^* \in S_{||\cdot||}*,$$

is uniformly bounded on bounded set s . Therefore , if we consider

$$\phi k(x^*) = \sup\{\phi_y * (x^*)^2 + 1k\psi_y * (x^*)^2 : \quad y^* \in S_{|\cdot|^*}\},$$
$$\phi(x^*) = \|x^*\| *2+ |Q(x^*)|^{*2} + \sum_k 2^{-k}\phi k(x^*),$$

the Minkowski functional $|||\cdot|||^*$ of the set $\{x^* \in X^* : \phi(x^*) + \phi(-x^*) \leq 4\}$ i s an equivalent dual norm on $X^*$.

Consider the mapping ( not necessarily linear ) $S : X^* \to Y^\perp$, defined as $S(x^*) = x^* - B(Q(x^*))$. It i s proved in [ 2 1 ] that $x^*$ is a lo cally uniformly rotund point for $|||\cdot|||^*$ provided $Q(x^*)$ i s lo cally uniformly rotund for $|\cdot|^*$
and $S(x^*)$ is lo cally uniformly rotund for $\|\cdot\|^*$ in $Y^\perp$. To conclude , observe that the mappings $S$ and $Q$ are continuous and open . Then , the set s

$$L_{|\cdot|^*} = \{x^* \in X^* : |\cdot|^* \text{ i s locally uniformly rotund at } Q(x^*)\},$$
$$L_{\|\cdot\|} * = \{x^* \in X^* : S(x^*) \text{ i s weak } * \text{ denting of } \|\cdot\|^* \text{ in } Y^\perp\}$$

and therefore $L = L_{|\cdot|} * \cap \Rightarrow notdef - asteriskmath \quad notdef$ a e − notdef$notdef -$ $G\infty$d − element$notdef - infinity - n_{notdef-s}$ e $notdef$s − notdeft − infinity s $notdef -$ $notdef$o − braceleft$notdef - infinity X$period − asteriskmath − notdef $\in$ e − notdef − $\mathrm{n_{c-union}}e notdef - commat$ e s ace

$X \| \cdot \|)$ h sth e M a − $\mathrm{z_{urin}}$ter$^{\|}$ section p operty .

Haydon gave in [ 28 ] an example of an Asplund space admitting no equi - valent G $\hat{a}$ teaux differentiable norm , namely the space $C_0(L)$ of all continu - ous functions vanishing at the infinity over the following tree $L$ : denote by $\omega_1$ the smallest uncountable ordinal , $\alpha$ an ordinal number and consider $L = \bigcup_{\alpha<\omega_1} \omega_1^\alpha$ which i s called the full uncountable branching tree of height

$\omega_1$. Therefore , it i s a natural question t o ask whether the space $C_0(L)$ admits an equivalent norm with the Mazur intersection property [ 1 1 , Ch . VII ] . The answer i s affirmative . Moreover , for every tree $T$, the space $C_0(T)$ admits a norm with the Mazur intersection property .

LEMMA 2 . 1 0 . *Let K be a compact Hausdorff scattered space such that* card $K$ = card $I, I$ *being the set of isolated points of* $K$. *Then , the Banach space* $C(K)$ *admits an equivalent norm with the Mazur intersection property .*

*Proof .* The space $C(K)$ is an Asplund space , so it s dual space i s identifiable with $\ell_1(K)$. For every $\omega \in K' = K \setminus I$, we can consider disj oint subsets of different points $\{t_n^\omega\}_{n=1}^\infty \subset I$ and $A = I \setminus \{t_n^\omega : \omega \in K', \quad n \in \mathbb{N}\}$. Denote by $\delta_t \in \ell_1(K)$ the evaluation at the point $t \in K$ and by $\chi t$ the characteristic function at the point $t$. Clearly $\chi t \in C(K)$ if and only if $t$ i s an isolated point in $K$. Let us consider the biorthogonal system $\{y_n^\omega, f_n^\omega\}_n \in x4e, \omega \in K' \subset$

$C(K)^* \times C(K)$, where $y_n^\omega = (1/n)\delta_{t\omega_n}$ and $f_n^\omega = n\chi t\omega_n$ . Then , the system

$$\mathcal{S} = \{\delta_\omega + y_n^\omega, f_n^\omega\}_{n\in} x4e, \omega \in K' \cup \{\delta_t, \chi t\} t \in A \subset C(K)^* \times C(K)$$

is a fundamental biorthogonal system in $C(K)^*$. We apply now Corollary 2 . 6 to finish the proof .

*Remark* 2 . 1 1 . The above tree $L = \bigcup_{\alpha<\omega_1} \omega_1^\alpha$ equipped with the order topology i s a locally compact scattered Hausdorff space such that the cardinal of it s isolated points i s equal t o card $(L)$. Hence , it s Alexandrov compactifica - tion $\alpha L$ is a compact Hausdorff scattered space such that card $(\alpha L)$ = card $(I)$, $I$ b eing the set of i solated points of $\alpha L$. So , the Banach space $C(\alpha L)$ veri - fies Lemma 2 . 10 . As $C_0(L)$ is isomorphic t o $C(\alpha L), C_0(L)$ also verifies this Lemma .

Every tree $T$ equipped with the order t opology is a locally compact scattered Hausdorff space with card $(T) \geq$ card $(I), I$ being the set of i solated points of $T$. When card $(T) >$ card $(I)$ we cannot apply Lemma 2 . 1 0 but , in spite of this

fact , next proposition shows that $C_0(T)$ admits an equivalent norm with the MIP .

PROPOSITION 2 . 1 2 . *The Banach space* $C_0(T)$ *admits a norm with the Mazur intersection property whenever* $T$ *is a tree space .*

*Proof .* For any $t \in T$ we denote by $t^+$ the set of immediate successors of $t$
and consider the subset of $T$

$$H = \{t \in T' \quad : \quad t^+ = \varnothing\},$$

where $T'$ i s the set of all accumulation points of $T$ and the closed subspace of

$$C_0(T)$$
$$Y = \{f \in C_0(T) \quad : \quad f(t) = 0, \quad \text{if} \quad t \in H\}.$$

The space $T \setminus H$ i s lo cally compact , Hausdorff , scattered and verifies that the
cardinal of it s i solated points is equal t o card $(T \setminus H)$. Hence , the Alexandrov compactification $\alpha(T \setminus H)$ of $T \setminus H$ i s scattered and verifies that card $(\alpha(T \setminus H)) =$ card $(I), I$ being the set of i solated points of $\alpha(T \setminus H)$. Observe that $Y \approx C_0(T \setminus H)$ is isomorphic to the space of all continuous functions on
$\alpha(T \setminus H)$. Then , by Lemma 2 . 1 0 , we obtain a norm on $Y$ such that its dual norm has a dense set of locally uniformly rotund points . On the other hand , it can be easily verified using the fact that $H$ i s an antichain and the Tietze ' s extension theorem that $C_0(T)/Y$ i s i somorphic to $c_0(H)$, and then $, C_0(T)/Y$ admits a norm such that it s dual norm has a dense set of locally uniformly rotund points . Now the assertion follows from Proposition 2 . 9 .

## 3 . MIP , MIP $*$, ASPLUND AND ALMOST ASPLUND SPACES

The results obtained in the previous section provide a wide range of Banach spaces with an equivalent MIP norm . This could induce to think that this class of Banach spaces is larger than the class of Asplund spaces . This is not the case . There are Asplund spaces which cannot b e renormed with the MIP ( [ 3 1 ] and [ 22 ] ) . An example t o this assertion is the Kunen space [ 35 ] , a $C(K)$
Banach space where $K$ is a scattered compact set ( and thus $C(K)$ i s Asplund ) constructed assuming the continuum hypothesis . The Kunen space i s a non - separable Asplund space satisfying that for every uncountable set $\{x_i\} i \in I$ in the space , there exists $i_0 \in I$ such that

$$x_{i_0} \in \text{conv}(\{x_i\} I \setminus \{i_0\}^{)}. \tag{4}$$

The first example of a non - separable Banach space satisfying ( 4 ) was con - structed by Shelah assuming the diamond principle for $\aleph_1$[54].

PROPOSITION 3 . 1 . *The Kunen and Shelah spaces do not admit an equi - valent norm with the Mazur intersection property . Analogously , the duals of the previous spaces do not admit a dual norm with the MIP* *.

*Proof .* First , if a Banach space $X$ with a norm $|\cdot|$ has the Mazur intersec - tion property , then , by Proposition 1 . 3 ( iii ) , the dual norm $|\cdot|^*$ has a dense set of weak $*$ denting points in it s unit sphere . Consider $0 < \delta < 1$ and find a fam - ily of weak $*$ denting points $(f_\alpha)_\alpha \in I \subset S_{|\cdot|*}$ with card $I =$ dens $X^* =$ dens $X$
such that

$$| f_\alpha - f_\beta | \geq \delta, \quad \text{for} \quad \alpha \neq \beta. \tag{5}$$

Then , there i s a family of slices $S(B_{|\cdot|^*}, y\alpha, \rho\alpha)$, for $\alpha \in I$, with $| y\alpha | =$

$$1,$$

$$S(B_{|\cdot|*}, \quad y\alpha, \quad \rho\alpha) \cap arrowdblright - Bnotdef^{asteriskmath-notdef}, notdef \quad notdef - y \quad comma - notdef_{\infty\rho} \quad e \tag{6}$$

We denote $x_\alpha = (1/\rho\alpha)y\alpha$ for every $\alpha \in I$. It follows from ( 6 ) that $f_\alpha(x_\alpha) > 1$ and $| f_\alpha(x_\beta) | \leq 1$ for $\alpha, \beta \in I, \quad \beta \neq \alpha$. Consequently ,

$$x_\alpha element - slash\text{conv}(\{x_\beta\}_{\beta \in} I \setminus \{\alpha\}^{)}. \tag{7}$$

Therefore , if $X$ i s a non - separable Banach space with the MIP , there i s an uncountable subset $\{x_\alpha\}_\alpha \in I \subset X$ satisfying ( 7 ) . This implies that that the Kunen and Shelah spaces does not admit an equivalent norm with the MIP .

For the second assertion , consider the Banach space $(X^*, |\cdot|^*)$ with the
weak $*$ Mazur intersection property . Then , by Proposition 1 . 6 , the norm $|\cdot|$ has a dense set of denting points in it s unit sphere . Take $0 < \delta < 1$ and find a family of denting points $(x_\alpha)_\alpha \in I$ in $X, | x_\alpha | = 1$, with card $I =$ dens $X$ such that

$$| x_\alpha - x_\beta | \geq \delta, \quad \text{for} \quad \alpha \tag{8}$$

$$\mathrm{F - r_{om}getthat}^{\mathrm{the}}_{,}{}^{\mathrm{for}^{\mathrm{fact}}\mathrm{that}_{\mathrm{every}}\mathrm{the}_{\alpha,x}^{\mathrm{points}}}\alpha slash - element\mathrm{conv}(x_\alpha)^{I}_{(^{\alpha}\{^{\in}{}^{\mathrm{are}}_{x\beta\}}{}^{\mathrm{denting})}}{}^{\mathrm{in}B_{|}}_{\{\alpha\}.\mathrm{Thus},}{}^{\cdot|}{}_{\mathrm{the}}\mathrm{and}_{\mathrm{duals}}\mathrm{condition}_{\mathrm{ofthe}}{}^{(}_{\beta\in I\setminus}$$

and Shelah spaces do not admit an equivalent dual norm with MIP $*$.

