Active Sums I

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ABSTRACT

Given a generating family $\mathcal{F}$ of subgroups of a group G, closed under conjugation and with partial order compatible with inclusion , a new group S can be constructed , taking into account the multiplication in the subgroups and their mutual actions given by conj ugation . The group S is called the active sum of $\mathcal{F}$, has G as a homomorph and is such that $S/Z(S) \simeq G/Z(G)$, where Z denotes the center .

The basic question we investigate in this paper is : when is the active sum S of the family $\mathcal{F}$ isomorphic to the group G?

The conditions found to answer this question are often of a homological nature . We show that the following groups are active sums of cyclic subgroups : free groups , semidirect products of cyclic groups , Coxeter groups , Wirtinger approximations , groups of order p^3 with p an odd prime , simple groups with trivial Schur multiplier , and special linear groups $\mathrm{SL}_n(q)$ with a few exceptions .

We show as well that every finite group G such that G/G' is not cyclic is the active sum of proper normal subgroups .

Key words : active sums , active sums of cyclic groups , regularity and independence , atomic and molecular groups .

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Introduction

A motivating question for the concept of active sum is : *to what extent can one re - cover a group G from a generating family* $\mathcal{F} = \{F_i\}i \in I$ *of subgroups , clos ed under conjugation , and their natural actions* ?

To be more precise , consider the disj oint union $U = \coprod_i F_i$, and the (partial)
multiplication on U induced by the multiplication in the subgroups (xy is defined if and only if x and y belong to the same subgroup in the disj oint union) . One defines
a second (this one global) operation , conjugation , in U by $x^y = y^{-1}xy \in F_k$ where $x \in F_i$, $y \in F_j$, and $F_i^y = F_k$ (we are assuming here that the subgroups are all
different ; see however Remark 1 . 4) . Thus U is what we will be calling an *active add* , that is , a partial algebra with two binary partial operations . We think of U as *the family of subgroups* $\{F_i\}i \in I$ *and their mutual actions* , and we call it the active add determined by $\mathcal{F}$. Every active add determines a group called the active sum of U. This group S is the image of U under the left adjoint to the inclusion functor from
the category of groups to the category of active adds . We show that the natural epimorphism φ from S onto G is such that $\varphi^{-1}(Z(G)) = Z(S)$, where Z denotes the center . We want to know under what conditions the active sum is isomorphic to the group G. The fact that ker $(\varphi) \subset Z(S)$ allows us to use homology of groups ,
in particular the five - term exact sequence and Ganea ' s extention to six terms [2 1] , [4] , [1 2] .

The concept of active sums of groups originated in papers by F . Tom à s [43] , [44] and P . Ribenboim [35] . Tom à s considered what in Ribenboim ' s terminology are discrete active families of normal subgroups and proved , for example , that any finite group G is the active sum of the family $\langle W_p \rangle$, where p runs over the primes dividing the order of G, and W_p is the subgroup generated by the $p - hyphen$ Sylow subgroups of G. Ribenboim proved that any finite group is the active sum of the family $\mathcal{F}$ of the subgroups of G of prime power order with $\mathcal{F}$ ordered by inclusion . A . D í az - Barriga and L . Rom á n obtained a family (also ordered by inclusion) more economical than $\mathcal{F}$ whose active sum is G, namely the family of tame intersections of subgroups of prime
power order .

Perhaps another motivating question for the concept of active sum , related to the original one , is the following : Is there an analogue for (not necesarily abelian) groups of the theorem stating that any finitely generated abelian group is a direct sum of
cyclic subgroups ?

The concept of active sum is proposed as an analogue for groups of the concept of direct sum for abelian groups (in fact Tom à s sometimes uses the term direct sum for this construction) .

Should the ' building blocks ' be cyclic groups ? Ribenboim proposes the atomic subgroups (also called groups of weight at most one [24]) as building blocks . These are the groups (important in knot theory) normally generated by one element . It is known (see [26] , [27]) that the finite atomic groups are the finite groups with cyclic abelianization .

Ribenboim defines a molecular group as one which is an active sum of atomic groups .

Non - molecular groups are not exhibited in Ribenboim ' s paper . Tom á s and Belana [3] have constructed a 2 - group which is not an active sum of any discrete family of proper normal subgroups . We defer to the sequel of this paper the proof that it is not molecular . Two intriguing questions are : which finite groups are molecular ?; which finite groups are active sums of cyclic subgroups ? We will present in the sequel of this paper families of molecular and non - molecular groups .

Besides the applications in Galois groups and finite group theory that appear in [43] , [42] , [36] , [7] , [6] , [34] , active sums (of groups) appear in the study of subgroups
of amalgamated free products (see [30] and Section 2 . 2 . 2) , knot groups (see [39] and
Section 2 . 2 . 5) , and Coxeter groups (see [22] and Section 2 . 2 . 4) . Other articles dealing
with the active sum are [8] , [32] , [33]

Lyndon and Schupp in [30] , Section I . 1 1 , consider a free product with amalgama - tion , $G = *_A H_i$, and describe what in our terminology is an active family of subgroups of G. They prove (in Proposition 1 1 . 4) that the active sum of this family is G, though the proof is marred with many errors ; use of our Corollary 2 . 5 provides an easy con - ceptual proof . More generally , for any subgroup G^* of G, they describe an active family of subgroups of G^*, with active sum N, a normal subgroup of G^* such that

$$G^*/N \text{is free.}$$

Simon [39] considers finitely presented groups G with infinite cyclic abelianization and normally generated by one element t, and takes the active sum S of the family of conjugates of the subgroup generated by t. He called S a best Wirtinger approxima - tion of G; the group G is the fundamental group of a closed orientable surface in $\mathbb{R}^4$
if and only if S is isomorphic to G for some suit able choice of t.

Here is an example considered in [1 6] and [1 3] , which will not be pursued in this paper . Let M_g be the mapping class group of the closed orientable surface F_g of genus g, that is , the quotient of the group of orientation - preserving homeomorphisms
of F_g by the subgroup of homeomorphisms isotopic to the identity . Let t be a Dehn twist along a non - separating curve , that is , an element of M_g which is not a proper power and is represented by a homeomorphism of F_g which is the identity in the complement of a non - separating annulus . All such t are conjugate up to inversion . By [28]t normally generates M_g. Again , consider the active sum S_g of the family of
conjugates of the subgroup generated by t. When $g \geq 3$ it turns out that $H_1(M_g) = 0$, $H_2(M_g) = \mathbb{Z}[16], H_1(S_g) = \mathbb{Z}$, and the (central) kernel of the natural epimorphism from S_g onto M_g is $\mathbb{Z}^2$, that is , S_g is the direct product of $\mathbb{Z}$ with the universal central extension of M_g. This example is important in the obtention of a finite presentation
of $M_g[17, 45]$, and in the proof of Kirby ' s surgery theorem [25 , 29] . A related example with Dehn twists along all curves is considered in [1 3] .

In Section 2 . 2 . 4 , we show that any Coxeter group is the active sum of a family of subgroups of order 2 . A 3 - transposition group $G[1]$ yields an additional example of a generating active family of subgroups of order two . In this case two different

subgroups of the family generate a subgroup of order at most six .

The concept of active sum , or that of active add , is a common , though not always explicit , ingredient in many seemingly unrelated papers , some of them with topological applications . Aside from those mentioned above , we mention [1 0] , [40] , and [24] .

The paper is organized as follows . In Section 1 , we give a construction of the active sum of groups using partial algebras (see [5]) . We find necessary and suffi - cient conditions on a generating active family $\mathcal{F}$ of subgroups of a group G, for the abelianization of the active sum of $\mathcal{F}$ to be isomorphic to the abelianization of G. The conditions found here are called *regularity* and *independence* .

In Section 2 we observe that it suffices to split the canonical epimorphism from the active sum to the group in question , to prove that they are isomorphic , provided that the family under consideration is regular and independent . We use this result in a few examples : free groups , free amalgamated products , semidirect products of cyclic groups , Coxeter groups , and best Wirtinger approximations . We also introduce atomic and molecular groups , showing that a finite class 2 group is molecular if and only if it is the active sum of cyclic subgroups . In Section 2 . 4 we show that the pullback of an active sum of groups is the active sum of the inverse image of the active family of the original group . We use this result to show that every finite , non - cyclic group is the active sum of proper normal subgroups , and that every finite group that is not a cyclic $p-$ group is the active sum of proper subgroups . We conclude the section with the study of the active sum via homology . We show here , that , if
the Ganea map [1 2] in certain exact sequence is an epimorphism , then regularity and independence of the generating active family of subgroups suffice for the active sum to be isomorphic to the group G in question . In particular , the result applies when the second homology group of G is trivial . A less trivial corollary of the aforementioned result is used to show that any group of order p^3 with p an odd prime , is the active sum of cyclic subgroups .

In section 3 we point out that independence is automatic for perfect groups . We deal next with some examples of perfect groups , and some examples of simple groups . We show that every finite simple group with trivial Schur multiplier is the active sum of cyclic subgroups . We investigate the finite special linear groups $\mathrm{SL}_n(q)$. When $n \geq 3$, previous results show that $\mathrm{SL}_n(q)$ is the active sum of cyclic subgroups , with a few exceptions . The analysis for the case $n = 2$ involves direct computations . The main result of section 3 . 4 is that , for a finite group G, and any generating regular and independent active family $\mathcal{F}$ of subgroups of G, if the order of any element of $\mathcal{F}$ is relatively prime to the order of the second homology group of G then the active sum of $\mathcal{F}$ is a covering group of G.

Some of the experimental evidence for this paper was gathered using the computer programs Maple and Gap .