The property exhibited in ( 4 ) shared by the spaces contructed by Shelah and Kunen , that i s , for every uncountable family of points in the space there is one point in the closed convex hull of the rest , has b een extensively studied
in [ 22 ] . Let us denote this property by $KS$. The following result was proved
for the Kunen space in [ 3 1 ] and for the general case in [ 22 ] .

THEOREM 3 . 2 . *Let $X$ be a Banach space . The following assertions are*

*equivalent* :

( i ) *$X$ has the KS property .*
( ii ) *Every weak $*-$ closed convex subset $K \subset X^*$ is weak $*-$ separable .*
( iii ) *Every convex subset $K \subset X^*$ is weak $*-$ separable .*

Let us mention that there are still a number of open problems concerning the MIP , as the existence of points of $\mathrm{F} - \mathrm{r}_{\acute{e}}$ chet differentiability in spaces with this property . While spaces with $\mathrm{F} - \mathrm{r}_{\acute{e}}$ chet differentiable norm satisfy the Mazur intersection property , it is unknown if it i s also the case of spaces with a ($^{\mathrm{F}-\mathrm{r}}\acute{e}$ chet ) differentiable bump function . In this setting , it was proved in [ 1 0 ] the following result .

THEOREM 3 . 3 . [ 1 0 ] *If a Banach space has the Radon - Nikod ý m property and a $F - r_{\acute{e}}$ chet differentiable bump function , then i t has an equivalent norm with the MIP .*

We are concerned now with the connections between Mazur intersection property on $X$ or weak $*$ Mazur intersection property on $X^*$ and the gen - eric differentiability of " most " equivalent ( dual ) norms defined on $X^*$ or $X$, respectively . Let $F$ be the space of all sublinear , positively homogeneous , continuous functionals on a Banach space $X$, furnished with the metric $\rho$ as - sociated t o the uniform convergence on bounded set s . Analogously , let $F^*$ be the space of all sublinear , positively - homogeneous , continuous and $w^*-$ lower semicontinuous functionals on $X^*$. The spaces $(F, \rho)$ and $(F^*, \rho)$ are complete metric spaces and thus Baire spaces .

A Banach space $X$( resp . the dual $X^*$ of a Banach space $X$) i s called *almost*
*Asplund* ( resp . *almost weak $*$ Asplund* ) space , if there exists a dense $G_\delta$ subset $F_0$ of $F$ ( resp . $F_0^*$ of $F^*$) such that every $f \in F_0$ ( resp . every $f^* \in F_0^*$)
is $\mathrm{F} - \mathrm{r}_{\acute{e}}$ chet differentiable on a dense $G_\delta$ subset of $X$( resp . of $X^*$). The first
author t o consider this class of Banach spaces was P . Georgiev [ 1 5 ] . He proved that MIP in $X$ and MIP $*$ in $X^*$ imply that $X$ i s almost Asplund and $X^*$ i s almost weak $*$ Asplund . More connections between differentiability of convex

functions and Mazur ( weak $*$ Mazur ) intersection properties were investigated by Kenderov and Giles [ 34 ] and J . P . Moreno [ 41 ] , among others . Later on , following the ideas of [ 1 5 ] , it was proved in [ 1 7 ] that the dual of a Banach space with the MIP i s a *almost wea* $k - asteriskmath$ *Asplund* space and , analogously , the predual of a dual space with the MIP $*$ intersection property is an *almost Asplund* space . We will focus here on this last result and it s geometrical derivations .

Some interesting consequences are obtained by considering *norms* instead of sublinear functionals . Among them , we can mention that " almost all in the Baire sense " ( we shall detail this lat er ) equivalent norms on a Banach space with a fundamental biorthogonal system are $\mathrm{F} - \mathrm{r}_{\acute{e}}$ chet differentiable on a dense $G_\delta$ subset . This i s the case , for instance , of spaces $\ell_1(\Gamma)$ and $\ell_\infty(\Gamma)$, for every $\Gamma$, whose bad differentiability behavior i s well known . Moreover , there are only few examples of spaces without fundamental biorthogonal system ( [ 49 ] , [ 44 ] ) so this result applies for most Banach spaces .

Denote by $\mathcal{H}_X$, or just $\mathcal{H}$ if there is no ambiguity on the space we are considering , the set of all bounded , closed , convex and nonempty subsets of a real Banach space $X$. The Hausdorff distance between $C_1, C_2 \in \mathcal{H}$ i s given by

$$d(C_1, C_2) = \inf\{\varepsilon > 0 : C_1 \subset C_2 + \varepsilon B, \quad C_2 \subset C_1 + \varepsilon B\},$$

where $B$ i s the unit ball of $X$. The space $(\mathcal{H}, d)$ is a complete metric space [ 36 ] and , hence , a Baire space . Denote by $\mathcal{H}^*$ the elements of $\mathcal{H}_{X*}$ which are weak $*$ closed . The space $(\mathcal{H}^*, d)$ is also a complete metric space . The mappings $I : (\mathcal{H}, d) \to (F^*, \rho)$, where $I(K) := \sigma_K$ the support functional on

$$K : \quad \sigma_K(x^*) = \sup_{x \in K} \langle x, x^* \rangle, \text{and} \hat{I} : (\mathcal{H}^*, d) \to (F, \rho), \text{where} \hat{I}(K^*) \quad := \sigma_K *,$$

the support functional on $K^*$ defined on $X$, $\sigma_K * (x) \quad = \sup_x *_\in K * \langle x, x^* \rangle$, are
homeomorphisms . The existence of the homeomorphisms $I$ and $\hat{I}$ and the duality between $\mathrm{F} - \mathrm{r}_{\acute{e}}$ chet differentiability and strong exposition can b e t ied together in the following Lemma 3 . 4 whose proof i s omitted .

LEMMA 3 . 4 . *A Banach space $X$ is almost Asplund if and only if there is a dense $G_\delta$ subset $\mathcal{H}_0^* \subset \mathcal{H}^*$ such that every element of $\mathcal{H}_0^*$ has a dense $G_\delta$ set of weak $*-$ strongly exposing functionals in $X$. A dual Banach space $X^*$ is almost weak $*$ Asplund if and only if there is a dense $G_\delta$ subset $\mathcal{H}_0 \subset \mathcal{H}$ such that every element of $\mathcal{H}_0$ has a dense $G_\delta$ set of strongly exposing functionals in $X^*$.*

Let $b : X \to S^*$ b e a selection of the subdifferential mapping of the norm , i . e . $\langle x, b(x) \rangle = \| x \|$ for every $x \in X$. Given $C \subset X$, $f \in X^*$ and $\alpha > 0$,
we will denote by $S(C, f, \alpha)$ the slice $\{x \in C : f(x) \quad > \sup \quad f(C) - \alpha\}$. The

following lemma i s a key tool in the proof of the result cit ed above . There is an analogous version for a dual Banach space with the weak $*$ Mazur Intersection
Property .

LEMMA 3 . 5 . *Let $X$ be an infinite dimensional Banach space with the Mazur intersection property . Then , for every $n \geq 2$, there is a subset $X_n \subset X$ such that :* ( i ) $\cup_{n=2}^{\infty} b(X_n)$ *is dense in* $S^*$,
( ii ) $\langle x, b(x)\rangle > \sup_{z \in X_n \setminus \{x\}} \langle b(x), z\rangle$, *for every* $x \in X_n$,
( iii ) $\| b(x) - b(y) \| > 1_n$, *for every* $x, y \in X_n$, $x \neq y$.

*Proof .* By Proposition 1 . 3 , the dual norm has a dense set $X_0^*$ of weak $*$ denting points in it s unit sphere . Consider for every $n \geq 2$, a maximal subset $X_n^* \subset X_0^*$ satisfying $\| x^* - y^* \| > 2/n$, for every $x^*, y^* \in X_n^*, x^* \neq y^*$. Then , $F_0^* = \cup_{n=2}^{\infty} X_n^* \subset X_0^*$ i s dense in $S^*$, and for every $x^* \in X_n^*$ there i s a slice
$S(B^*, yn(x^*), \gamma n(x^*)), yn(x^*) \in B^*$ and $\gamma n(x^*) \in (0, 1_n)$ so that ,

$$x^* \in S(B^*, yn(x^*), \gamma n(x^*)), \quad \text{diam} S(B^*, yn(x^*), \gamma n(x^*)) < 2^1 n$$

and

$$S(B^*, yn(x^*), \gamma n(x^*)) \cap arrowdblright - Bnotdef^{notdef-comma-asteriskmath} y_{notdef-n} \text{parenleft} - \text{notdef} znotdef - \tag{9}$$

for every $x^*, z^* \in X_n^* . x^* \neq z^*$. By ( 9 ) it follows that $yn(x_1^*) \neq yn(x_2^*)$ for $x_1^* \neq x_2^*$, i . e . the mapping $yn : X_n^* \to S$ i s an inj ection . We have $\| x^* -$
$b(yn(x^*)) \| < 2^1 n$, for every $x^* \in X_n^*$ and

$$\| b(yn(x_1^*)) - b(yn(x_2^*)) \| > 1_n$$

for each $x_1^*, x_2^* \in X_n^*$, $x_1^* \neq x_2^*$. If we define $X_n = \{1_{-\gamma n(x^*)}^{yn(x^*)} : x^* \in X_n^*\}$, then it i s easy t o check the conditions ( i ) , ( ii ) , ( iii ) and the proof i s completed .

THEOREM 3 . 6 . *Consider a Banach space $X$ with dual $X^*$ .*

( i ) *If $X$ has the Mazur intersection property then $X^*$ is almost weak $*$ Asplund .*

( ii ) *If $X^*$ has the weak $*$ Mazur intersection property then $X$ is almost Asplund .*

*Sketch of the proof .* The idea of the proof i s contained in Theorem 4 of [ 1 5 ] . In order t o prove ( i ) , it i s enough t o show the existence of a dense $G_\delta$ subset $\mathcal{B}_0 \subset \mathcal{H}$ such that every element of $\mathcal{B}_0$ has a dense $G_\delta$ set of strongly exposing functionals in $X^*$. Let $\{X_n\}_n \geq 2$ b e the sequence we have found in Lemma 3 . 5 and for every $x \in X_n$ define :

$$\alpha_n(x) = \langle x, b(x)\rangle - \sup_{y \in X_n \setminus \{x\}} \langle y, b(x)\rangle,$$

For integers $n \geq 2$ and $m \geq 1$ denote :

$$H_{n,m} = \{x \in X_n : \alpha_n(x) > 1/m\}$$

and define $\mathcal{B}_{n,m,k}$ as the set of all $Z \in \mathcal{H}$ for which there are $\alpha > 0$ and $\gamma > 0$ such that diam $S(Z, b(x), \alpha) < 1k - \gamma$ for each $x \in H_{n,m}$ if $H_{n,m} \neq \varnothing$ and
$\mathcal{B}_{n,m,k} = \mathcal{H}$ if $H_{n,m} = \varnothing$. It can b e proved that $\mathcal{B}_{n,m,k}$ i s a dense and open subset of $\mathcal{H}$ for every $n \geq 2$ and $m, k \in \mathbb{N}$. We omit the rather t echnical and cumbersome proof that can be found in [ 1 7 ] . Finally , it i s easy to see that every element of $\mathcal{B}_0 := \cap \Rightarrow notdefB{-}notdef^{comma-notdef-n}_{m-notdef} notdef{-}k{-}comma$ i $\infty$ s $negationslash{-}notdef{-}notdef{-}notdef{-}r$o $-$ notdefn $-$ notdef $-$ notdef $-$ notdef$^{g-braceleft}$ y $notdef{-}notdef{-}notdefelement{-}enotdef{-}notdef{-}notdef{-}x$ p o $-$ infinity$^{e-snotdef-d}$b $-$ notdef $-$ notdef$not$ $notdef - y_{e-notdef}$a $-$ unionh $-$ notdef $- cbackslash - x\varnothing^{\in} M - bardbl_{,}$b e n g $M =_{\cup}$ $nm\{b - parenleftw)$ : $w \in H$ $comma - n_m\}$. yt heB are c a t $-$ e goryt heo e $-$ r m $,_{\mathcal{B}}$0 i d en s $-$ e $\delta$ i n $\mathcal{H}_{\cdot S}$ n ce $M$ i d en s $-$ e i n $S*$ a nds n cet hes r ongy e xpo$^{i-s}$ n gf un c $-$ t io nas orma $G\delta$ s ub s $-$ e$_{t,t}$ hep roofi c ompete d $._T$ hep roofo f ( i ) i s m 1 $-$ i ar .