1 . The active sum of groups

1 . 1 . The active sum via partial algebras

We give a construction of the active sum of groups using partial algebras . This con - struction is economical notationally and conceptually . For the definition and proper - ties of partial algebras we refer the reader to [5] . We start by defining the categories Add and Actad . In the terminology of [5] , Add is the category of partial algebras with one operation ' · ' of arity two ; and Acta d is the category of partial algebras with two operations ' · ' , and ' ∧ ' , both with arity two . What this means is that an object of the category Add , called an add , consists of a set A, together with a partial binary

$$\begin{array}{r} \text{operation :} \\ M_{line-A-d127-d95} \cdot \text{———} A \\ A^2 \end{array}$$

called multiplication . A morphism of adds $f : A \to B$ is a function $f : A \to B$, such that $f \times f(M_A) \subseteq M_B$, and the diagram

$$\begin{array}{cc} & M_A f \overset{\times f}{\_} M_B \\ \cdot d15 - d15 - line & d15 - d15 - line \cdot \\ & A\text{-}fB \end{array}$$

commutes . Composition is composition of functions . An object A in the category Acta d , also called an *active add* , consists of a set A, together with two partial binary functions

$$\begin{array}{cc} & \wedge \\ M_{line-A-d127-d95} \cdot \text{———} A & C_{d127-line-d15-d15^A} \text{———} A \\ A^2 & A^2 \end{array}$$

that we call multiplication and conjugation . A morphism $f : A \to B$ between active adds is a function $f : A \to B$ such that $f \times f(M_A) \subseteq M_B$ and $f \times f(C_A) \subseteq C_B$, and the diagrams

$$\begin{array}{cccc} & & M_A f \overset{\times f}{\_} M_B & C_A f \overset{\times f}{\_} C_B \\ \cdot line - d15 - d15 & d15 - d15 - line \cdot & \wedge line - d15 - d15 & d15 - d15 - line \wedge \\ & & A\text{-}fB & A\text{-}fB \end{array}$$

commute . Given $a, b \in A$, we write $\cdot(a, b) = a \cdot b$ and $\wedge_{(a,} b) = a^b$. Adds can be traced back to [2] .

Denote the category of groups and homomorphisms by G rp . We have a full and faithful functor $F :$ G rp $\rightarrow$ Acta d defined as follows : Given a group G, the active add $F(G)$ has the same underlying set as G, $M_{F(G)} = C_{F(G)} = G^2$; the function

$$\text{that}^{M_{F(G)}}_{(h,} \rightarrow g) G_{\mapsto} \text{is}^{\text{multiplication}}_{g^-} {}_{1_{hg.}} \text{If we keep in the group only}_{\text{the first}} G, \text{and the partial}_{\text{function}} \text{function} C^{F\ (G)}_{\text{we obtain}} \rightarrow G_{\text{a}} \text{func}$$

G rp $\rightarrow$ Add

[5] gives an explicit construction of the product in a category of partial algebras of a given similarity type . It is not hard to see that this category also has equalizers of pairs of arrows : given $f, g : A \rightarrow B$ the set $\{a \in A \mid f(a) = g(a)\}$ is a closed subset of the partial algebra A, and it determines a closed subalgebra that is the equalizer of f and g. What this means is that the category of partial algebras of a given similarity type is complete [3 1] . Analyzing these constructions in the case of Acta d , it is not hard to see that the functor $F :$ G r p $\rightarrow$ Acta d defined above preserves limits .

Proposition 1 . 1 . *The functor* $F :$ G rp $\rightarrow$ Acta d *has a left adjoint .*

Proof . Consider the following commutative diagram :

G rp F_- Acta d

$$d68 - d68 - d68 - d68 - d68 - d68_U \quad d120 - d120 - d120 - d120 - d120 - d120_V$$

Set

where Set is the category of sets and U and V are forgetful functors . As we mentioned before , F preserves limits . It is well known that the functor U has a left adjoint . V is clearly faithful . Furthermore , G rp is complete , well powered and well co - powered . These conditions suffice for the existence of a left adjoint to F[18], pg .213. $line - line$

Denote the left adjoint of F by S : Acta d $\rightarrow$ G rp . Explicitly , given an active add A, we have that $S(A)$ is the free group generated by A modulo the normal subgroup generated by elements of the form

$$ab(a \cdot b)^{-1} \quad \text{with} (a, b) \in M_A;$$
$$b^{-1}ab(a^b)^{-1} \quad \text{with} (a, b) \in C_A.$$

Explicit constructions for coproducts and directed colimits in the category of par - tial algebras for a given similarity type are also given in [5] . Furthermore , the anal - ysis given there of congruences makes it clear that coequalizers also exist : Given $f, g : A \rightarrow B$, consider the smallest congruence θ on B that contains the pairs $(f(a), g(a))$ for all $a \in A$, then the coequalizer is the quotient morphism $B \rightarrow B/\theta$. This means that the category of partial algebras for a given similarity type is co - complete .

As examples , if R is a rack where we view R as an active add with empty domain of the multiplication and global conjugation , then $S(R)$ is the associated group defined

in [1 0] ; if R is a pregroup [40] , chapter 3 , viewed as an active add where the domain for the conjugation is empty , then $S(R)$ is the universal group of R. **Definition 1 . 2 .** (i) An *active family of adds* consists of a functor $\Gamma : \mathbf{I} \to \mathsf{Add}$,
where $\mathbf{I}$ is a small category , together with an active add A whose underlying

$$\text{addis} \lim_{\to I \in \mathbf{I}} \Gamma I.$$

(i i) We call $S(A)$ the *active sum* of the active family of adds .

The situation we consider in this paper is as follows . Let G be a group and $\mathbf{I}$ a partially ordered set . Denote by Sub (G) the partially ordered set of subgroups of G, where the order is given by inclusion . Consider an injective , order preserving map $\mathbf{I} \to$ Sub (G), and denote by Γ the functor

$$\mathbf{I} \to \text{Sub } (G) \to \mathsf{G\ rp} \to \mathsf{Add} .$$

We then have the add $\lim_{\to \mathbf{I}} \Gamma$(the colimit is t aken in the category of adds) . To use the conjugation in the group G to define a total operation $(\lim_{\to \mathbf{I}} \Gamma)^2 \to \lim_{\to \mathbf{I}} \Gamma$, making it an
object of $\mathsf{Acta\ d}$, we would need the family $\langle \Gamma I \rangle I \in \mathbf{I}$ to satisfy the following conditions :

(a) It is closed under conjugation (i . e . for every $I, J \in \mathbf{I}$ and $g \in \Gamma J$, there exists a

$$K \in \mathbf{I} \text{ such that } \Gamma K = \Gamma I^g).$$

(b) The order in $\mathbf{I}$ is compatible with conjugation (i . e . If $I, I', J, J', K \in \mathbf{I}$ with

$$I \leq J, \text{ and } g \in \Gamma K \text{ is such that } \Gamma I'^g = \Gamma I \text{ and } \Gamma J'^g = \Gamma J, \text{ then } I' \leq J' \text{ in } \mathbf{I} .$$

More generally , we have

Lemma 1 . 3 . *Let C be an active add where th e operation $\wedge : C \times C \to C$ is total . Let*
$S \subseteq C^2$ and θ be the congruence generated by S in the underlying add of C. Then θ is a congruence in the categ ory $\mathsf{Acta\ d}$, if the following conditions are satisfied for al l

$$a, b, h, k \in C :$$

(i) *If $(a, b) \in S$, then $(a^h, b^h) \in \theta$.*
(i i) *If $(h, k) \in S$, then $(a^h, a^k) \in \theta$.*
(i ii) *$((a \cdot b)^h, a^h \cdot b^h) \in \theta$ if $a \cdot b$ is defined in C.*
(iv) *$(a^{h \cdot k}, (a^h)^k) \in \theta$ if $h \cdot k$ is defined in C.*

Proof . For every $h \in C$ define $\varphi h : C \to C/\theta$ such that $\varphi h(a) = [a^h]$, for every $a \in C$. Using condition (i ii) , it is not hard to see that φh is a morphism of adds . Condition (i)
tells us that $S \subseteq \ker \varphi h$, where $\ker \varphi h = \{(a, b) \in C^2 \mid \varphi h(a) = \varphi h(b)\}$. From [5] we know that $\ker \varphi h$ is a congruence . Therefore $\theta \subseteq \ker \varphi h$. That is to say , for every $(a, b) \in \theta$ we have $(a^h, b^h) \in \theta$. Similarly , for every $a \in C$, we define $\psi_a : C \to C/\theta$ such that for every $h \in C$ we have $\psi_a(h) = [a^h]$. We have then that $\theta \subseteq \ker \varphi a$. Therefore , for every $(h, k) \in \theta$ we have that $(a^h, a^k) \in \theta$. We thus conclude that for every $(a, b), (h, k) \in \theta$ we have $(a^h, b^k) \in \theta$. $line - line$

The situation we were t alking about before the lemma corresponds to $C = \coprod_{I \in \mathbf{I}} \Gamma I$, and θ the congruence of adds such that $C/\theta = \lim_{\rightarrow\, I \in \mathbf{I}} \Gamma I$.

It is worth mentioning that the definition of active quiver of groups given in [35] produces an active add in our context , and the active sum , in the sense of [35] , of the active quiver and the left adjoint S given here applied to the active add are isomorphic .

1 . 2 . Regularity and independence

Let G be a group and let $\mathcal{F} = \langle F_i \rangle i \in I$ be a family of pairwise different subgroups of G closed under conjugation . What we mean by closed under conjugation is , that for every element $F_i \in \mathcal{F}$ and every element $g \in G$ we have that $F_i^g \in \mathcal{F}$, where $F_i^g = g^{-1} F_i g$. For $h \in F_i$ we denote the element $g^{-1} h g \in F_i^g$ by h^g. Assume we have a partial order $\leq$ on $\mathcal{F}$ compatible with group inclusion and conjugation . That is to say, $\text{forall}^{\text{for}} k^{\text{all}} \in i^{,j}_{F^{\ell}} \in^{\in} I \mathcal{F}^{\text{we}} \text{have}^{\text{that}}_{\text{wehave}} \text{that}^{F_i}_{F_i^k \leq F_j^k} \text{implies}^{j}_{\leq F_j^k}$. A $\text{that}\text{family} F_i \mathcal{F} \text{is}_{\text{with}}$ a $\text{subgroup}_{\text{theseproperties}}$ of $F_j, \text{and}_{\text{will}} \text{that}_{\text{be}}$ called an *active family* of subgroups of G. If the family generates the group G, we will call it a *generating active family* . When the order is the trivial one we say that the family $\mathcal{F}$ is *dis cre te* .

Such a family $(\mathcal{F}, \leq)$ is an active family in the sense of [35] . Therefore we can consider its active sum S. Explicitly , we have that $S = minus - multiply \mathcal{F}/R$, where $multiply - minus \mathcal{F}$ is the free product of the elements of $\mathcal{F}$, and R is the normal subgroup of $minus - multiply \mathcal{F}$ generated by elements of the form :

(i) $h^{-1} g h (g^h)^{-1}$ with $g \in F_i$ and $h \in F_j$.

(i i) $g c_{ij}(g)^{-1}$ with $g \in F_i, F_i \leq F_j$ in $\mathcal{F}$ and $c_{ij} : F_i \rightarrow F_j$ the inclusion map .

By the universal property of the active sum , there exists a unique homomorphism $\varphi : S \rightarrow G$ such that for every $i \in I$ the diagram

$$\begin{array}{r} F_i \gamma^i_{-} S \\ d64 - d64 - d64 - d64 - d64 d15 - d15 - line \varphi \\ G \end{array}$$

commutes , where γi is the canonical injection of F_i into the active sum . If the family $\mathcal{F}$ generates G then $\varphi : S \rightarrow G$ is an epimorphism .

Remark 1 . 4 . It is of course possible to consider a more general definition allowing for the family of subgroups to have repetitions by introducing a function

$$\begin{array}{r} k : I \times (\amalg F_i) \rightarrow I \\ i \\ \text{suchthat} F_{k(i,y)} = y^{-1} F_i y. \end{array}$$

We assume for the rest of the section that $\mathcal{F}$ is a generating active family of subgroups of G. We will make use of homology theory ; the following lemma will allow us to do so . **Lemma 1 . 5 .** *If $Z(S)$ is th e center of S and $Z(G)$ is the center of G, then* $\varphi^{-1}(Z(G)) =$

$$Z(S).$$

Proof . Since φ is an epimorphism , $Z(S) \subseteq \varphi^{-1}(Z(G))$. For the converse , consider an element of S represented by $g \in F_k$ and let $w \in S$. We can write $w = f1f2 \cdots f_n$ where $fi \in F_{ji}(i = 1, ..., n)$. It is easy to show , by induction on n, that the element $w^{-1}gw$ of S is represented by the element $\varphi(w)^{-1}g\varphi(w)$ of $F_k^{\varphi(w)}$. In particular , if $\varphi(w) \in Z(G)$, then $w^{-1}gw = g$ for any $g \in F_k$ and any k, and therefore $w \in Z(S)$. *line − line*

Consider the following short exact sequence

$$1 \longrightarrow \ker(\varphi) \longrightarrow S\varphi\_G \longrightarrow 1.$$

As a consequence of the previous lemma , we have that ker $(\varphi) \subseteq Z(S)$. Therefore ([2 1] , pg . 203) we have an exact sequence

$$H_2(S)\varphi^*\_H_2(G) \longrightarrow \ker(\varphi) \longrightarrow H_1(S)\varphi^*\_H_1(G) \longrightarrow 0.$$

We have that φ is an isomorphism if and only if $\varphi*$ is a monomorphism in $H_1($ and thus , an isomorphism) , and an epimorphism in H_2. We will find necessary and suffi - cient conditions on the family $\mathcal{F}$ for $\varphi* : H_1(S) \to H_1(G)$ to be an isomorphism .

Define the following graph $\mathbf{I}$. The objects of $\mathbf{I}$ are the elements of I. There is in $\mathbf{I}$ at most one arrow from one object to another . There is an arrow $i \to j$ in $\mathbf{I}$ if either of the following two conditions is satisfied :

(i) $F_i \leq F_j \text{in} \mathcal{F}.$

(i i) There is a $k \in I$ and a $g \in F_k$ such that $F_j = F_i^g$.

Consider the following diagram $\Gamma : \mathbf{I} \to \mathsf{G\,rp}$: For every $i \in I$ define $\Gamma(i) = F_i/R_i$, where R_i is the normal subgroup of F_i generated by elements of the form $x^{-1}x^g$,

the$^{\text{where}} x \in F_j \text{condition}'_{F_i}{}^g \in_{\leq} F_j^{G,} \text{then}^{\text{and}} F_{j\text{define}} \leq F_{i\Gamma(i} \geq_{\to} j_{F^g}^{j)} \text{in}_{\text{as}} \mathcal{F}_{\text{the}}.$ If the arrow$i \to$ homomorphism$j_{i_F/}^{\text{con}}$

induced by the inclusion $F_i \to F_j$. If , on the other hand , we have that the arrow $i \to j$ in $\mathbf{I}$ comes from $F_j = F_i^g$, then define $\Gamma(i \to j)$ as the homomorphism $F_i/R_i \to F_j/R_j$ induced by the homomorphism $x \mapsto g^{-1}xg$ from F_i to F_j. It is not hard to see that the definition of Γ does not depend on the choice of g such that $F_j = F_i^g$.

Proposition 1 . 6 . *With the a bove notation , we have that $S/S' \simeq \lim_{\to \mathbf{I}} F_i/R_i$, where the colimit is taken in the category of a be lian groups .*

 Proof . It is not hard to see that R_i is contained in the kernel of the composition

$$F_i \to S \to S/S',$$

where the first map is the canonical inclusion of F_i in the active sum S, and the second is the projection of S onto its abelianization . Therefore we have a map $F_i/R_i \to S/S'$. For every $i \to j$ in $\mathbf{I}$ we have that the diagram

$$\begin{array}{d119} & & & & & & & & & & & & & & F_i/R_i \\ d71-d71-d71-d71-d71-d71-d71 & & & & & & & & & & & & & & \Gamma_{-}^{(i\to j)} \\ & & & & & & & & & & & & & & F_j/R_j \\ & & & & & & & & & & & & & & S/S' \end{array}$$

commutesthe diagram There exists then a unique homomorphism $\lim_{\to \mathbf{I}} F_i/R_i \to S/S'$ such that

$$\begin{array}{d75} & & & & & & & & & & & & & & F_i/R_i \\ d75-d75-d75-d75-d75-d75-d75 & & & & & & & & & & & & & & \lim_{\to \mathbf{I}} F_i/R_i \\ & & & & & & & & & & & & & & S/S' \end{array}$$

commutes for every $i \in I$.

On the other hand , the family of homomorphisms

$$F_i \to F_i/R_i \to \lim_{\to \mathbf{I}} F_i/R_i$$

indexed by I satisfies the appropriate conditions to apply the universal property of that theactive the $\sum S$.diagram Therefore , we have a unique homomorphism $S \to \lim_{\to \mathbf{I}} F_i/R_i$ such

$$\begin{array}{d73} & & & & & & & & & & & & & & F_i \\ d73-d73-d73-d73-d73-d73-d73 & & & & & & & & & & & & & & S \\ & & & & & & & & & & & & & & \lim_{\to \mathbf{I}} F_i/R_i \end{array}$$

commutes . Since $\lim_{\to \mathbf{I}} F_i/R_i$ is abelian , we have an induced homomorphism $S/S' \to$

$$\lim_{\to \mathbf{I}} F_i/R_i.$$

other The two morphisms thus induced between S/S' and $\lim_{- \mathbf{I}} F_i/R_i$ are inverses of each$line-line$ Observe that we have , for every $i \in I$, the following short exact sequence of abelian
groups

$$0 \longrightarrow \frac{F_i \cap G'}{R_i} \eta_{-}^{i} \frac{F_i}{R_i} \psi_{-} i \quad \frac{F_i G'}{G'} \longrightarrow 0.$$

 where ηi is the inclusion . We obtain the following commutative diagram of abelian
groups with exact rows [38] , pp . 46 – 47 :

$$0 \longrightarrow (F_i \cap G')/R_i \eta^i F_i/R_i \psi_i \quad F_i G'/G' \longrightarrow 0$$
$$\alpha_i d15 - d15 \quad \beta i d15 - d15 \quad d15 - d15 \gamma i$$
$$\lim_{\rightarrow \mathbf{I}} (F_i \cap G')/R_i \text{-} \eta \lim_{\rightarrow \mathbf{I}} F_i/R_i \text{-} \psi \lim_{\rightarrow \mathbf{I}} F_i G'/G' \longrightarrow 0.$$

In this context , we give the following definition : **Definition 1 . 7 .** We say that $\mathcal{F}$ is *regular* if $\beta i((F_i \cap G')/R_i) = 0$ for every $i \in I$. Next theorem follows immediately from the definition :

Theorem 1 . 8 . *The fo l lowing s tatements are equivalent .*

(i) $\mathcal{F}$ *is regular .*

(ii) $\eta(\lim_{\rightarrow \mathbf{I}} (F_i \cap G')/R_i) = 0.$

(i ii) ψ *is an is omorphism .*

We have the following result for discrete families .

Lemma 1 . 9 . *Assume that the family* $\mathcal{F}$ *is dis crete . The family* $\mathcal{F}$ *is regular if and only if* $[F, N_G(F)] = F \cap G'$ *for every* $F \in \mathcal{F}$, *where* $N_G(F)$ *is th e normalizer of* F *in* G.

Proof . If the family $\mathcal{F}$ is discrete , then $R_i = [F_i, N_G(F_i)]$ for every i. Then clearly $F_i \cap G'/R_i = 0$. Therefore $\mathcal{F}$ is regular . Suppose now that $\mathcal{F}$ is regular . Then , two elements i, j in the same connected component of the diagram $\mathbf{I}$ satisfy $i \leq j$ and $j \leq i$. The induced morphism $F_i/R_i \rightarrow F_j/R_j$ is an isomorphism . If we take one representative in every connected component of $\mathbf{I}$, and call T the set of representatives , we have that $\lim_{\rightarrow \mathbf{I}} F_i/R_i \simeq \bigoplus_{t \in T} F_t/R_t$. Given i, we may assume that $i \in T$. Then βi is basically the coprojection $F_i/R_i \rightarrow \bigoplus_{t \in T} F_t/R_t$. Since $\mathcal{F}$ is regular , we have that $F_i \cap G' \subseteq R_i = [F_i, N_G(F_i)]$. The inclusion $[F_i, N_G(F_i)] \subseteq F_i \cap G'$ is always satisfied . $line - line$

In some cases it is easier to work with what we call a transversal . We define the equivalence relation $\sim$ on I as follows . For $i, j \in I$ we say that $i \sim j$ if and only if there is a $g \in G$ such that $F_i^g = F_j$. A subset $T \subseteq I$ that contains exactly one representative for each class of equivalence of $\sim$ is called a *transversal* . It is not hard
to see that

Lemma 1.10. $\mathcal{F}$ *is regular if for e very* $t \in T$, *there exists an* $i \in I$ *with* $i \sim t$, *such*

$$that \beta i((F_i \cap G')/R_i) = 0.$$

We define the graph $\mathbf{T}$ from T and $\sim$ as follows : The objects of $\mathbf{T}$ are the elements of T. There is at most one arrow in $\mathbf{T}$ between any two objects s and t in $\mathbf{T}$. There is an arrow $s \to t$ in $\mathbf{T}$ if we can find $i \sim s$ and $j \sim t$ such that there is an arrow

$$i \to j \text{in} \mathbf{I}.$$

We now define a diagram from $\mathbf{T}$ to the category of abelian groups : for every $t \in \mathbf{T}$ let $A_t = F_t G'/G'$. If there is an arrow $s \to t$ in $\mathbf{T}$, then we can induce a canonical homomorphism $A_s \to A_t$. We can then consider the abelian group $\lim_{\to \mathbf{T}} A_t$.