An interesting corollary i s now at hand , as a direct consequence of the above result and the results in section 2 .

COROLLARY 3 . 7 . *Consider a Banach space $X$ with dual $X^*$.*

( i ) *If $X$ has a fundamental biorthogonal system then $X$ is almost Asplund .*

( ii ) *If $X^*$ has a fundamental biorthogonal system $\{x_i, x_i^*\} i \in I \subset X^* \times X$ then*
*$X^*$ is almost weak $*$ Asplund .*

Let $N$ be the set of all equivalent norms on a Banach space $X$ furnished with the metric $\rho$, defined in this way ,

$$\rho(n_1, n_2) = \sup\{|\, n_1(x) - n_2(x)\,|; x \in B_{||\cdot||}\}, \quad \text{where} n_1, n_2 \in N,$$

and $N^*$ the set of all equivalent dual norms on $X^*$. Since $N$ i s an open subset of the complete metric space of all continuous seminorms on $X$ under the distance $\rho$ and the map $\pi : \|\cdot\| \to \|\cdot\|^*$ is an homeomorphism b etween $N$ and
$N^*$, both are Baire spaces . If the space $\mathcal{H}(\mathcal{H}^*)$ i s replaced by the set of all

unit balls of equivalent norms ( dual norms , resp ectively ) , we obtain analogous
results replacing $F(F^*)$ by $N(N^*)$.

There are few known Banach spaces without fundamental biorthogonal systems . In fact , the question whether every Banach space i s almost Asplund remains open . According to Corollary 3 . 7 , a possible counterexample should have no fundamental biorthogonal system . This is the case of Kunen space mentioned above , but it is Asplund . On the other hand , it is worth t o mention that the duals of the Kunen and Shelah spaces are not almost weak $*$ Asplund . In fact , there is no equivalent dual norm being $\mathrm{F} - \mathrm{r}_{\acute{e}}$ chet differentiable on a dense set in the preceding spaces . Otherwise , the unit ball of the associated ( predual ) norm in the Kunen or Shelah spaces would b e the closed convex hull of it s strongly exposed points . This would produce in the Kunen and Shelah spaces , by imitating the proof of Proposition 3 . 1 , an uncountable family satisfying the separation property given in ( 7 ) , thus a contradition . Plichko proved that $\ell^c_\infty(\Gamma)$ ( b eing card $\Gamma$ strictly bigger than the cardinal of the continuum ) does not admit a fundamental biorthogonal system . We do not know if this space and the Shelah space are almost Asplund .

Next theorem illustrates , under a different point of view , the relation - ship b etween convexity and Mazur intersection properties . As an application , analogies and differences between these properties and the Radon - Nikod $\acute{y}$ m property are exhibited . Our aim here is t o point out that Mazur intersec - tion properties seem to b e a good alt ernative t o Radon - Nikod $\acute{y}$ m property when some convexity conditions are required [ 1 7 ] , [ 23 ] and [ 29 ] . Recall that a Banach space $X$ is said t o have the Radon - Nikod $\acute{y}$ m property if every element of $\mathcal{H}$ i s the closed convex hull of it s strongly exposed points . A Banach space $X$ i s Asplund if and only if $X^*$ has the Radon - Nikod $\acute{y}$ m property .

THEOREM 3 . 8 . ( A ) *Let $X$ be a Banach space whose dual $X^*$ has the*
*weak $*$ Mazur intersection property . Then*

( i ) *there exists a dense $G_\delta$ subset $\mathcal{B}_0 \subset \mathcal{H}$ such that every element of $\mathcal{B}_0$ is the closed convex hull of i ts strongly exposed points .*

( ii ) *there exists a dense $G_\delta$ subset $\mathcal{B}_0^* \subset \mathcal{B}_{X*}^*$ such that every element of $\mathcal{B}_0^*$*
*is the weak $*$ closed convex hull of i ts weak $*$ strongly exposed points .*

( B ) *Let $X$ be a Banach space with the Mazur intersection property . Then there is $\mathcal{B}_0$ satisfying ( i ) and there exists a dense $G_\delta$ subset $\mathcal{B}_0^* \subset \mathcal{B}_{X*}^*$ such*
*that every element of $\mathcal{B}_0^*$ is the weak $*$ closed convex hull of i ts weak $*$ denting points .*

## 4. Intersection of closed balls and porosity

4.1. Distance of two sets. Given a normed space $X$, and two closed and bounded subsets $C, D \subset X$, denote by $\rho(C,D) = \inf \{\| x - y \| : x \in A, y \in B\}$. F. Hausdorff calls $\rho(A,B)$ the lower distance between $A$ and $B$, though it is clear that it is not a metric, since the triangle inequality is not fulfilled. How to define then a distance between closed and bounded sets? Here is the most accepted formula, namely the *Hausdorff distance*, that we have already used in section 3:

$$d(C,D) = \sup\{\rho(x,D), \rho(y,C) : x \in C, y \in D\}$$
$$= \inf\{\varepsilon > 0 : C \subset D + \varepsilon B \quad \text{and} \quad D \subset C + \varepsilon B\}$$

being $B$ the unit ball. A well known theorem of H. Hahn establishes that the family of all closed and bounded sets of $X$, endowed with the Haudorff distance, is a complete metric space when $X$ is complete [36]. Recall that $\mathcal{H}_X$ (or simply by $\mathcal{H}$, when it causes no confusion) denotes the family of all closed, bounded and convex subsets of $X$. To prove that $\mathcal{H}$ is also a complete metric space with the Hausdorff metric, when $X$ is complete, it just suffices to prove that, given a convergent sequence $\{C_n\} \subset \mathcal{H}$, the limit $C$ also is a convex set. We may assume that $d(C_n, C) < 1/n$, for every $n$. Defining $D_n = C_n + (1/n)B$, we know that $C \subset D_n$ and $d(D_n, C) < 2/n$, so $\lim_n \{D_n\} = C$. Now, take $x, y \in C$ and suppose that $z$ lies in the segment $[x,y] = \{tx + (1-t)y : t \in [0,1]\}$. If $z \; element - slash \; C$, there is $m \in \mathbb{N}$ satisfying $(z+(2/m)B) \cap \Longrightarrow notdef.notdef$ T $\mathrm{i-notdef-hs-infinity_{im}} \in \mathrm{l-p-infinity-notdefi-notdef-e^{s-notde}}$ t $infinity - a^{notdef-notdef-t}_{\{z} element - slash - notdef notdef - notdef - D notdef - notdef - notdef - notdef - notdef_{notdef}$wi − notdef − h c h − notdef c ntradicts t e a − f ct $\in D$, f o rev ery $n$ T h − u s $z \in C$ a d $C$s − i c nvex. T h − e refore, $\mathcal{H}$ e dowed w th e H a − u sdorff d stanceis a c $\mathrm{om^{l-p}}$ ete m e − t ric s ace a d h nce a B $\mathrm{i - a_{re}}$ s ace.

2. P o $r - o$ $s - u$ s ts. Motivated by problems in Real Analysis and, espe - cially, in differentiation theory, several authors considered what came to be known as *porosity*, a notion which concerns the size of holes of a set near a point. Topologically sp eaking, porous sets are smaller than merely being a countable union of nowhere dense closed sets [62]. Consequently, porosity has been usually used to describe smallness in a topological sense. Precisely, let $M$ be a metric space, $P$ a subset of $M$, $B(x,R)$ the closed ball centered at $x$ with radius $R$ and $\gamma(x,R,P)$ the supremum of all $r$ for which there exists $y \in M$ such that $B(y,r) \subset B(x,R) \setminus P$. The number

$$\rho(x,P) = 2 \limsup_{R \to 0} \quad \gamma(x, RR, P)$$

is called the porosity of $P$ at $x$. We say that $P$ is porous at $x$ whenever $\rho(x, P) > 0$ and , when $P$ is porous at every point of $M$, we simply say that $P$ is a porous set . If there is $\varepsilon > 0$ satisfying $\rho(x, P) > \varepsilon$ for every $x \in M$, then $P$ i s said t o b e uniformly porous . Finally , replacing " ) − one − six6(i − l m su " by " ) - 1 in the above definition , we encounter the notions of *very porosity* and *very porous s et* , respectively . The unit sphere of a normed linear space i s an easy example of an uniformly very porous set .

In convex geometry , the use of porosity received in recent years a great deal of attention . Several t opics as smoothness , strict convexity , diameters , nearest points and others have b een investigated by using porosity . We refer to the works of Zamfirescu [ 63 ] , [ 64 ] and Gruber [ 25 ] , [ 26 ] for more information about this rich line of research .

In Banach space theory , porosity has b een used t o describe t opological properties of the set of points of F − r echet nondifferentiability [ 48 ] , [ 50 ] and also in relation with questions of best approximation [ 5 ] and variational principles [ 9 ] . For these and other applications of porosity , we refer t o Zajicek ' s survey [ 62 ] and Phelps ' book [ 48 ] .

Let $\mathcal{M}$ b e the collection of all intersections of balls , considered as a subset
of $\mathcal{H}$ furnished with the Hausdorff metric . The space has the Mazur inter - section property or MIP if $\mathcal{M} = \mathcal{H}$[39]. We will prove that $\mathcal{M}$ i s uniformly very porous if and only if the space fails the MIP . To this end , we need a handy description of the elements of $\mathcal{H} \setminus \mathcal{M}$, obtained as a consequence of Proposition 4 . 1 , whose proof i s partially based in Proposition 1 . 3 . The only difficulty li es in ( iii ) implies ( i ) ( see [ 30 ] for the details of the proof ) . In what follows , given $f \in X^*$, we denote $K_f = \ker f \cap \Rightarrow_{L-notdef} notdef\text{equal} - \text{notdef} notdef x \infty \in B\text{colon} - \text{notdef} notdef f f x - notdef) \infty greaterequal - notdef - notdef\text{zero} - \text{notdef} - \text{notdef}_{\{} notdef notdef$

$$Mf = \{x \in B : fx) \leq 0_\}. \qquad \text{d}$$

$r - o \quad o - p$ SITION 4 1 . *Given a Banach space , the following conditions are*

*equivalent* :

( i ) *The space has the Mazur intersection property .* ( ii ) *There is a dense set* $F \subset S^*$ *satisfying* $M_f \in \mathcal{M}$ $(L_f \in \mathcal{M})$ *for each*

$$f \in F.$$

( iii ) *There is a dense set* $F \subset S^*$ *satisfying* $K_f \in \mathcal{M}$ *for each* $f \in F$.