For every $i \in I$ there is a $t \in T$ such that $i \sim t$. We therefore have an isomorphism $F_i G'/G' \to A_t$. These isomorphisms induce a homomorphism

$$\lim_{\to \mathbf{I}} F_i G'/G' \to \lim_{\to \mathbf{T}} A_t.$$

It is not hard to see that the above homomorphism is an isomorphism .

Now , for every $t \in \mathbf{T}$ we have an obvious monomorphism $A_t \to G/G'$. Furthermore , if $s \to t$ in $\mathbf{T}$, the diagram

$$\begin{array}{rr} & A_s \text{———} A_t \\ d69-d69-d69-d69-d69-d69 & d15-d15-line \\ & G/G' \end{array}$$

commutes . We can induce then a unique homomorphism $\varepsilon : \lim_{\to \mathbf{T}} A_t \to G/G'$ such that for every $t \in \mathbf{T}$ the diagram

$$\begin{array}{rr} & A_t \varepsilon_t \text{———} \lim_{\to \mathbf{T}} A_t \\ d70-d70-d70-d70-d70-d70-d70 & d15-d15-line \varepsilon \\ & G/G' \end{array}$$

commutes , where ε_t is the canonical homomorphism into the colimit . Since the homomorphism $A_t \to G/G'$ is mono , we conclude that every ε_t is also mono . Observe furthermore that ε is an epimorphism .

Definition 1.11. $\mathcal{F}$ is called *independent* if the homomorphism $\varepsilon : \lim_{\to \mathbf{T}} A_t \to G/G'$ is an isomorphism .

Theorem 1.12. *$\varphi*$: $S/S' \to G/G'$ is an is omorphism if and only if $\mathcal{F}$ is regular and independent .*

Proof . The composition

$$S/S' \text{———} \lim_{\to \mathbf{I}} F_i/R_i \text{———} \lim_{\to \mathbf{I}} F_i G'/G' \text{———} \lim_{\to \mathbf{T}} A_t \text{———} G/G'$$

is the homomorphism $\varphi* : S/S' \to G/G'$ induced by $\varphi : S \to G$. $line - line$

2 . First conditions for recovering a group as an active sum

2 . 1 . Generating sets for the active sum

We show in this section that , for any regular and independent family of subgroups of a group G, a subset of the active sum generates the active sum if its image under the canonical projection to G generates G. An interesting corollary in such a case is that , to prove that the active sum is isomorphic to the given group , it suffices to split the canonical projection from the active sum to the group . **Lemma 2 . 1 .** *Let H be a subgroup of a group G. If $HG' = G$ and $HZ(G) = G$ then*

$$H = G$$

Proof . If $HZ(G) = G$, we have that for every $g1, g2 \in G$, we can find $h_1, h_2 \in H$ and $z_1, z_2 \in Z(G)$ such that $gi = h_i z_i$. Clearly $[g1, g2] = [h_1 z_1, h_2 z_2] = [h_1, h_2]$. Therefore

$$G' = H'.\text{Then} G = HG' = HH' = H. \quad line - line$$

Corollary 2 . 2 . *Let G be a group and $X \subset G$ a subset . If the images of X under the canonical homomorphisms $G \to G/G'$, and $G \to G/Z(G)$ generate G/G' and $G/Z(G)$ respectively , then X generates G.*

Corollary 2 . 3 . *If $\varphi : S \to G$ is an epimorphism of groups , such that $\ker \varphi \subseteq S' \cap Z(S)$, then a subset X of S generates S if and only if $\varphi(X)$ generates G.*

Proof . Assume that $\varphi(X)$ generates G. Consider

$$S \_\_\_\_\_ G \simeq \_\_\_\_\_ S/\ker\varphi \_\_\_\_\_ S/S' \cap Z(S) d103 - d103 - d103 - d103 - d103 S/S'$$
$$d87 - d87 - d87 S/Z(S).$$

Observe that every arrow is an epimorphism . Since $\varphi(X)$ generates G, we have that its images under $S \to S/S'$ and $S \to S/Z(S)$ generate S/S' and $S/Z(S)$ respectively . According to the previous corollary , we have that X generates S. $line - line$

We can now show :

Proposition 2 . 4 . *Let G be a group , and le t $\mathcal{F}$ be a generating active family of subgroups of G. Let S be th e active sum of th e family $\mathcal{F}$, and le t $\varphi : S \to G$ be the canonical homomorphism . If $\mathcal{F}$ is regular and independent , then a subset $X \subseteq S$*
generates S if and only if $\varphi(X)$ generates G.

Proof . The fact that $\mathcal{F}$ is regular and independent means that the bottom arrow of the commutative diagram

$$\begin{array}{rr} & S \varphi\_ G \\ line - d15 - d15 & d15 - d15 - line \\ & S/S' \text{-} \varphi_* G/G' \end{array}$$

is an isomorphism . It is easy to see then , that ker $\varphi \subseteq S'$. On the other hand , according to Lemma 1 . 1 . 5 , ker $\varphi \subseteq Z(S)$. Now apply the previous corollary . *line − line*

Corollary 2 . 5 . *Let $\mathcal{F}$ be a regular and independent generating active family of sub - groups of a group G. Let S be the active sum of th e family $\mathcal{F}$, and $\varphi : S \to G$ the canonical projection . The epimorphism $\varphi : S \to G$ splits if and only if φ is an is omorphism .*

2 . 2 . Examples

We use the above results to prove that several groups are active sums of certain families of subgroups , cyclic subgroups in most cases .

2 . 2 . 1 . Free groups

As a first example , let F be a free group on the set X. Let $\mathcal{F} := \{\langle x^w \rangle \mid x \in X, w \in F\}$. It is clear that $\mathcal{F}$ is a generating active family of cyclic subgroups of F.

Since $\langle x^w \rangle \cap F'$ is trivial for every element of $\mathcal{F}$, we have that the family is regular .

It is also easy to see that we can choose the family $\{\langle x \rangle\}_{x \in X}$ as representa - tive groups for the transversal . Then the colimit $\lim_{\to x \in X} \langle x \rangle F'/F'$ in the category of abelian groups is isomorphic to the free abelian group in X, and isomorphic to it by the homomorphism induced by the inclusions $\langle x \rangle F'/F' \to F/F'$. Thus the family is independent .

Let S be the active sum of the family $\mathcal{F}$. According to Corollary 2 . 5 , to show that S is isomorphic to F it suffices to split the canonical projection $\pi : S \to F$. This is easily done since F is free .

2 . 2 . 2 . Free amalgamated products

Let $\langle H_\nu \rangle_\nu \in N$ be a family of groups and A another group . Assume that , for every $\nu \in N$ we have an inj ective homomorphism $\varphi\nu : A \to H_\nu$ that is not an isomorphism . Let G be the amalgamated product of the groups H_ν with amalgamated subgroup A. For details of this construction we refer the reader to [30] . Let $\psi_\nu : H_\nu \to G$ be the usual homomorphism . It is well known that ψ_ν is an inj ection , so that we can identify H_ν with a subgroup of G. It is also well known that G is the colimit in G rp of the diagram

$$Ad80 - d80 - d80 - d80 \overset{d55 - d55^{d123 - d110 - d123 - d110 - d123 - d110 - d123 - d110 - d123 - d110 - d123^{\varphi\nu}}_{\varphi_\nu \prime}}{\overset{H_\nu}{\underset{\cdot\cdot H_\nu \prime\prime}{H_\nu \prime}}}_{\varphi_\nu \prime\prime}$$

with inj ections $\psi_\nu : H_\nu \to G$.

We want to express G as the active sum of the active family of subgroups formed by all the conjugates of A in G plus all the conjugates of H_ν in G, for all $\nu \in N$, ordered by inclusion . We denote by S the active sum of this family , and let $\varphi : S \to G$ be the usual projection . Observe that the diagram

$$Ad55 - d55^{d111-d111-d111-d111}_{\lambda_{\nu'}} {}_{\lambda_{\nu''}} {}_{d123-d110-d123-d110-d123-d110-d123-d110-d123-d110-d123-prime^{\varphi\nu d55-d55_{\varphi}}_{d80-d80-d80-d80^{\nu}_{\varphi_{\nu''}}}} {}_{H_{\nu'} d79-d6}$$

commutes , where $\lambda_\nu : H_\nu \to S$ is the usual homomorphism to the active sum . By the universal property of the colimit , there is a unique homomorphism $\pi : G \to S$ such that $\pi \circ \psi_\nu = \lambda_\nu$ for every $\nu \in N$. It is clear that the diagram

$$\begin{array}{r} G \pi \_\_\_\_\_ S \\ d64 - d64 - d64 - d64 - d64_{Id} d15 - d15 - line \varphi \\ G \end{array}$$

commutes . By Corollary 2.5, φ is an isomorphism .

As mentioned in the introduction , [30] also considers a subgroup G^* of G, and constructs an active family of subgroups of G^* whose active sum is a normal subgroup N of G^* such that G^*/N is free . In this case , as in the previous one , the families considered allow repetitions of the subgroups , see Remark 1 . 1 . 4 .

2 . 2 . 3 . Semidirect products of cyclic groups

We show now that the semidirect product $G := \mathbb{Z}_s \propto^t \mathbb{Z}_r$ of $\mathbb{Z}_s$ and $\mathbb{Z}_r$ by t can be expressed as an active sum of cyclic subgroups . We recall that G has two generators x and y with the following properties : $x^r = y^s = 1$ and $y^{-1}xy = x^t$. Since x has order r, we have $t^s \equiv 1 \bmod r$. This in particular means that $(r, t) = 1$.