THEOREM 4 . 2 . *The set* $\mathcal{M}$ *is uniformly very porous if and only if the space fails the Mazur Intersection Property .*

*Proof .* We find it convenient t o isolate from the argument the following observation : *consider* $C \in \mathcal{H}$ *and* $\lambda > 0$ *so that* $D = \{x \in C : d(x, \partial C) \geq \lambda\} = \varnothing$; *every set* $E \in \mathcal{H}$ *with* $d(C, E) < \lambda$ *contains also* $D$. The proof i s fairly

easy : if $x \in D \setminus E$, there i s a norm one functional $f$ separating $x$ and $E$. Say , for instance , that $f(x) > \sup\ f(E)$. Clearly , $\sup\ f(C) \geq f(x) + \lambda > \sup\ f(E) + \lambda$, so $d(C, E) > \lambda$, a contradiction .

By Proposition 4 . 1 , if $X$ fails the Mazur Intersection Property there i s a norm one functional $f$ such that $M_f\ element - slash\ \mathcal{M}$. It means that there i s also
$x_0 \in B \setminus M_f$ such that every ball containing $M_f$ contains also $x_0$. Denote by $\alpha = f(x_0) > 0$ and consider an arbitrary subset $C \in \mathcal{B}$. We will prove that

$$\rho(C, \mathcal{M}) = 2 \lim_{R \to 0} \inf\ \gamma(C, R^{R,} \mathcal{M}) \quad \geq \quad 1\alpha + \alpha \quad .$$

and the proof will b e accomplished by looking at two cases .

Case 1 . The functional $f$ attains it s maximum over $C$, say at $y0 \in C$. Define the sets $C_R = C + RB$ and $D_R = \{x \in C_R : f(x) \leq \sup\ f(C)\}$. Notice that $D_R element - slash \mathcal{M}$ since $D_R$ contains $y0 + RM_f$ and misses the point $y0 +$
$Rx_0$. However , we do not know the existence of $r > 0$ such that $B_d(D_R, r) \subset \mathcal{H} \backslash \mathcal{M}$, which is necessary to compute the porosity of $C$. It is then convenient to select a suitable modification of $D_R$, namely the set $E_R = D_R + \alpha R 2B$. We claim that the ball $B_d(E_R, \alpha R/2 - 1/n)$ satisfies

$$B_d(E_R, \alpha R/2 - 1/n) \cap arrowdblright - notdef \quad = notdef emptyset - notdef$$

for $n \in \mathbb{N}$ large enough so that $\alpha R/2 - 1/n > 0$. Indeed , if $G \in \mathcal{H}$ and $d(G, E_R) \leq \alpha R/2 - 1/n$ then $y0 + Rx_0\ element - slash\ G$ but , due to the first remark , $y0 + RM_f \subset G$ so every ball containing $G$ should contain also $y0 + Rx_0$.

Now , since $d(E_R, C) \leq R + R\alpha/2$, then $B_d(E_R, \alpha R/2 - 1/n) \subset B(C, R+\ R\alpha)$. It means that $\gamma(C, R + R\alpha, \mathcal{M}) \geq \alpha R/2 - 1/n$, for $n$ large enough , so $\gamma(C, R + R\alpha, \mathcal{M}) \geq \alpha R/2$, thus implying that

$$2 \lim_{R \to 0} \inf\ \gamma(C, R^R +^+ R^{R\alpha,}_{\alpha} \mathcal{M}) \quad \geq \quad \lim_{R \to 0} \inf\ R\alpha R +_{R\alpha} \quad = \quad 1\alpha + \alpha \quad .$$

Case 2 . The functional $f$ does not attain it s maximum over $C$. Given $R > 0$, we take $ym$ so that $f(ym) = \sup\ f(C)$ and $d(ym, C) < R/m$. Consider now $C_m = \text{conv}\ (\{ym \cup C\})$. Since $C_m$ satisfies the condition of Case 1 ,

$$\gamma(C_m, R + R\alpha, \mathcal{M}) \quad \geq \alpha R/2 \text{and}, \quad \text{consequently}, \gamma(C, R + R\alpha + R/m, \mathcal{M}) \quad \geq$$
$$\alpha R/2. \quad \text{Therefore}$$
$$2 \lim_{R \to 0} \inf \gamma(C, R^R +^+ R^{R\alpha}_{\alpha} +^+ R^{R/m,}_{/m} \mathcal{M})$$
$$\geq \lim_{R \to 0} \inf\ R + \alpha R R\alpha + R/m = 1 + \alpha^{\alpha} + 1/m$$

for every $m \in \mathbb{N}$ and the theorem is proved .

Notice that , if $Celement - slash\mathcal{M}$, then $x + \lambda Cslash - element\mathcal{M}$ for every $x \in X$ and $\lambda \in \mathbb{R}$. It means that $\mathcal{M}$ is porous in a much stronger sense than stated in Theorem 4 . 2 , and close t o the notions of cone meager and angle - smallness introduced by Preiss and Zajicek ( see [ 50 ] and [ 48 ] ) .

## 5 . Stability of the sum in $\mathcal{M}$.

Two of the most important ways of combining two convex sets $C, D$ t o produce a third one are the vector sum $C + D$ and the convex hull conv $(C \cup D)$,
together with the operations $C\hat{+}D = (C + D)$ and conv $(C \cup D)$ of forming the respective closures . The stability of $\mathcal{M}$ with respect t o the usual set operations is very easy t o check : $\mathcal{M}$ i s stable under translations , dilations and intersections and it i s not stable under unions , convex hulls and the closure of convex hulls . For instance , if you consider in $\mathbb{R} \oplus_\infty \mathbb{R}$ the sets $C = \{(0, 0)\}$ and $D = \{(1, 1)\}$, then conv $(C \cup D)$ i s not an intersection of balls . However , the situation with respect t o the sum and the closure of the sum seems t o b e more complicated . The present note i s concerned with the extent t o which the property of b eing an intersection of balls i s preserved by the operations $+$ and $\hat{+}$. We will concentrate our attention also in a modest but quite relevant question : let $B$ be the unit ball of $X$, $\lambda > 0$ and $C \in \mathcal{M}$; is it true that $C + \lambda B \in \mathcal{M}$? An affirmative answer t o this question would provide the following topological consequence for $\mathcal{M}$.

Proposition 5 . 1 . *The set $\mathcal{M}$ is a closed subset of $\mathcal{H}$ provided $C\hat{+}\lambda B \in \mathcal{M}$ for every $C \in \mathcal{M}$ and each $\lambda > 0$.*

*Proof .* Let $\{C_n\}$ be a sequence in $\mathcal{M}$ and let $C \in \mathcal{H}$ be such that $\lim_n d(C_n, C) = 0$. To prove that $C \in \mathcal{M}$, take $xelement - slashC$ and let $\delta =$ dist $(x, C) > 0$.
We may assume that $d(C_n, C) < \delta/4$, for every $n \in \mathbb{N}$. On the one hand , $C \subset C_n\hat{+}2d(C_n, C)B$ and , on the other hand $, xslash - elementC_n\hat{+}2d(C_n, C)B$. Now , as the set $C_n\hat{+}2d(C_n, C)B$ i s an intersection of balls , there is a ball $D$ such that

$$xelement - slashD \text{and} C \subset C_n\hat{+}2d(C_n, C)B \subset D.$$

The stability of $\mathcal{M}$ under the operation $+$ implies , in particular , that $C + D$ is a closed set whenever $C, D \in \mathcal{M}$. Therefore , in this case $, C\hat{+}\lambda B = C + \lambda B \in \mathcal{M}$ and , by the above proposition $, \mathcal{M}$ i s closed . Incidentally , let us mention that the stability under $\hat{+}$ does not imply the stability under the vector sum , as the following remark shows . Recall that many non - reflexive Banach spaces
can be renormed t o satisfy the MIP . The space $c_0(\mathbb{N})$ i s the simplest example

since every separable space with separable dual admits a $\mathrm{F}-\mathrm{r}_{\acute{e}}$ chet differentiable
( and thus MIP ) norm [ 1 1 ] .

*Remark* 5 . 2 . When $X$ i s a nonreflexive Banach space with the MIP and $C \in \mathcal{M}$, the set $C + \lambda B$ need not be closed . Consequently , $\mathcal{M}$ need not be stable under vector sums , even if it i s stable under $\hat{+}$.

*Detail .* Indeed , when $X$ is nonreflexive , there i s a functional $f \in S^*$ which does not attain it s norm . Since $X$ has the MIP and $\mathcal{M} = \mathcal{H}$, the set $C = \{x \in B : f(x) \leq 0\}$ i s an intersection of balls . However , this is not the
case for $C + \lambda B$ b ecause this set is not closed when $0 < \lambda < 1/2$. Indeed , there
is $x \in (1/2)B$ for which $f(x) = \lambda$. Hence for all $n$ with $1/n < 1/2 - \lambda$ we have

$x+(\lambda+1/n)B \subset B.$ Then$\varnothing = D_n = (x+(\lambda+1/n)B) \cap minus-arrowdblright-onenotdef(minus-notdefnotdefn$

$x \quad n_{\in} D,$ c l early $(n \quad +\lambda B) intersection-x$plus $-$ arrowdblright$notdef$parenleft $-$ notdef$slash-$
$notdefnotdef^{\text{parenright}-\text{notdef}} B$parenright $-$ infinity$notdef^{equal-negationslash-infinity-notdef} \varnothing$a $-$ notdef$notdef-$
$d$s $-$ infinity$^{\text{o}-\text{notdef}-\text{notdef}}\{notdef - notdef$s $-$ i $-$ notdef$element - inotdef - notdef -$
$notdef$nt $-$ notdef $-$ notdef$notdef - hunion - e$c $-$ notdef$^{\text{o}-\text{l}}_{notdef-s}$ ure of
$\lambda B$ H o ev e $-$ r, $C \cap$one $-$ arrowdblright $-$ minusparenleft $-$ notdef $-$ lambda$notdef-$
$parenrightnotdef = \varnothing notdefinfinity-a \in$ d s $-$ notdef $-$ infinity$notdef^{notdef-x} notdef \infty C$plus $-$ braceleft$notde$
$notdef - lambdanegationslash - notdef - B - notdefnotdef - notdefnotdef \cup notdef$

T H E $b - i_{na}$ Y INTE SECTION PROP $r - e$ TY . When $B$ is the unit ball and $C = intersection - Bi - arrowdblright_{\text{inotdef}} notdef$ a n $-$ notdef i n $-$ notdeft $-$ infinity $-$ e r s $-$ negationslash $-$ notdef $-$ notdef $-$ notdef ec $notdef - t$i $-$ notdef $-$ notdef $-$ notdef o {n $-$ notdef $-$ notdef $-$ notdef o $notdef -$ $notdef - notdef^{\text{b}} infinity - a$ ls $,_{notdef_{\text{i}}} negationslash - notdefnotdef - notdef - inotdef$ t $\cup notdef - m^{\text{p}-\text{angbracketright}}$i $-$ t$^{\text{n}}$ gt o w r $-$ i te

$$+\lambda B = \cap B - arrowdblright - inotdef - plus_{\lambda notdef} notdef - Bnotdefnotdef - equal \Rightarrow_B notdef - notdef - no \tag{10}$$

and , as a consequence , t o conclude that $C + \lambda B \in \mathcal{M}$. However , ( 1 0 ) i s false
in general . To be convinced of this , consider $(\mathbb{R}^2, \quad \| \cdot \quad \| 2)$ and define $B_1$ as the Euclidean unit ball , $B_2 = B_1 + (2, 0)$ and take $\lambda = 1$.