Observe that the conjugates of y are $y, yx^{t-1}, yx^{2(t-1)}, ..., yx^{(r-1)(t-1)}$. Let $F_i = \langle yx^{i(t-1)} \rangle$ for $i = 1, 2, ..., r$, and let $F_0 = \langle x \rangle$. Consider the discrete generating active family $\mathcal{F} = \langle F_i \rangle_{i=0}^{r}$. Let us see that $\mathcal{F}$ is regular and independent .

It is not hard to see that $G' = \langle x^{t-1} \rangle$. Also , $R_0 = \langle x^{t-1} \rangle$. Thus $(F_0 \cap G')/R_0$ is trivial . It can be shown that

$$(yx^{i(t-1)})^m = y^m x^{i(t^m-1)}.$$

In particular $(yx^{i(t-1)})^s = y^s x^{i(t^s-1)} = 1$. We conclude that $F_i \cap G'$ is trivial if $i \neq 0$. Therefore $\mathcal{F}$ is regular .

Now we see that $\mathcal{F}$ is independent . As representatives for the transversal we can consider F_0 and F_1. Observe that $F_0 G'/G' \simeq \langle x \rangle / \langle x^{t-1} \rangle$. On the other hand

$F_1G'/G' \simeq \langle y\rangle$. If $\mathbf{T}$ is the transversal , $\lim_{\rightarrow \mathbf{T}} A_t \simeq \langle x\rangle/\langle x^{t-1}\rangle \times \langle y\rangle$, and the morphism $\lim_{\rightarrow \mathbf{T}} A_t \to G/G'$ is clearly and isomorphism . Therefore $\mathcal{F}$ is independent .

Denote the active sum of the family $\mathcal{F}$ by S. To avoid confusion , denote by a the generator in S that corresponds to x, and b_i the one that corresponds to $yx^{i(t-1)}$.

It is clear that we can define a homomorphism $f : G \to S$ such that $f(x) = a$ and $f(y) = b_1$. Clearly the diagram

$$\begin{array}{ccc} & & S \\ d118-d118-d118-d118-d118-d118-d118^f & & d72-d72-d72-d72-d72-d72-d72^{\varphi} \\ & & \mathbb{Z}_s \propto^t \mathbb{Z}_r\text{-}i_d\mathbb{Z}_s \propto^t \mathbb{Z}_r \end{array}$$

commutes . Thus , by Corollary 2.5, G is isomorphic to the active sum S.

2 . 2 . 4 . Coxeter groups

Let W be a Coxeter group . We show that every W is the active sum of subgroups $\text{of}^{\text{order}}_{\text{every}} s2^{\text{Following}}_{\in S}$ has order $\text{the}_{2.\text{The}}\text{notation}_{\text{relations}}$ of [22are], wehavegiven by $\text{a}(_{ss}\text{set}_{\prime)}{}_{m}\prime^{s,s}_{S(\text{of})}\text{generators}_{=1.\text{Define}}\text{of}_{\text{th}}$

$$\mathcal{F} := \{\langle s^w\rangle \mid w \in W, s \in S\}.$$

Clearly , $\mathcal{F}$ is an active generating family of subgroups of W of order 2 . We show first that $\mathcal{F}$ is regular and independent .

There is a homomorphism $W \to \mathbb{Z}_2$ sending every $s \in S$ to 1 . This means that for every $w \in W$ and $s \in S, \langle w^{-1}sw\rangle \cap W'$ is trivial . Thus , $\mathcal{F}$ is regular .

Let Γ be the graph obtained from the Coxeter graph of (W, S) by deleting the edges labeled by ∞, or by an even number . Then $W/W' \simeq \bigoplus_{\pi_0(\Gamma)} \mathbb{Z}_2$, where $\pi_0(\Gamma)$ denotes the connected components of Γ. Choose an s_c in every element $c \in \pi_0(\Gamma)$. Then $T := \{\langle s_c\rangle \mid c \in \pi_0(\Gamma)\}$ is a transversal for $\mathcal{F}$. It follows that $\lim_{\rightarrow c\in\pi_0\ (\Gamma)}\langle s_c\rangle \simeq \bigoplus^{\mathbb{Z}}_{c\in\pi_0(\Gamma)\text{Let}_H\ \overset{2.}{\text{be}}\text{the}^{\text{Thus}}\mathcal{F}^{\text{is}}_{\text{active}}\text{independent}_{\sum \text{ofthefamily}}}$. $\mathcal{F}$. To make the notation easier , assume that for every $F \in \mathcal{F}$ we have chosen an $s \in S$ and a $w \in W$ such that $F = \langle s^w\rangle$. Furthermore , if $F = \langle s\rangle$, assume that we have chosen $w = e$. Thus , we may suppose that the family $\mathcal{F}$ is of the form $\{\langle s^{w_j} j\rangle\}_j \in J$ for some indexing set J. We have that H

$$sw^l_l = sw^k_k s^{w_j} jsw^k_k{}^{\text{to}s_j\text{foreveryj}}_{\text{canbepresented}}\text{with}J_0 \subseteq J, \text{andthat}x_j\text{corresponds}^{\varphi(s_j)=x_j.\text{Tobeabletoextend}}_{\text{in}W.\text{Assume}S=\{s_j\}_{j\in J_0^{\in J_0.\text{Define}\varphi:S\to H\text{such}}}\text{bygenerators}\{x_j\}_j}$$

φ to a homomorphism $W \to H$, we must show that the relations satisfied in W by $\text{since}^{\text{theelements}}$ the $\text{condition}^{\text{of}S\text{are}}(\text{also}^{\text{satisfied}}_{ss')^{m(s}},{}_{s\prime}) = 1\text{by}^{\text{theirimages}}_{\text{isequivalent}}$ to under φ in H. This is not hard ,

$$s^{(s's)^{line-parenleft-prime-parenright-minusms,s\frac{1}{2}}} = s'$$

 if $m(s,s')$ is odd , and equivalent to

$$s'(ss')^{\frac{m(s,s')-2}{2}}s = s'$$

if $m(s,s')$ is even . Therefore we obtain a homomorphism $\varphi : W \to H$ that is clearly a section of the usual projection $H \to W$. By Corollary 2 . 5 we have that H and W are isomorphic .

2 . 2 . 5 . Best Wirtinger approximations

According to [39] , a Wirtinger presentation is a presentation

$$\langle \langle x_i \rangle i \in I; \langle r_k \rangle k \in K \rangle$$

where every relator r_k is of the form $x_i^{-1} w^{-1} x_j w$ where $i, j \in I$ and w is a word on the generators $\langle x_i \rangle i \in I$. We will assume that there is a distinguished element $i_0 \in I$. Given a group G and an element $t \in G$, a Wirtinger approximation is a group X given by a Wirtinger presentation $X = \langle \langle x_i \rangle i \in I; \langle r_k \rangle k \in K \rangle$, and an epimorphism $\psi : X \to G$ such that $\psi(x_{i_0}) = t$, and ψ induces an isomorphism $X/X' \to G/G'$. A best Wirtinger approximation is a Wirtinger approximation $\varphi : Y \to G$, with $Y = \langle \langle y_p \rangle_p \in P; \langle r_q \rangle_q \in Q \rangle$, such that for every Wirtinger approximation $\psi : X \to G$, there exists an epimorphism $\widehat{\psi} : X \to Y$ such that $\widehat{\psi}(x_{i_0}) = y_{p0}$ and $\psi = \varphi \circ \widehat{\psi}$.

Let G be a group , and $t \in G$. Assume that $G/G' \simeq \mathbb{Z}$ and that $G/\langle\langle t \rangle\rangle \simeq 1$, where $\langle\langle t \rangle\rangle$ denotes the normal closure of the subgroup generated by t. Theorem 1 . 3 in [39] shows that in such a case (G, t) has a best Wirtinger approximation . We show that the active sum of the family $\mathcal{F} := \{\langle t^w \rangle \mid w \in G\}$ together with the canonical projection is a best Wirtinger approximation . Since $G/\langle\langle t \rangle\rangle \simeq 1$, we have that $\mathcal{F}$ is a generating active family of subgroups of G. It is easily seen that $\mathcal{F}$ is regular and independent .

We may assume that $\mathcal{F} = \{\langle t^w p \rangle\}_p \in P$ for some indexing set P, and furthermore , that $w_{p0} = e$. Let S be the active sum of the family $\mathcal{F}$, and $\varphi : S \to G$ the

isregularand$^{\to G/G'}_{\text{whenever}(t^w p)^{t^{wq}} = t^{wr} \text{in} G}$. We Since$\mathcal{F}^{:S \to G \text{inducesanisomorphism} S/S'}_{p,q \in P \text{Relations} s_r^{-1} s_q^{-1} s_p s_q^{\text{independent}, \text{wehavethat} \varphi}}$then canonicalprojection. Wecan presentSwithgenerators

know that φ is an epimorphism , and $\varphi(s_{p0}) = t$. Thus $\varphi : S \to G$ is a Wirtinger

approximationof(G, t).

Assume that $\psi : X \to G$ is another Wirtinger approximation , with $X = \langle \langle x_i \rangle i \in I; \langle r_k \rangle k \in K \rangle$, as above . We want to construct an epimorphism $\gamma : X \to S$.

Since $X/X' \simeq \mathbb{Z}$, we have that every x_i is conjugate of x_{i_0}. Therefore $\psi(x_i)$ is a conjugate of t. There is a unique $p \in P$ such that $\psi(x_i) = t^w p$. We can then define $\gamma(x_i) = s_p$ if $\psi(x_i) = t^w p$. To be able to extend γ to all of X, we must show that the relations in the presentation of X are also satisfied by their images under γ.

 Assume$\psi(x_j) = \text{that}_{t^{w_q},} x_k^{-1} \text{and}^{x} - j\psi_{(x_k)}^{1_{x_i x_j}} \text{is}_{=t^{w_r}}^{\text{a}}{}_{.}^{\text{relator}}$ Then in$_{\text{we}}^{\text{the}}$presentation$_{\text{have}}$ of X, and that $\psi(x_i) = t^{w} p_{,}$

$$1 = \psi(x_k^{-1} x^{-} j^{1} x_i x_j) = (t^{w_r})^{-1} (t^{w_q})^{-1} t^{w_p} t^{w_q}.$$

Therefore $s_r^{-1} s_q^{-1} s_p s_q$ is a relator in the presentation of S. We conclude that $\gamma : X \to S$ is a homomorphism . Clearly , the diagram

$$\begin{array}{r} X \psi \_ G \\ \gamma d15 - d15 d126 - d126 - d126 - d126 - d126 - d126_{\varphi} \\ S \end{array}$$

commutes , and $\gamma(x_{i_0}) = s_{p0}$. It remains to show that γ is an epimorphism . Since the morphisms $S/S' \to G/G'$ and $S/\ker \varphi \to G$, induced by φ are isomorphisms , it is not hard to see that $X \to S \to S/S'$ and $X \to S \to S/\ker\varphi \to S/Z(S)$ are epimorphisms , where the first arrow in both is γ and the rest are projections . According to Corollary 2 . 2 , we have that γ is an epimorphism . Thus , S is a best Wirtinger approximation of (G, t).