As an easy example , notice that ( 10 ) holds in $(\mathbb{R}^n, \quad \| \cdot \quad \|_\infty)$. Sine [ 55 ] proved that ( 1 0 ) i s satisfied in those normed spaces with the so called *binary intersection property* ( BIP ) : every collection of mutually intersecting closed balls has nonempty intersection . However , we will prove in Section 5 . 2 that the validity of ( 1 0 ) for every $\lambda > 0$ does not characterizes spaces with the BIP . This property plays a major role in questions of extendability of general continuous linear maps , as proved by Nachbin and Goodner ( see [ 45 ] and references therein ) . We note that normed spaces with the BIP are complete . Moreover , a Banach space $X$ has the BIP if and only if $X = C(K, \mathbb{R})$ with the supremum norm , where $K$ i s a extremally disconnected , or *Stonean* , compact Hausdorff space ( Nachbin [ 43 ] ,

Goodner [ 2 7 ] and Kelley [ 32 ] ) . The following proposition improves that above mentioned result of Sine .

PROPOSITION 5 . 3 . *If a normed space* $X$ *has the BIP then every ( nonemp -*

$ty)C = intersection - B_{\Rightarrow}i \in notdefnotdefnotdef - Mn^{infinity-d}D - negationslash - notdef - notdef - notd$

*Proof .* Recall that , as noted above , we have $X = C(K, \mathbb{R})$. Given an extreme point $e$ of the unit ball of $X$, there i s only one way of making $X$ into a complete vector lattice having $e$ as an order unit such that the norm deduced from the order relation and $e$ i s identical to the sup norm [ 43 ] . For instance , we can choose $e = 1_K$ if the canonical order induced by $\mathbb{R}$ in $C(K, \mathbb{R})$ is desired . Every closed ball is identical to a segment and , in particular ,

$$B_i = B(x_i, r_i) = [x_i - r_i e, x_i + r_i e].$$

$C = intersection - Bi - arrowdblright_{=notdef}[_{notdef}unotdef - pbraceleft - notdefx\infty notdef - notdef - notde$

and , analogously $, D = [\alpha, \beta]$. Indeed , given any bounded family $\{fi\} \subset C(K)$ both $\inf_i fi$ and $\sup_i fi($ taken in the order of $C(K))$ are continuous functions on $K($ see [ 38 ] , Prop . 1 . a . 4 ) . Consequently ,

$$intersection - B_{\Rightarrow} i + notdef D - notdef notdef \quad \infty^{=} \quad unotdef \mathrm{p} - \mathrm{notdef} - \mathrm{notdef} - \mathrm{notdef} braceleft - bracele$$

5 . 2 . The case of $c_0(I)$. The geometry of the unit ball of the space $\ell_\infty(I)$ is quite close t o that of the unit ball of $c_0(I)$. Thus , it seems quite natural to ask about the stability of $\mathcal{M}$ in this latter space . ( Recall that for a ( not necessarily countable ) set $I$, a point $x = (x_i)$ i s in $c_0(I)$ provided $x_i \to 0$ in the sense that for any $\varepsilon > 0$, there are only finitely many indices $i \in I$ for which $| x_i | > \varepsilon$.) First of all , we must try t o obtain an easy - to - use description of set s which are intersection of balls . Denote by $\{e_i\}$ and $\{fi\}$ the canonical basis of $c_0(I)$ and the associated functionals , respectively . Since the unit ball for the supremum norm on $c_0(I)$ i s $B = intersection - f_i minus - arrowdblright^1_{notdef}[notdef - minusnotdef, 1_{notdef})\infty$ i $notdef - notdef - notdef - negationslash$ i $notdef notdef - notdef - notdef$ e $notdef - notdef - notdef - asy_{element-t}$o $-$ notdef $-$ notdef $-$ howt hat $B\prime$ i a c ose db al w i – t h r adu s $\lambda > 0$ i a ndo ny i i h ast hef orm

.

$$\mathrm{angbracketright}^{\mathrm{d}}_{\in_{c_{notdef-notdef-notdef}}} notdef - notdef - notdef \overset{bi}{\mathrm{c}}\infty^{\mathrm{o}} \; -\overset{i=}{}_{notdef} \mathrm{no}^{notdef-negationslash_{notdef}} \; 2^{\cup^{\mathrm{s}}}_{\lambda} \mathrm{fo}^{\mathrm{a}^{1-1}}_{notdef\rangle} \quad \mathrm{bal-lw}^{i}_{\mathrm{h}} \mathrm{it}^{\in}$$

e m$^{\mathrm{p}}$t$^{\mathrm{y}}$i$^{\mathrm{t}}_{\mathrm{e}}$srec$_{\mathrm{t}}$ i o n w e h v

$$notdef - notdef^{notdef-intersection-notdef}_{=} \_i^{\infty} notdef - notdef - notdef - negationslash_{notdef} notdef - notdef -$$

Moreover , fixing an index $\alpha_0$, for each $i$ we have $a_{\alpha_0 i} \leq \sup_\alpha a_{\alpha i} \leq \inf_\alpha b_{\alpha i} \leq b_{\alpha_0 i}$ and , as a consequence , there exists $k > 0$ such that $-k \leq \sup_\alpha a_{\alpha i} \leq \inf_\alpha b_{\alpha i} \leq k$ for every $i \in I$. Conversely , *a set* $C = intersection - f_i minus - arrowdblright^1_{notdef} anotdef - icomma - notdef b_{notdef}]_{notdef}$ *i* $\infty$ *a n notdef − it − braceleftnotdef − notdef − notdef − e r* $- \in notdef - notdef - notdef$ *ction o fb lls p* $r - o^{vided}$ *t ere e ists* $k > s$ *ucht at* $-k \leq_a i \leq b \leq k$ *f r l i* os eet hs , l t $x \in$ *C* nds uppo$^{s-e}$, f ori n t − s a nc e − comma t hat $f$ 0( $x$) $<$ *i.* ec aim t hatt heb al $(ai + k$parenright $-$ e$i + kB$ c ontan s $C$ b utn ot $x._{\mathrm{T}}$ ot hs e nd , o t − efis − r$_{\mathrm{t}}$ t hat $f0($ $x - (a0 + k$parenright $-$ e$i0) < -k$ s o $x \in ($ $a0 + ke - parenrighti0 + kB._{\mathrm{C}}$ l − e a l − r y ,

$$\subset kB \cap minus - arrowdblrightnotdef - parenleft_{[}anotdefnotdef - zeronotdef - com$$

$$\cap minus - arrowdblrightnotdef - parenleftnotdef^0_{a-notdef-inotdef^{,bnotdef]}_\infty}) \quad element - propersubset_{notdef-infi}$$

Indeed , if $y \in kB \cap minus - arrowdblright^1_{notdef}(notdef - ainotdef, notdefbnotdef_{infinity-bracketright}), \in$ t $infinity - notdef - h^{\mathrm{e-n-notdef}} f - notdefnotdef - zero_{\mathrm{parenleft-infinity}y} notdef - notdef - minus\{i - notdef - notdef - notdefnotdef - notdef - notdef_{notdef-notdef-plus}$ $union - k - notdef^{\mathrm{parenright-e}} i - notdef0_{braceleft-parenright} = f0_{(y)} - (i0 + k) \geq$ $k$ a nda so $f0_{(y)} - (i0 + k) \leq b0 - ai - k \leq k - k - ai = -ai \leq k.$ F o − r nyo h e − r i ndex $i \neq i,$ w eh v − a e $|\, i(y - (i0 + k$parenright $-$ e$i0_{)|} =$ $\int\limits_f i(ybar - parenright \leq k.$ W ea e a dyn owt os ate t hen extp r − o poition .

ROPO *s − i* TION 5 4 . *Given C and D two ( nonempty ) intersections of balls in* $c_0(I)$, *the set* $C + D$ *is also an intersection of balls . Precisely , if* $C =$

$$\in notdef$$

$$tdef - notdef - notdef - negationslash^+_{notdef} d - notdef - notdef - notdef_i notdef - notdef - notdef - period$$

*f − period* The inclusion $C + Df - propersubset \cap^1_{\lfloor} arrowdblright - a_{notdef} i$ *notdef − plusnotdef − c_i b − notdefnotdef +_\infty d_{negationslash-notdef-notdef-notdef} i$bracketright − notdefs − notdef − notdef − s *notdef − infinity − r*a − notdefi − notdef gh t − notdef − notdef$^{notdef-notdef-f}$ o$^{infinity-r}$ wa *notdef* $- r_{\mathrm{period-negationslash-notdef-d}} notdef - notdef^{\mathrm{T}} notdef - o_\cup$ *notdef* e the rev rseinc usion , we wl as s ume t − h a t $0 \in [$ *comma − i di*$_]$ o r eve y $i \in I$ erwise , we wo l − d rep ace $\sum C$ n $D$ y $C\prime = C\_$ u a n $D\prime = D -$ som e $u \in D$ . L$_\sum$ $z = ii - e_i \in \{^f_\cap \Rightarrow a - bracketleft - notdef_i notdefnotdef^{+c}_{notdef} notdef, \infty b - i$ *notdef − plus − notdef − notdef − notdef − negationslashd − notdef − notdef − notdef*$^{\{}_{i]} braceright - infinity - notdef.notdef - notdef$ W e $\infty$ w − notdef$_{\mathrm{a}}$ t − infinity *notdefnotdef − negationslashnotdef − notdefnotdef*

$xe_i \in C$ a n d $= iye_i \in D$ s u h t − h at $zi = xi + \frac{y}{i}$ o r e e r y $i \in I.$

$+ ci \leq i \leq b - i \leq i_b +$ comma − i$_d$ e a c − h$i \in I$ f ll i n o o n e ( a n don l yon parenright − e of ow$_{\mathrm{i-n}}$ g s ubs$^{\mathrm{t-e}}$ s $I1 = \{ \in I a + c \leq z - i <$

$a_{i} comma - brace right^{I} \quad 2 = \{_{i} \quad \in \quad I \quad a \quad \leq , I \quad =_{3} \{_{i} \quad \in I \quad b < z i \leq \quad b - i + \quad i_{d}.$ W d e$_{\text{fi}}$e $-$ n$xi = \quad a$ i n a$_{\text{c}}$ s e $i \quad \in I 1, i - n \begin{matrix} \text{a} \\ \text{c} \end{matrix} e \quad i \in$ two $-$ I a n d $x - i = bi$ n a$_{\text{c}}$ e $\quad i \in$ three $-$ I. O b io$_{\text{u}}$ l $-$ y, $ai \leq xi \leq i - b$ a n d $= \quad i_{z} - \quad x_{i} \leq \quad i$ S n$_{\text{c}}$e$_{|^{x}}$ $\quad i - bar \leq | \, i^{|\text{f}}$ r$_{\text{o}}$ a l $-$ l$i \quad \geq m$ f o s m e $m \quad \in \mathbb{N}$, w a r $-$ e that $x($ and h $-$ e nce $y)$ isan e l em en o f $c(I)$.

Corollary 5 . 5 . *If* $\quad C = \cap \Rightarrow \alpha notdef \quad notdef$ a $\quad notdef - ninfinity - on^{e - notdef - notdef - notdef - negationslash} m^{p - notdef} notdef - notdef - notdef - t - y\{i - notdef - notdef - notdef \in t - er - notdef - notdef - notdef_{s} infinity - e$ c$t$ $i - notdef$ o $\quad notdef - negationslashn^{notdef - notdef - o} \cup^{b} l - a - notdef_{notdef - l}$ s i n $0, \| \quad \cdot \quad \| \quad \infty_{)}$ a nd $\lambda > 0$ t hen $\cap B - arrowdblright_{\alpha}^{notdef - plus - notdef} {}_{notdef} notdef - Bnotdefinfinity - equal \cap_{\Rightarrow} B - notdef - notdef - notdef - notdef - notdef - alpha_{\{} \text{plus} - \text{notdef}_{\in}^{notdef - notdef - notd} infinitynotdef - C - negationslash - notdef - notdef - notdefo_{n} \quad braceleft - notdef - notdef - notdef - e_{notdef - notdef - notdef - notdef} union - element$ n $\quad notdef - notdef - notdef - notdefl^{notdef \infty notdefth} negationslash - notdef \quad notdef \quad \cup i \quad t^{y}$ o f $\quad (1^{\text{zero} - \text{parenright}}$ d oe n $\quad$ ot c h r $\quad a^{c} t^{e}$ rz $\quad e_{s} \quad$ t he $\quad BI_{P}$

$\text{im} f - period_{\text{pi} - \text{le} - \text{s}} \text{Since}_{\text{that}} \quad {}_{B\alpha} +_{\lambda}^{B_{\alpha}} = B = \cap_{f - intersection_{i} notdef_{arrowdblright - minus^{1}}^{f}}^{i} \begin{matrix} (^{a - notdef notdef alpha - i_{notdef notde}^{b}} \\ \Rightarrow (notdef a - notdef - bracketleft^{not} \end{matrix}$

 again Proposition 5 . 4 we obtain

$$\cap \Rightarrow \alpha notdef notdef - lambda B - notdef notdef - equal \cap_{\infty} i - arrowdblright - notdef - notdef - notdef - n$$

5 . 3 . Polyhedral norms . Recall that a Banach space is *polyhedral* [ 33 ] if the unit ball of any of it s finite dimensional subspaces i s a polyhedron . The typical example of a polyhedral space is $c_0$ endowed with the usual supremum norm . Is it true that $\mathcal{M}$ is stable under vector sums in every polyhedral space ? We will answer this question in the negative , despite the fact that the geometry of the unit ball of these spaces i s quite close t o that of $(c_0, \quad \| \cdot \quad \|_{\infty})$.

Most of the knowledge that we have about polyhedral spaces is due t o the work of V . Fonf ( see [ 1 3 ] and [ 14 ] ) . Among many other things he proved
that , given a polyhedral Banach space $X$ with unit ball $B$, there is a set ( not necessarily countable )$\{fi\} i \in I$ of norm – one functionals such that :
( 1 3 ) For every $x \in X$, there is $i_0 \in I$ such that $\| x \| = fi_0(x)$
4 ) For every $i \in I, \text{elative}_{)\text{in}} f_{i\text{teriorin}}^{-1}(\{1\}) \cap \Rightarrow minus - ones_{(\{}^{notdef\text{n-notdef}notdef - ne - \text{infinity}}_{1\}})$ $\text{m} - \text{element}^{\text{t-inf}}$
y $notdef notdef$
With this t ool in our hands , we easily obtain a description of the sets in $\mathcal{M}$ which i s just a generalization of the one obtained for $c_0(\Gamma)$. In the following proposition , we keep the above notation ( see [ 24 ] ) .

Proposition 5 . 6 . *A bounded convex set $C$ in a polyhedral Banach space*
*is an intersection of balls if and only if* $C = intersection - f_i minus - arrowdblright^1_{notdef}[notdef - i \text{ n } notdef notdef - ff notdef - parenleft t_{\infty} C^{comma - parenright} negationslash - notdef - notdef - notdef - notdef notdef \text{u} - \text{notdef} - \text{notdef} - \text{notdef} braceleft - pf notdef - notdef - notdef - parenleft t_{\in} C^{\text{bracketright} - \text{parenright} - \text{notdef} - \text{notdef} - \text{notdef}}) . \infty notdef$

The Proposition above implies that in a finite dimensional Banach space with polyhedral norm , every set in $\mathcal{M}$ i s a *finite* intersection of balls . The first question p ertaining t o the stability of $\mathcal{M}$ in a polyhedral space i s whether ,
given two sets $C = intersection - f_i minus - arrowdblright^1_{notdef} a notdef - icomma - notdef b_{notdef}]_{notdef}$ a $infinity - n^{negationslash - notdef - notdef - notdef - d} D notdef - notdef - notdef - equal \cap_{notdef - notdef - notdef} i - arrowdblright - element^1_{notdef - notdef - notdef} \infty^{notdef}_{notdef - bracketleft^c} i notdef not$
$d^i_{notdef]} negationslash - notdef - notdef_{o notdef - notdef} \infty_{notdef} union - negationslash - notdef - notdef - notdef^{\text{h-notdef}} notdef - notdef - notdef - a - notdef braceleft - angbracketright_{notdef - notdef - notdef \in}$

$$+ D = i_{\cap}[^{notdef} a - notdef_i^{notdef +} notdef notdef - c^i infinity - b\text{plus} - \text{notdef} - \text{notdef} - \text{notdef} - \text{negationslash} n$$
$$)$$

As the next proposition shows , the answer t o this question can b e negative , even if we reformulate the question in a slightly different way : Is ( 1 5 ) true if we assume , in addition , that $a_i = \inf fi(C), b_i = \sup fi(C), c_i = \inf fi(D)$ and
$d_i = \sup fi(D)$? The answer is again *no* , since a positive answer would imply the stability of $\mathcal{M}$ under vector sums in every polyhedral Banach space , and this i s not the case even in finite dimensional spaces .

PROPOSITION 5 . 7 . *The set $\mathcal{M}$ is not stable under vector sums in* $(\mathbb{R}^n, \| \cdot$

$$\| 1), n > 3 orin \ell_1(I).$$

*Proof .* The segment $C$ j oining the point ( 1 / 2 , 1 / 2 , 0 ) with $(-1/2, -1/2, 0)$ is an intersection of exactly two balls of radius 1 . This i s also the case of the segment $D$ j oining the point $(-1/2, 1/2, 0)$ with $(1/2, -1/2, 0)$. However , the set $C + D$ is not an intersection of balls . Indeed , denote by $\{f1, f2, f3, f4\}$ the norm one functionals satisfying ( 13 ) and ( 14 ) and by $B$ the unit ball . Since
$C+D = B \cap \Rightarrow (notdef - x, y - element, notdef - z \text{parenright} - \text{element} notdef - element \mathbb{R} - \setminus \ni \ll \mho \approx^3 :$
$z = 0\}$, we have that $\inf fi(C+D) = -1$ and $\sup fi(C+D) = 1$ for every $i = 1, ..., 4$. According t o Corollary 6 . 4 , if $C + D$ were an intersection of balls then

$$C+D = intersection - f_i minus - arrowdblright^1_{notdef}(notdef - \text{in} notdef notdef - ff notdef - parenleft_\infty C^{\text{plus-nega}}$$

which is a contradiction .

The spaces $(\mathbb{R}^n, \| \cdot \| 1)$ and $\ell_1(I)$ are particular cases of $X = Y \oplus_1 Z$ where
$Y = (\mathbb{R}^3, \| \cdot \| 1)$ and $\oplus_1$ denotes that the sum i s endowed with the $\ell_1-$ norm . The intersection of every ball in $X$ with the subspace $Y$ is an $\ell_1-$ ball . As a consequence , if a closed , bounded and convex subset of $Y$ is an intersection of $X-$ balls , it is also an intersection of $Y-$ balls . Finally , the sets $C$ and $D$ considered in the above paragraph are intersection of $X-$ balls but this i s not the case for the set $C + D$. For instance , t o see that $D$ i s the intersection of the two balls $B_1 = (1/2, 1/2, 0) + B$ and $B_2 = (-1/2, -1/2, 0) + B$, just
take into account that , for every $x = (x_1, x_2, x_3) + z \in B_1 \cap \Rightarrow_{\text{w-notdef}} notdef$ h $notdef - v_{\text{e}\infty\in}$

$$-/2 \mid +x2 \quad -/2 \mid +x3 \mid +bardbl - z \parallel \leq \quad 1\text{ad} \mid x1 \quad + slash - one2 \mid +x2 \quad + \text{one} - \text{slash2} \mid +x3 \mid +z - bardb$$

1

nsequent $1 - \text{y}$,

$$1 \quad -12 \mid + \quad bar - x2 \quad -12 \mid \leq 1 - \mid 3 \mid - \quad \parallel \quad \parallel$$
$$1 \quad +12 \mid + \quad bar - x2 \quad +12 \mid \leq 1 - \mid 3 \mid - \quad \parallel \quad \parallel$$

and the only solution i s when $\mid x_3 \mid = \quad \parallel z \parallel \quad = 0$ and $(x_1, x_2) \in D$.

It has been proved in [ 24 ] that in $(\mathbb{R}^3, \quad \| \cdot \quad \| 1)$ the family $\mathcal{M}$ i s stable under
adding balls . As a consequence , we get that this property i s different from

being stable under the closure of vector sums . Though the result i s also
true for $(\mathbb{R}^n, \|\cdot\ \|1)$ with $n > 3$, the arguments of the proof are those of the tridimensional case , which has the advantage of great simplicity .

In Remark 5 . 2 , we observed the existence of spaces for which $\mathcal{M}$ i s not stable by adding balls . However , *we have no example of a normed space for which* $\mathcal{M}$ *is not stable under the operation* $C\hat{+}\lambda B$, $C \in \mathcal{M}$ *and* $\lambda > 0$. On the other hand , the set of norms for which $\mathcal{M}$ is stable under vector sums is not closed in the space of all equivalent norms , endowed with the uniform metric . Indeed , in a finite dimensional Banach space , the set of norms with the Mazur intersection property i s dense .

## 6 . Mazur sets and Mazur spaces

As we mentioned in the introduction , a set $C$ is an intersection of balls if it satisfies the following separation property : For every $xelement - slashC$, there i s a
closed ball $B$ such that $C \subset B$ but $xelement - slashB$. This property can be strengthened by simply replacing the point $x$ by a hyperplane . We say that $C$ i s a *Mazur*
*set* if given any hyperplane $H$ with dist $(C, H) > 0$, there i s a ball $D$ such
that $C \subset D$ and $D\cap \Longrightarrow notdefnotdefnotdef$ N t − infinity − $o_{e\in t} notdef - infinity - a^{t-notdef}_{notdeft}$ $i^{infinity-s}$ i s − notdef − notdef$\{notdef - infinity - enotdef - unotdef - iv^{a-infinity}$ l e − element n t − notdef − notdef$notdef - notdefnotdef - t \cup notdef - s$ y $H - i$ ng t at $C$

a M a − $z_{ur}$ e − s tif gi ven $f \in X*$ w t − h s p $fCnotdef) < \lambda$ th en h − t ereex i − s t s a b l − $l_D$ ch t − h at $C \subset D$ a d u − s p $fD) < \lambda$ ( $C^{o-n}$ siderth e h perplane $H = f1(\lambda))$. e − $n_{ote}$ b y $\mathcal{P}$ t e c lec tion o al M z − $a_{ur}$ s − e ts o a n r m d − e s ace .