2 . 3 . Atomic and molecular groups

In this subsection we introduce a new class of subgroups , called *atomic* (or of weight 1 [24]) . We also introduce the concept of molecular [35] .

Definition 2 . 6 . A group G is called *atomic* if there exists an element $g \in G$, such that G is equal to the normal closure of $\langle g \rangle$ in G. A subgroup H of a group G is called an *atomic subgroup* if H is atomic in its own right . A group G is called *molecular* if G is the active sum of atomic subgroups .

We denote the normal closure of $\langle g \rangle$ in G by $[g]G$. Thus , G is atomic if there is a

$$g \in G \text{suchthat} [g]G = G.$$

Clearly , an abelian group is atomic if and only if it is cyclic . We also have that every simple group is atomic . It is not true in general that G/G' cyclic implies that G is atomic [24] . However , it has been shown that for finite groups , G is atomic if and only if G/G' is cyclic [26] . Thus symmetric groups are atomic . Every atomic group is of course molecular , as is any group that is the active sum of cyclic subgroups .

Since we know that for $n \geq 4$, the alternating group A_n is not an active sum of cyclic subgroups , A_n gives us an example of a molecular group that is not the active sum of cyclic subgroups .

2 . 3 . 1 . All atomic class 2 groups are cyclic

A class of groups in which being molecular is the same as being the active sum of cyclic subgroups is the class 2 groups .

Lemma 2 . 7 . *If G is a class 2 atomic group , th en G is cyclic .*

Proof . Assume $G = [a]G$. Then we have that $G/G' \simeq \langle aG' \rangle$. Since G is class 2 , we have $G' \subseteq Z(G)$. Therefore we can include $G/Z(G)$ in G/G', which is cyclic . We then use an exercise in [20] that says that $G/Z(G)$ cyclic implies G abelian . $line - line$

It is not hard to show that every subgroup of a class 2 group is a class 2 group . Thus we have :

Proposition 2 . 8 . *Let G be a class 2 group . G is molecular if and only if G is th e active sum of cyclic subgroups .*

2 . 3 . 2 . All finite atomic p - groups are cyclic

Another class for which being molecular is the same as being the active sum of cyclic subgroups is the class of p groups .

Lemma 2 . 9 . *Let p be a prime number . If G is a finite atomic p− group , then G is cyclic .*

Proof . Since G is atomic , $G/\Phi(G)$ is also atomic . Thus $G/\Phi(G)$ is cyclic . According to Corollary 1 . 2 , page 1 73 [1 4] this means that G is cyclic . $line - line$

Thus we have :

Proposition 2 . 10 . *A p− group G is molecular if and only if it is the active sum of a family of cyclic subgroups .*

2 . 4 . The pullback of an active sum is an active sum

We show that the pullback of a group that is the active sum of a family of subgroups is the active sum of the family of inverse images of the given family of subgroups .

2 . 4 . 1 . Inverse image of a family of subgroups

Let $\varphi : G \to H$ be an epimorphism of groups , and $\mathcal{H} = \langle H_i \rangle i \in I$ a generating active family of subgroups of H. Assume that $0 \in I$, that H_0 is the trivial subgroup , and that $H_0 \leq H_i$ for all $i \in I$. We can produce a generating active family of subgroups of G by considering the inverse images of the subgroups H_i. Specifically , let $F_i = \varphi^{-1} H_i$ and consider the family $\mathcal{F} = \langle F_i \rangle i \in I$. The order is given by $F_i \leq F_j$ if and only if $H_i \leq H_j$ in $\mathcal{H}$. Clearly , $F_0 = \ker(\varphi)$. Let S be the active sum of the family $\mathcal{F}, S'$ the active sum of the family $\mathcal{H}$ and $\xi : S' \to H$ the canonical epimorphism . We will show that the pullback of ξ along φ is isomorphic to the active sum of the family $\mathcal{F}$. We will need the following lemma .

Lemma 2 . 1 1 . *Suppose that* $h_1 \cdot h_2 \cdots h_n = k_1 \cdot k_2 \cdots k_m \in minus - multiply - H$, *with* $h_i \in H_{r_i}$ *and* $k_j \in H_{s_j}$. *If for e very* $i = 1, ..., n$, *we have* $gi \in F_{r_i}$ *such that* $\varphi(gi) = h_i$, *then* $weg_1 \cdot cang_2 \cdot ..^{find,}_{\cdot gn=}{}^{for} g'_1 e^{very}_{\cdot g'_2} \ldots^{j} g'^{=}_{n} 1^{;...}_{inthe}, m$ *elementsactive sum of* $g'_j \mathcal{F}. \in F_{s_j}$ *such that* $\varphi(g'_j) = k_j$ *and*

Proof . If $h_1 \cdot h_2 \cdots h_n = k_1 \cdot k_2 \cdots k_m \in minus - multiply \mathcal{H}$, then there is a sequence $h_1 \cdot h_2 \cdots h_n = w_0 \sim w_1 \sim \cdots \sim w_t = k_1 \cdot k_2 \cdots k_m$ of words in the elements of the family $\mathcal{H}$, where the symbol $w \sim w'$ means that w can be obtained from w' by adding the identity of some H_i between two letters of w, or by multiplying together two consecutive letters of w that belong to the same group , or by the opposite operations (deleting an identity that appears in w or considering a product in some H_i as two letters) . It suffices then , to show that the claim of the lemma is true if $h_1 \cdot h_2 \cdots h_n \sim k_1 \cdot k_2 \cdots k_m$. We do the case where $h_\ell = xy$ in H_i and leave all the other cases to the reader . We have

$$h_1 \cdot h_2 \cdot h_{\ell-1} \cdot (xy) \cdot h_{\ell+1} \cdots h_n \sim h_1 \cdot h_2 \cdot h_{\ell-1} \cdot x \cdot y \cdot h_{\ell+1} \cdots h_n . \text{Assume} \varphi(gi) = h_i .$$

Then $\varphi(g\ell) = xy$. Since φ is onto , we can find $\hat{x}, \hat{y} \in F_{r_\ell}$ such that $\varphi(\hat{x}) = x$ and $\varphi(\hat{y}) = y$. Notice that $g\ell\hat{y}^{-1_{\hat{x}}-1} \in \ker \varphi \subseteq F_{r_\ell}$. Therefore $g\ell\hat{y}^{-1} = g\ell\hat{y}^{-1_{\hat{x}}-1_{\hat{x}}} \in F_{r_\ell}$. We have $\varphi(g\ell\hat{y}_{)}^{-1} = x, \varphi(\hat{y}) = y$ and $(g\ell\hat{y}_{)}^{-1}\hat{y} = g\ell$. Therefore , the elements we can

$$\text{choosearе} g1, ..., g\ell - 1, g\ell\hat{y}_{,}^{-1}\hat{y}, g\ell + 1, ...gn. \quad line - line$$

2 . 4 . 2 . The pullback of an active sum

Theorem 2 . 12 . *With the same definitions , consider the pullback*

$$\begin{array}{rr} P\pi_2 & \text{———} S' \\ \pi_1 line - d15 - d15 & d15 - d15 - line \xi \\ & G \text{-} \varphi H. \end{array}$$

Then P *is is omorphic to the active sum of the family* $\mathcal{F}$. *Proof .* Let $\zeta : S \to G$ be the canonical epimorphism . Observe that the family $\langle \varphi \mid F_i : F_i \to H_i \rangle$ induces a homomorphism $\psi : S \to S'$ such that the diagram

$$\begin{array}{rr} F_i \varphi^{|F}_{-} i & H_i \\ line - d15 - d15 & d15 - d15 - line \\ & S \text{-} \psi S' \end{array}$$

commutes , where the vertical arrows are the canonical inj ections . It is easy to see

 that the outer square of the diagram

Table ignored!

 account that $\ker \varphi = F_0 \leq F_i$ for all $i \in I$, we have

$$\begin{aligned} g'_1 g'_2 \cdots g'_n R &= (x_1 g 1) \cdots (x_{n-1} gn - 1)(x_n gn)R \\ &= (x_1 g 1) \cdots (x_{n-1} gn - 1^x n) gn^R \\ &= (x_1 g 1) \cdots (x_{n-1} g^{-1}{}_{-} n_x n \quad 1_{gn-1) gn^R} \\ &= (x_1 g 1) \cdots (x_{n-2} gn - 2^x n - 1^{x g_{n-} n}{}_{-1} 1_{)gn-1 gn^R} \\ &= (x_1 g 1) \cdots (x_{n-2} x^{g_{n-2} n^{-1}_{-1}} x^{g_{n-} n}{}_{-1} 1^{g^{-1}}{}_{n-} 2_{gn-2) gn-1 gn^R} \end{aligned}$$

. . . . .

$$= (x_1 x^{g_1 2}{}_{-}{}^{1} x^{g_2 3}{}_{-}{}^{1_g} - 1^1 \cdots x^{g_{n-} n}{}_{-1} 1^{\cdots g} - 2^1 g_1^{-1}{}_{)g1g2} \cdots gn^R_{\cdot}$$