By the separation theorem , every Mazur set i s an intersection of balls and so $\mathcal{P} \subset \mathcal{M} \subset \mathcal{H}$. However , we will show that the converse i s not always true , even if the norm i s F − $r_{é}$ chet differentiable . There are mainly two reasons connecting Mazur set s with the subj ect of this paper : On the one hand , $\mathcal{P}$ i s
always stable under ( the closure ) of vector sums ; on the other hand , sometimes

$$\mathcal{P} = \mathcal{M} = \mathcal{H}.$$

Proposition 6 . 1 . *Given two Mazur sets* $C$ *and* $D$, *the set* $C\hat{+}D$ *is always a Mazur set . However* , $C\cap arrowdblright - inotdefnotdef - nnotdef - tnotdefnotdef - ninfinity - c$ e $element - s$ s $infinity - notdef - a$ r $l - notdef - i$ y $notdefnotdef - aM - infinitya - braceleft - z_{u-notdef-notdefrnotdef^{s-notdef-e}} t - notdefnotdef - notdefnotdef - notdef$ C $union - n_{s-notdef}$ e $q - notdefuent^{y-l}$, *i a ways s able u derth e c osure o ve ctor s m sb tit i s n t n cessar i − l y able w th r spectto i − n tersections .*

*oo f − period* Let $C$ and $D$ be two Mazur subsets of a Banach space $X$. Consider a functional $f \in X^*$ and $\lambda \in \mathbb{R}$ such that $\sup f(C\hat{+}D) < \lambda$. Denote by

$\alpha = \sup\ f(C)$ and $\beta = \sup\ f(D)$. Clearly , $\sup\ f(C\hat{+}D) = \sup\ f(C) + \sup\ f(D)$ and so $\alpha + \beta < \lambda$. Therefore , there are two real numbers $\alpha'$ and $\beta'$ satisfying $\alpha < \alpha', \beta < \beta'$ and $\alpha' + \beta' < \lambda$. Now , since $C$ and $D$ are Mazur set s , there

are two closed balls $B_1$ and $B_2$ such that $C \subset B_1$ and $D \subset B_2$ satisfying sup $f(B_1) < \alpha'$ and sup $f(B_2) < \beta'$. The sum of the two balls $B_1$ and $B_2$ i s again a ball $B_3$ that obviously contains $C \hat{+} D$ and satisfies

$$\sup f(B_3) = \sup f(B_1) + \sup f(B_2) < \alpha' + \beta' < \lambda.$$

Since we know that there exist Banach spaces for which $\mathcal{M}$ i s not stable under
the closure of vector sums ( we proved that $(\mathbb{R}^3, \| \cdot \| 1)$ is such an example ) , the first part of this proposition implies that $\mathcal{P}$ can actually b e different from $\mathcal{M}$. The two segments $C$ and $D$ of Proposition 5 . 7 are the intersection of two balls ( which are , obviously , Mazur sets ) but they themselves are not Mazur
set s .

DEFINITION 6 . 2 . Spaces in which every element of $\mathcal{M}$ i s a Mazur set $(\mathcal{P} = \mathcal{M})$ will b e called *Mazur spaces* .

In an analogous way , we can define a subset $C$ of a dual Banach space $X^*$ t o b e a weak $*$ Mazur set if it can b e separated by balls from weak $*$ closed hyperplanes $H$ with dist $(C, H) > 0$. We can denote the family of all weak $*$ Mazur set s by $\mathcal{P}^*$ and we can say that $X^*$ i s a weak $*$ Mazur space if $\mathcal{P}^* = \mathcal{M}$. Proposition 6 . 1 can b e formulated for weak $*$ Mazur set s and proved in essentially the same way . We do not know , however , an example of a weak $*$ Mazur set which i s not a Mazur set . Therefore , we know no example of a weak $*$ Mazur *space* which is not a Mazur space ( that i s , a dual space for which $\mathcal{P} \subsetneq \mathcal{P}^* = \mathcal{M}$). Going back to Mazur spaces , the next proposition shows that the case $\mathcal{P} = \mathcal{M} = \mathcal{H}$ has a nice geometric characterization , in terms of weak $*$ denting points of the dual unit ball . Recall that a Banach space satisfies the MIP if and only if the set of weak $*$ denting points of the dual ball i s a residual set of the dual sphere [ 1 8 ] ( see also Proposition 1 . 3 ) .

PROPOSITION 6 . 3 . *A Mazur space $X$ satisfies the Mazur intersection prop - erty if and only if every norm one functional in $X^*$ is a weak $*$ denting point of $B^*$.*

*Proof .* Chen and Lin proved in [ 6 ] that $f$ is a weak $*$ denting point of the dual unit ball $B^*$ if , and only if , for every bounded subset $A \subset X$ with inf $f(A) > 0$ there i s a ball $D$ containing $A$ such that inf $f(D) > 0$. Suppose that $\mathcal{P} = \mathcal{H}$ and consider $f \in B^*$ and a bounded subset $A$ such that inf $f(A) > 0$ . Then $C \equiv \text{conv } (A) \in \mathcal{P}$ and thus there i s a ball $D$ satisfying $A \subset C \subset D$ with inf $f(D) > 0$. Conversely , let $C \in \mathcal{H}$ and $H$ b e a closed hyperplane such

that dist $(C, H) > 0$. We may assume that $H$ is the kernel of a norm - one
functional $f \in B^*$ and inf $f(C) > 0$. The existence of the desired ball i s due to the fact that $f$ i s a weak $*$ denting point .

In Remark II . 7 . 6 of [ 1 1 ] , there is an example of a dual norm on $\ell_1(\mathbb{N})$ with
the property that every point of the unit sphere is a weak $*$ denting point . Consequently , Mazur spaces with the MIP need not be reflexive , although they are certainly Asplund spaces . Indeed , their dual spaces admit dual LUR norms [ 5 1 ] and , therefore , they admit F – r$_é$ chet differentiable norms . Spaces for which every point of the unit sphere i s a denting point can be characterized as those satisfying a weaker notion of lo cal uniform rotundity introduced by T – r oyanski in [ 58 ] and called *average locally uniform rotundity* ( see also [ 1 1 ] ) . On the other hand , there is a wide family of Banach spaces which are not Asplund spaces , even though they can b e renormed t o satisfy the MIP [ 3 1 ] . Obviously , these ( renormed ) spaces cannot b e Mazur spaces . The next corollary contains an example of an Asplund space satisfying the MIP but failing to be a Mazur
space .

COROLLARY 6 . 4 . *A reflexive space with a $F - r_é$ chet differentiable norm is always a Mazur space . However , spaces with $F - r_é$ chet differentiable norms need not be Mazur spaces . Finally , Mazur spaces with the MIP are always smooth
spaces .*

*Proof .* In a reflexive space with a F – r$_é$ chet differentiable norm , every norm one functional of the dual is the differential of the norm at some point . Con - sequently , it i s a weak $*$ strongly exposed point ( and thus a weak $*$ denting
point ) of the dual unit ball .

On the other hand , it is well known that there is only a partial duality between smoothness and convexity . As a matter of fact , from the pioneer - ing results about renormings on spaces of continuous functions on scattered compact spaces due t o Talagrand [ 5 7 ] , we know that there are spaces with F – r$_é$ chet differentiable norm whose dual space admits no rotund norm . This is the case , for instance , for $C([0, \omega_1])$. Since every weak $*$ denting point i s also an extreme point , the proposition above implies that the dual norm of a F – r$_é$ chet norm in a Mazur space must be rotund . As a consequence , $C([0, \omega_1])$, endowed with an equivalent F – r$_é$ chet differentiable norm is not a Mazur space . The previous proposition shows that , in particular , a Mazur space with the MIP has a dual rotund norm and thus the norm of the space itself is G $\hat{a}$ t eaux differentiable .

To finish our discussion on Mazur spaces and the MIP , notice that the condition of $\mathrm{F}-\mathrm{r}_{\acute{e}}$ chet differentiability in Corollary 6 . 4 i s essential . Indeed , there are even finite dimensional Banach spaces with the MIP which are not
Mazur spaces . Take , for instance , a norm in $\mathbb{R}^3$ with a dense set of denting points which contains a segment in it s unit sphere . The predual norm has the
MIP but $\mathbb{R}^3$ endowed with this predual norm is not a Mazur space .

6 . 1 . EXAMPLES OF MAZUR SPACES . This section i s devoted t o present - ing some examples of Mazur spaces which are not merely reflexive spaces with a $\mathrm{F}-\mathrm{r}_{\acute{e}}$ chet differentiable norm . We will prove that this i s the case for $c_0(I)$ and $\ell_\infty(I)$ with their usual norms . These spaces are natural candidates t o b e Mazur spaces in view of the results obtained in Sections 2 and 3 . It is a bit surprising that every two dimensional space i s a Mazur space . This result
distinguishes dimension $d \leq 2$ from dimension $d \geq 3$ : Note that $(\mathbb{R}^3, \|\cdot\|_1)$ i s not a Mazur space , since $\mathcal{M}$ i s not stable under vector sums ( Proposition 5 . 7 ) .

PROPOSITION 6 . 5 . *For every set $I$, the space $(c_0(I), \|\cdot\|_\infty)$ is a Mazur space .*

*Proof .* Consider $C = intersection - f_{i-arrowdblright}{}^{-1}_{notdef}\,(notdef - ai - notdef,_{notdef} b \quad notdef - bracketright_{)}, \infty_{\mathrm{a}}\mathrm{n} - \mathrm{negationslash} - \mathrm{notdef} - \mathrm{notdef} - \mathrm{notdefo} - \mathrm{notdef} - \mathrm{notdef} - \mathrm{no}$
$r\,\mathrm{m}\ notdef - notdef - notdef - o\mathrm{n} - \mathrm{element}^{notdef-notdef-notdef-e}\ \mathrm{f}\ \mathrm{u} - \mathrm{infinity}_{\mathrm{n-notdef^{t-c}}} negationslash - notdef - ion$
$equal_{)}iyei^{*} \in 1_{\mathrm{a}}\,\mathrm{ndt\ wor\ al\ \ n\ umbes}\ \alpha > \beta\ \mathrm{s\ ucht\ hati\ nf}\ f(C) = \alpha > \beta.{}_{\mathrm{T}}\ \ \mathrm{h}$
$\mathrm{e}-\mathrm{r_e}\ \mathrm{i}\ \ \mathrm{n\ ol\ ss\ ng\ en^{e-r}\ ality\ i\ na\ s}-\mathrm{sum^{i-n}\ gt\ hat}\ 0 \in C.{}_{\mathrm{W}}\ \mathrm{em\ u\ t}-\mathrm{sfi\ nda}$
b al $D$ s ucht hat $C \subset D$
ndi nf $f(D) > \beta$. ek nowf romS e $\mathrm{c}-\mathrm{t\ io\ n\ 5\ 2\ t\ hat}\ D = \cap i - arrowdblright^{1} notdef^{[}_{notdef-parenleft^{c}} icomma - notdef notdef_{i}^{d} notdef_{]}\infty_{\mathrm{i}}\ \mathrm{s}\ notdef - notdef - notdef - negationslash\ \mathrm{a}\ notdef\{\ \mathrm{lo\ f\ ra\ ius}\ \lambda > \mathrm{i\ f\ an\ don\ yif}\ i \to -, d_i \to$
$\lambda\ \mathrm{n\ d}\ di -_{c} i = \lambda_2\ \ \mathrm{S_i\ \ ce\ wa\ n}-\mathrm{t}C \subset D\mathrm{w_e\ ne\ d}\ [a\ \ b_i^{]} \subset i - c\ \ di_{]}\ \mathrm{f\ o\ r}$
ev ry $i \in$ a n dac ordingly we ose $\lambda = \mathrm{s_u\ p}\ \{\ \mathrm{ma\ x}-\mathrm{braceleft}\mid ai, \mid bi_{|}\}\}.$
T est ategy wi lb e t o d e ne $ci = -\lambda\ \mathrm{a\ n\ d}\ = \overset{\mathrm{e}}{\lambda}\ x\ \mathrm{ept\ f}-\mathrm{o^{r}\ \ afi\ n\ te}$
$\mathrm{nu\ mb^{r-e}}$ of co rdinates . M $\mathrm{r}-\mathrm{e}$ pr $\mathrm{ci^{e-s}\ ly\ ,le_t}\ F \subset \mathrm{b\ e\ n\ tes\ e}$
$\mathrm{su\ h\ \ th\ t^{\Sigma}}\ element - slash - F\,|^{i} < \alpha - \beta 2\lambda.\ \mathrm{F_o\ rev\ ry}\ i \in F, \mathrm{w_e\ de\ ne}$