Applying $\zeta : S \to G$ to both sides we obtain

$$\begin{aligned} g = g'_1 g'_2 \cdots g'_n &= (x_1 x^{g_1 2}{}_{-}{}^{1} x^{g_2 3}{}_{-}{}^{1_g} - 1^1 \cdots x^{g_{n-} n}{}_{-1} 1^{\cdots g} - 2^1 g_1^{-1}{}_{)g1g2} \cdots gn \\ &= (x_1 x^{g_1 2}{}_{-}{}^{1} x^{g_2 3}{}_{-}{}^{1_g} - 1^1 \cdots x^{g_{n-} n}{}_{-1} 1^{\cdots g} - 2^1 g_1^{-1}{}_{)g} \end{aligned}$$

in G. Therefore $x_1 x^{g_1 2}{}_{-}{}^{1} x^{g_2 3}{}_{-}{}^{1_g} - 1^1 \cdots x^{g_{n-} n}{}_{-1} 1^{\cdots g} - 2^1 g_1^{-1}$ is the identity . We conclude that

$$g'_1 g'_2 \cdots g'_n R = g1g2 \cdots gn^R_{\cdot}$$

We show now that the definition of β does not depend on the choice of

$$h_1 h_2 \cdots h_n R'.$$

Assume that $(g, h_1 h_2 \cdots h_n R') = (g, h'_1 h'_2 \cdots h'_n R')$. Therefore we have that

$$h_1 h_2 \cdots h_n {h'_m}^{-1} \cdots {h'_1}^{-1} \in R'.$$

This means that we can write $h_1 \cdot h_2 \cdots h_n \cdot {h'_m}^{-1} \cdots {h'_1}^{-1}$ in $minus - multiply\mathcal{H}$ as a product of conjugates of generators of R'. It is clear that for every factor in this last product
we can find a preimage along φ in such a way that the resulting product in $minus - multiply\mathcal{F}$ is an element of R. Using Lemma 2 . 1 1 we can find elements $gi \in F_{r_i}$ and $g'_j \in F_{s_j}$ for all
$i^{,j,}_{\text{say}}$such$_{g1g2}$.that$_{..gn^R}\varphi_{=}(g\prime^2_{i)g^{=}_{i1g}} h^{i}_{..g}{}^{,\varphi(g'_j)}{}_{,mR.} = h_j$ and $g1g2 \cdots gng'^{-1}_m \cdots g'^{-1}_1 R = R$. That is to$line - line$
2 . 4 . 3 . Finite groups as active sums of proper subgroups
We draw now some conclusions of the pullback theorem above . For any group G, we can decompose G/G' as a direct sum of cyclic groups $G/G' \simeq \bigoplus_{i \in I} C_i$. Let $\varphi : G \to \bigoplus_{i \in I} C_i$ be the projection . We may assume that $0 \in I$ and C_0 is trivial . Notice then that G/G' is the active sum of the family $\langle C_i \rangle i \in I$, where $C_0 \leq C_i$ for all $i \in I$. Theorem 2 . 1 2 asserts that G is isomorphic to the active sum of the family

$\langle \varphi^{-1}(C_i) \rangle i \in I$. This produces a non - trivial decomposition of G as an active sum of normal subgroups provided that G/G' is not cyclic .

As pointed out in Section 2 . 3 , a finite group G is atomic if and only if G/G' is cyclic . Thus we have :

Theorem 2 . 13 . *Every non - atomic finite group is the a ctive sum of proper normal subgroups .*

The following theorem states that , except for cyclic groups of prime power order , every finite group is decomposable .

Theorem 2 . 14 . *If G is a finite group that is not a cyclic p− group , th en G is th e active sum of proper subgroups .*

Proof . The previous theorem allows us to assume that G is atomic . By Lemma 2.9, G cannot be a $p-$ group . Then G is the active sum of it s subgroups of order a

$$\prime[35]. \quad line - line$$

2 . 5 . Central extensions and homology

2 . 5 . 1 . Homology , Ganea ' s map and regularity and independence

Recall that a *central extension* is a short exact sequence

$$K\kappa line - d47 - d47S\pi d47 - d47 - lineG,$$

where $K \subset Z(G)$. Such an extension gives rise to the five - term exact sequence [2 1] ,

$$H_2(S)\pi^* \text{———} H_2(G) \text{———} K \text{———} H_1(S)\pi^* \text{———} H_1(G) \text{———} 0. \qquad [4] :$$

Therefore , K is trivial , that is , $S \to G$ is an isomorphism , if and only if $H_1(S) \to H_1(G)$ is monic and $H_2(S) \to H_2(G)$ is epic .

Thus , if S is the active sum of a generating active family $\mathcal{F}$ of subgroups of G, the canonical map $S \to G$ is an isomorphism if and only if $\mathcal{F}$ is regular and independent , and $H_2(S) \to H_2(G)$ is surj ective . As a consequence we have :

Lemma 2 . 15 . *If G is a group such that $H_2(G)$ is trivial , then every generating active family of subgroups of G that is regular and independent has active sum G.*

Still assuming S to be the active sum of a family $\mathcal{F}$ of subgroups of G, the image of the homomorphism $H_2(S) \to H_2(G)$ contains , besides the images of the inclusion induced homomorphisms $H_2(F) \to H_2(G)$ with $F \in \mathcal{F}$, the image of the so called
Ganea map $\chi : H_1(G) \otimes Z(G) \to H_2(G)$ which we describe next .

If $g, g' \in G$ are such that $gg' = g'g$, an element $g \wedge g' \in H_2(G)$, the Pontrjagin product , is defined as follows . Define the homomorphism $\mathbb{Z} \oplus \mathbb{Z} \to G$ such that $(1, 0) \mapsto g$ and $(0, 1) \mapsto g'$. We thus obtain $\mathbb{Z} \simeq H_2(\mathbb{Z} \oplus \mathbb{Z}) \to H_2(G)$. One generator

of $H_2(\mathbb{Z} \oplus \mathbb{Z})$ goes to the element $g \wedge g'$ and the other to its inverse . This ambiguity of sign can be avoided using the Schur - Hopf formula [4] . However , the above description of $g \wedge g'$ will be enough for our purposes .

Assume now that we have a central extension

$$N\kappa d47 - d47 - lineG\pi d47 - d47 - lineQ. \tag{1}$$

Ganea [1 2] has added a term to the exact sequence of five terms :

$$H_1(G) \otimes N\chi\_H_2(G)\pi_*\_\_\_\_\_H_2(Q)\_\_\_\_\_N\_\_\_\_\_H_1(G)\pi_*\_\_\_\_\_H_1(Q)\_\_\_\_\_0. \tag{2}$$

We will call χ Ganea ' s map . Using the algebraic description of Ganea ' s map given in [9] , it follows that $\chi([g] \otimes x) = g \wedge x$. Clearly χ is an epimorphism if and only if $H_2(G)$ is generated by elements of the form $g \wedge x$ with $g \in G$ and $x \in N$. We can take $N = Z(G)$ in particular .

Theorem 2 . 16 . *If $\chi : H_1(G) \otimes Z(G) \to H_2(G)$ is an epimorphism , th en every generating active family of subgroups of G that is regular and independent has active sum is omorphic to G.*

Proof . Let $\mathcal{F}$ be a regular and independent generating active family of subgroups of G, and let S be the active sum of the family $\mathcal{F}$. Let $\varphi : S \to G$ be the induced epimorphism and let M be the kernel of φ. Consider the short exact sequence

$$0\_\_\_\_\_M\_\_\_\_\_S\varphi\_G\_\_\_\_\_0.$$

We obtain the five term exact sequence

$$H_2(S)\varphi^*\_H_2(G)\_\_\_\_\_M\_\_\_\_\_H_1(S)\varphi^*\_H_1(G)\_\_\_\_\_0.$$

Since $\mathcal{F}$ is regular and independent , we have that $\varphi*$ is an isomorphism at the H_1 level . To show that M is the trivial group it suffices then to show that $\varphi*$ is an epimorphism at the H_2 level . Since $H_2(G)$ is generated by elements of the form $g \wedge x$ with $g \in G$ and $x \in Z(G)$, it will be enough if we show that these elements are in the image of $\varphi*$. Let $\widehat{g},\widehat{x} \in S$ such that $\varphi(\widehat{g}) = g$ and $\varphi(\widehat{x}) = x$. Since $x \in Z(G)$, Lemma 1 . 1 . 5 tells us that $\widehat{x} \in Z(S)$. Therefore $\widehat{g} \wedge \widehat{x} \in H_2(S)$. We have that $\varphi * (\widehat{g} \wedge \widehat{x}) = g \wedge x$. We conclude that M is the zero group , that is , $\varphi : S \to G$ is an isomorphism . $line - line$

Recall that the short exact sequence (1) is called *s t em* if it is central and $N \subset G'$. As a corollary to our last result we have :

Corollary 2 . 17 . *Let $N\kappa line - d47 - d47G\pi d47 - d47 - lineQ$ be a s t em extension , with $H_2(Q)$ finitely gen - erated and N is omorphic to $H_2(Q)$. If $\mathcal{F}$ is a generating active family of subgroups of G that is regular and independent , th en the active sum of the family $\mathcal{F}$ is is omorphic to G.*

Proof . Since we have a central extension , we can induce the long exact sequence (2) . Since the extension is stem , we have that $\pi_* : H_1(G) \to H_1(Q)$ is an isomorphism [9] . This means that the next map to the left is trivial . Therefore , the homomorphism $H_2(Q) \to N$ is an epimorphism . Since $H_2(Q)$ is finitely generated and isomorphic to N, this map must be an isomorphism . This means that π_* at the level H_2 is trivial . It follows that Ganea ' s map χ is an epimorphism . Therefore , the map $H_1(G) \otimes Z(G) \to H_2(G)$ is surj ective and Theorem 2 . 1 6 finishes the proof . $line - line$

2 . 5 . 2 . Example : Groups of order p^3

As an application we will see that all groups of order p^3, with $p \neq 2$ prime , are active sums of cyclic groups . According to Theorem 5 . 1 , p . 203 of [1 4] , there are only two such groups for every $p \neq 2$, namely :

$$H = \langle x, y, z \mid x^p = y^p = z^p = 1, [x, z] = [y, z] = 1, [x, y] = z \rangle,$$

and

$$K = \langle a, b \mid a^{p^2} = b^p = 1, [a, b] = a^p \rangle,$$

and they are extra special . Actually $H' = Z(H) = \langle z \rangle$, and $K' = Z(K) = \langle a^p \rangle$. Since $H/H' \simeq K/K' \simeq \mathbb{Z}_p \oplus \mathbb{Z}_p$ and $H_2(\mathbb{Z}_p \oplus \mathbb{Z}_p) \simeq \mathbb{Z}_p$, we see that both groups can be fitted in Corollary 2 . 1 7 . Therefore , to show that they can be expressed as an active sum of cyclic subgroups , all we have to do is to find an active generating family $\mathcal{F}$ of cyclic subgroups that is regular and independent . In the case of H we consider the family

$$\mathcal{F} = \{\langle xz^i \rangle \mid i = 0, 1, ..., p-1\} \cup \{\langle yz^i \rangle \mid i = 0, 1, ..., p-1\},$$

and in the case of K we consider

$$\mathcal{G} = \{\langle a \rangle\} \cup \{\langle ba^{ip} \rangle \mid i = 0, 1, ..., p-1\}.$$

We leave it to the reader to show that $\mathcal{F}$ and $\mathcal{G}$ are regular and independent .