$$\begin{aligned} &= a - i - i - 2\lambda \leq^{>} 0_0 \\ &= a - i - i + 2\lambda \leq^{>} 0_0 \end{aligned}$$

and , for every $i$ $element - slash$ $F$, take $c_i = -\lambda$ and $d_i = \lambda$. It is easy to check that $D = intersection - f_{i-arrowdblright}{}^{-1}_{notdef}\,(notdef - ci - notdef, d_{notdef} notdef - bracketright_{)}\infty\,\mathrm{i}\ negationslash - notdef - notdef - notdef$
$\mathrm{a}\ notdef - notdef - notdef notdef - notdef - notdef - a_{\mathrm{l}} \in \mathrm{a}\ notdef - notdef - notdef - n^{infinity-d}\ \mathrm{t}\ notdef - h^{negationslash-notdef-a}_{\mathrm{t}notdef-notdef}\ notdef\ notdef\ .\ \mathrm{ej\ u^{s-t}}$
$\mathrm{n\ \ e}-\mathrm{e\ dt\ o\ c\ ompu^{t-e}}$

 inf $f(D)$. Let $F^+ = \{i \in F : yi > 0\}$. For every $x \in D$ we have

$$f-intersection_i arrowdblright-minus$$

$$\text{sincethendeed},_a \text{point}_{,b}^{\sum} \in {}_{[i}^{i\in F+}{}_{,b]} \text{w} a_i e_i + \text{hen}_i \sum_{\in F} i \in F \setminus F^+ \text{a}_{\text{nd},\text{fort her stofcoordnat}-\text{e}}^{b_i e\, i \text{is an} \in \text{of} C=}$$

eo ngst o $[i, b\ ]$ s n ce $0 \in C$.

PROPOSITION 6 . 6 . *Let $K$ be a Stonean compact Hausdorff space . Then $C(K)$ is a Mazur space .*

We finish this section with a result that distinguishes dimension $d \leq 2$. Indeed , we will see later that for normed linear space $X$ with dimension greater than 2 there is an equivalent norm $\| \cdot \|$ for which $(X, \| \cdot \|)$ i s not a Mazur
space .

THEOREM 6 . 7 . *Every two dimensional normed linear space is a Mazur*
*space .*

The following lemma i s a key t ool in proving Theorem 6 . 7 . We will denote by $B^*$ the dual unit ball of $B$. As usual , ext $C$ stands for the collection of all extreme points of $C$.

LEMMA 6 . 8 . *Suppose that $C \in \mathcal{M}, x \in \partial C$ and that there exists $f \in \partial B^* \setminus$ ext $B^*$ satisfying $f(x) = \sup f(C)$. Then there is $y \in B$ with $f(y) = \sup f(B)$*
*such that any $g \in \partial B^*$ with $g(y) = \sup g(B)$ satisfies $g(x) = \sup g(C)$.*

*Proof .* Since $f$ is not an extreme point of $B^*$, there i s a vertex $y \in \partial B$ such that $f(y) = 1$. Suppose that there is $g \in \partial B^*$ with $g(y) = 1$ but $g(x) < \sup g(C)$. Choose $h \in \partial B^*$ with $h(y) = 1$ such that $f$ lies in the
interior of the segment defined by $h$ and $g$. Let $x'$ be the intersection of the lines $\{s \in \mathbb{R}^2 : h(s) = h(x)\}$ and $\{s \in \mathbb{R}^2 : g(s) = \sup g(C)\}$. Since $x' element - slash C$,
the proof of the lemma will b e accomplished by showing that $x'$ i s in every ball containing $C$, which provides a contradiction .

 Indeed , let $a + \lambda B$ b e a ball such that $C \subset a + \lambda B$. Consider a point $z \in C$
satisfying $g(z) = \sup\ g(C)$. Necessarily

$$g(a + \lambda y) \geq g(a + \lambda B) \geq \sup g(C) = g(z)$$

and , analogously $, h(a + \lambda y) \geq h(x)$. Hence we have

$$x' \in \mathrm{conv}\{x, z, a + \lambda y\}$$
$$\mathrm{whichimplies} x' \in C.$$

Notice that the condition $f element - slash$ ext $B^*$ was essential in the above lemma . In fact , the statement i s not true for extreme points . Suppose , for instance ,
that $D$ i s the euclidean ball in $\mathbb{R}^2$ and $B = \{(2,0)+3D\} \cap \Rightarrow ({}^{notdef-zero}, \mathrm{two} - \mathrm{element}) \mathrm{plus} - \mathrm{notdef} - \mathrm{not}$
$intersection$ t he

$$\mathbb{R}2$$

onl

$$@y$$

On the other hand , the lemma i s not valid for higher dimensional spaces .
Consider , for instance , the space $(\mathbb{R}^3, \quad \| \cdot \| 1)$, the set $C = \{(t, t, 0), -1 \leq t \leq 1\}$, the point $x = (0,0,0) \in C$ and the functional $f(x_1, x_2, x_3) = x_3$ which i s not an extreme point of the dual unit ball .

*Proof of Theorem* 6 . 7 . It is enough to show that for any $C \in \mathcal{M}, f \in \partial B^*$ and $\varepsilon > 0$, there i s a closed ball $B_\varepsilon$ containing $C$ and satisfying sup $f(B_\varepsilon) = \sup\ f(C) + \varepsilon$. We split the proof into two cases .

Case 1 : $f \in \mathrm{ext}\ B^*$. There exists $y \in \partial B$ such that $f(y) = 1$ and the line
$L = \{s \in \mathbb{R}^2 : f(s) = 1\}$ is ( at least ) a one - sided tangent t o $B$ at $y$. Since $y$ defines two sides in $L$, it is convenient t o fix one which is tangent to $B$ and call
it the *positive* side ( with respect t o $y$). Let $L_C = \{x \in \mathbb{R}^2 : f(x) = \sup\ f(C)\}$ and $L_\varepsilon = \{x \in \mathbb{R}^2 : f(x) = \sup\ f(C) + \varepsilon\}$. We fix a point $z \in L_\varepsilon$ satisfying , first , that $\{z + sy : s \in \mathbb{R}\} \cap \Rightarrow$ a $notdef \mathrm{n} - \mathrm{notdef} notdef^{\mathrm{e-notdef-s}} infinity - c\mathrm{n} - \mathrm{elementd} - \mathrm{infinity} - \mathrm{notdef}, {}^{\mathrm{t-notdef}}\ notdef - h_{\mathrm{a}} infinity - t\ \mathrm{t}\ \ \mathrm{h} - \mathrm{notdef} - \mathrm{notdef}$ e $\mathrm{s} - \mathrm{infinity} - \mathrm{notdef}$ e $notdef - notdef - tnotdef - partialdiffC - notdefnotdef - notdef - notdef - notdef - intersectionarrowdblright - notdefnotdef - union - xelement - notdef - elementnotdef - notdef - Rnotdef - twonotdef\{\ ) = \mathrm{s\ pu}\ f(C)\}$ le s int he pos $\mathrm{t}^{\mathrm{i-v}}$ e sid of $LC$ w $\mathrm{t} - \mathrm{i}$ hres e ct tot h e p $\mathrm{o} - \mathrm{it} - \mathrm{n}$

$$= \{z + sys \quad \in \mathbb{R}\} intersection - period \Rightarrow notdefnotdefnotdefnotdef$$

Finally , for every $\lambda > 0$, we consider the point $a_\lambda = z - \lambda y$ and the ball $a_\lambda + \lambda B$. We just need t o show that there i s $\lambda_C > 0$ such that $C \subset a_{\lambda_C} + \lambda_C B$. To do

that , we first choose a point $b$ in the positive side of $L_C$ with resp ect to $z'$ and $\lambda_0 > 0$ such that

$$C \subset \text{conv}\{b, \quad a_{\lambda_0}, z = a_{\lambda_0} + \lambda_0 y\}.$$

We need only find $\lambda_C > 0$ satisfying $\lambda_C \geq \lambda_0$ and $b \in a_{\lambda_C} + \lambda_C B$. Consider the point $b' = L_\varepsilon \cap \Rightarrow a - notdef_{\lambda_0} element - plussnotdef - parenleftb\{a_{\lambda_0}) : s \in \mathbb{R}\}$ and define the sequence $\{x_n = z + (b' - z)/n\}$. Let $yn$ b e the corresponding point of $\partial(a_1 + B)$ such that the segment j oining $x_n$ and $yn$ i s orthogonal ( in the euclidean sense ) t o $L_\varepsilon$. If $x_n \in \partial(a_1 + B)$, in this case we define $yn = x_n$. Notice that $yn$ i s well defined for $n$ sufficiently large . Since the positive side of $L_\varepsilon$ is tangent t o $a_1 + B$ at $z$, we have

$$\| yn - x_n \| \quad \| x_n - z \| -1 \quad = n \| yn - x_n \| \quad \| b' - z \| -1 \quad n \to \infty \to \quad 0.$$

Therefore , there is an $n_0$ such that

$$n \| yn - x_n \| \quad \| b' - z \| -1 \quad < \varepsilon \| b' - z \| -1$$

for every $n \geq n_0$. As a consequence $, n \| yn - x_n \| < \varepsilon$ and hence $\partial(a_n + nB) intersection - b$

p

Case 2 : $felement - slash$ ext $B^*$. Let $\phi, \psi \in$ ext $B^*$ b e such that $f$ li es in the interior
of the segment $[\phi, \psi] \subset \partial B^*$. Let $y \in B$ be such that $f(y) = 1$. We have $\psi(y) = \phi(y) = 1$, since $\psi(y) \leq 1, \phi(y) \leq 1$ and there is $0 < t < 1$ satisfying
$1 = f(y) = t\phi(y) + (1 - t)\psi(y)$. Consider now $x \in C$ satisfying $f(x) =$
sup $f(C)$. By Lemma 6 . 8 we know that $\psi(x) =$ sup $\psi(C)$ and , analogously , $\phi(x) =$ sup $\phi(C)$. As in the preceding case , we will consider balls $a_\lambda + \lambda B$
for which $a_\lambda + \lambda y = x + \varepsilon y$. Now pick $z, w \in \mathbb{R}^2$ with $\phi(z) = \phi(x)$ and $\psi(w) = \psi(x)$ satisfying $C \subset$ conv $\{z, w, x\}$. The only question i s whether there is $\lambda > 0$ so that $z, w \in a_\lambda + \lambda B$. The existence of such a $\lambda$ can b e proved using an argument of differentiability , as in Case 1 , since $\psi$ and $\phi$ are extreme points of $B$ *
.

Corollary 6 . 9 . *A Banach space has dimension less than three if and only if is a Mazur space with respect to every equivalent norm .*

*Proof .* It i s clear that one dimensional spaces are always Mazur spaces and Theorem 6 . 7 states that this i s also the case of two dimensional spaces . To prove the reverse , suppose that the Banach space $X$ contains a three –
dimensional subspace $Y$, which can b e assumed ( after renorming ) to be $(\mathbb{R}^3, \| \cdot$
$\| 1)$. Letting $Z$( in it s inherited norm ) b e the complement of $Y$ in $X$, so we can assume that $X$ i s the $\ell_1$ – sum $Y \oplus_1 Z$. We proved in Proposition 5 . 7 that in this case $\mathcal{M}$ is not stable under the closure of vector sums and hence $X$
with this norm is not a Mazur space .

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