3 . Perfect and simple groups

3 . 1 . Regularity and independence for perfect groups

We will show that most of the groups $\mathrm{SL}_2(q)$, all the simple groups with trivial Schur multiplier and all of Suzuki ' s groups [41] , with the exception of the smallest one , are active sums of cyclic subgroups . The proofs consist on identifying regular and independent generating active families of cyclic subgroups (actually , one such subgroup and all its conjugates) , and considering the Schur multiplier , that turns out to be trivial in many of these cases .

If G is a perfect group , independence is not an issue :

Lemma 3 . 1 . *If $\mathcal{F}$ is a generating active family of subgroups of a perfect group G, then $\mathcal{F}$ is independent .*

Proof . If G is a perfect group , then G/G' is trivial . On the other hand , if we take any $F \in \mathcal{F}$, we have that FG'/G' is also trivial . Thus the corresponding colimit in the definition of independence is the trivial group . Since G/G' is also trivial , the family is independent . $line - line$

Next , we have a condition for regularity for cyclic subgroups :

Proposition 3 . 2 . *Let G be a group . If $a, b \in G, a \neq 0$, are such that $a^b = a^s$, and $s - 1$ and th e order of a are re lative prime , then the family*

$$\mathcal{F} = \{w^{-1}\langle a\rangle w \mid w \in G\}$$

is a regular (not necessarily generating) active family of subgroups of G. Proof . $\mathcal{F}$ is clearly an active family of subgroups of G. Observe that

$$a^{s-1} = a^{-1}b^{-1}ab \in [\langle a\rangle, N_G(\langle a\rangle)].$$

Since $(o(a), s - 1) = 1$, then $\langle a\rangle = \langle a^{s-1}\rangle$. Therefore $\langle a\rangle = [\langle a\rangle, N_G(\langle a\rangle)]$. Since every $F \in \mathcal{F}$ is a conjugate of $\langle a\rangle$, we have that $F = [F, N_G(F)]$ is true for every $F \in \mathcal{F}$. According to Lemma 1.1.9, $\mathcal{F}$ is regular . $line - line$

Remark 3 . 3 . Under the hypotheses of the proposition , if $\mathcal{F}$ is g enerating , then G is perfect (every e lement of the family is contained in G'). If G is s imple , th en the family $\mathcal{F}$ is a generating family .

It is clear that the conditions for the proposition are satisfied if a simple group G has as a subgroup a dihedral group of order $2m$, with m odd . In such a case G has a regular generating active family of cyclic subgroups of order m.

3 . 2 . Simple groups with trivial Schur multiplier

Theorem 3 . 4 . *Every finite s imple group G with trivial Schur multiplier is the active sum of cyclic subgroups .*

Proof . Assume G is a finite non - abelian simple group . Then any non - trivial active family of cyclic subgroups is generating , and by Lemma 3 . 1 , independent . If in addi - tion $H_2(G)$ is trivial , Lemma 2 . 2 . 1 5 tells us that G is the active sum of any non - trivial regular active family of cyclic subgroups of G. It suffices then to find elements in G that satisfy the conditions of Proposition 3 . 2 . Observe that it suffices to show that G contains a dihedral subgroup of order $2n$ with n odd . Since G is simple it is generated by a conjugacy class of involutions . According to Corollary 2 . 67 in [1 5] there must be involutions $s, t \in G$ such that the order of st is not a power of 2 . Assume the order of st is $2^k n$ with $n > 1$ odd . Then $\langle s, (st)^{2^k}\rangle$ is isomorphic to a dihedral subgroup of order $2n$. $line - line$

The above proof tells us nothing about the size of the cyclic subgroups that have to be considered .

3 . 3 . Special linear groups

3.3.1. $\mathrm{SL}_n(q), n \geq 3$ **Theorem 3 . 5 .** *Let* $n \geq 3$. *The group* $\mathrm{SL}_n(q)$ *is an active sum of subgroups of order*

3 , if it is not one of th e fo l lowing : $\mathrm{SL}_3(2), \mathrm{SL}_3(3), \mathrm{SL}_4(2)$ *and* $\mathrm{SL}_3(4)$.

Proof . According to [23] , pages 244 – 246 , the groups $\mathrm{SL}_n(q)$ are perfect and have trivial Schur multiplier (we are assuming that $n \geq 3$ and that the group is not one of $\mathrm{SL}_3(2), \mathrm{SL}_3(3), \mathrm{SL}_4(2)$ or $\mathrm{SL}_3(4)$). [11] in page 2 gives for any field K two matrices of order 2 in $\mathrm{SL}_3(K)$ that generate an S_3 :

$$A := \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad B := \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Furthermore , $\mathrm{SL}_{k-1}(q)$ is included in $\mathrm{SL}_k(q)$. Thus we conclude that S_3 is a subgroup of $\mathrm{SL}_n(q)$. Using Lemma 3 . 1 and Proposition 3 . 2 , we obtain a generating regular and independent active family of subgroups of order 3 . Since the Schur multiplier is trivial , the active sum of the family is isomorphic to $\mathrm{SL}_n(q)$, according to Lemma 2.2.15. $line - line$

We will see in the sequel to this paper , that none of $\mathrm{SL}_3(2), \mathrm{SL}_3(3), \mathrm{SL}_4(2)$ and $\mathrm{SL}_3(4)$ is an active sum of cyclic subgroups .

3.3.2. $\mathrm{SL}_2(q), q$ A POWER OF A PRIME

Let p be a prime number , and let $q = p^r$ be a power of $p, r \geq 1$. We denote by $\mathbb{F}_q$ the finite field of order q. We will consider the special linear groups $\mathrm{SL}_2(q)$ of 2×2 matrices with entries in $\mathbb{F}_q$ and determinant 1 . The theorem we want to prove is :

Theorem 3 . 6 . *If* $q \neq 4$ *then the group* $\mathrm{SL}_2(q)$ *is an a ctive sum of cyclic subgroups .*

Observe that $\mathrm{SL}_2(4) \simeq A_5$. We show in the sequel to this paper , that A_5 is not the active sum of cyclic subgroups . We begin with the case where $q - 1$ is not a power of 2 .

Proposition 3 . 7 . *If* $q - 1$ *is not a power of* 2 , *and* $q \neq 4$, *then the group* $\mathrm{SL}_2(q)$ *is an active sum of cyclic subgroups .*

Proof . According to [23] , pg . 244 and 246, $\mathrm{SL}_2(q)$ is a perfect group , except for $\mathrm{SL}_2(2)$,

and the Schur multiplier is trivial (with the exception of $\mathrm{SL}_2(4)$). $\mathrm{SL}_2(2) \simeq S_3$, and this last group is the active sum of cyclic subgroups . Assume that $q \neq 2$. It follows from Lemma 2 . 2 . 1 5 , that all we have to do is to find a regular and independent

generating active family of cyclic subgroups . According to Lemma 3 . 1 , independence is not an issue . We will find elements in $\mathrm{SL}_2(q)$ satisfying the conditions of Proposition 3 . 2 . Let t be a prime divisor of $q-1$ with $t \neq 2$, and let λ be a primitive root of F_q^*. Consider the following elements of $\mathrm{SL}_2(q)$:

$$A := \begin{pmatrix} \lambda^{\frac{q-1}{t}} & 0 \\ 0 & \lambda^{-q\_-\frac{1}{t}} \end{pmatrix}, \quad B := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Observe that the order of A is t, and that $A^B = A - 1$. Since t is a prime number different from $2, A$ and B satisfy the conditions of Proposition 3 . 2 . Since A is not in the center of $\mathrm{SL}_2(q)$, it follows from 3 . 2 . 8 in [37] that the family consisting of the cyclic subgroup $\langle A \rangle$ and its conjugates is a generating family . Thus , $\mathrm{SL}_2(q)$ is an active sum of cyclic subgroups of order t. $line - line$

Proposition 3 . 8 . *If $p \neq 3$ and $q-1$ is a power of 2 , then $\mathrm{SL}_2(q)$ is an active sum of cyclic subgroups .*

Proof . Again [23] tells us that $\mathrm{SL}_2(q)$ is perfect , and its Schur multiplier is trivial . Consider the elements

$$A := \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad B := \begin{pmatrix} cc2 & 0 \\ 0 & 2^{-1} \end{pmatrix}.$$

Since $p \neq 2, B$ is well defined . We have that $A^B = A^4$. Since $o(A) = p$ and 3 are rela - tive prime , we have that A and B satisfy the hypotheses of Proposition 3 . 2 . Therefore the family consisting of the subgroups that are conjugates of the one generated by A, is a regular and independent family of cyclic subgroups . Since A is not in the center of $\mathrm{SL}_2(q)$, that the family is generating follows as in the previous proposition . $line - line$

The above propositions , together with the observation about $\mathrm{SL}_2(4)$, leave us with the cases $\mathrm{SL}_2(3^r)$, where $3^r - 1$ is a power of 2 . According to [1 9] it was Leo Hebreus who showed that the equation $2^x - 3^y = -1$ has only two solutions , namely $2^1 - 3^1$ and $2^3 - 3^2$. Thus we are left with $\mathrm{SL}_2(3)$ and $\mathrm{SL}_2(9)$. We will show that both groups are active sum of cyclic subgroups .

Proposition 3 . 9 . *The group $\mathrm{SL}_2(3)$ is the active sum of subgroups of order 3 .*

Proof . The order of $G := SL_2(3)$ is $24.G'$ has order 8 . It follows that for any element A of order $3, \langle A \rangle \cap G' = 1$. It follows from Lemma 1 . 1 . 9 , that the family $\mathcal{F}$ consisting of all the subgroups of G of order 3 , is regular . Since $\mathcal{F}$ consists of 3 - Sylow subgroups of G, then a transversal for $\mathcal{F}$ has exactly one element , $\langle A \rangle$ say . Clearly , the order of $\langle A \rangle G'$ is 24 , therefore $\langle A \rangle G'/G' \simeq G/G'$, and the family is independent . [23] again says that G has trivial Schur multiplier . By Lemma 2.2.15, G is the active sum of $\mathcal{F}$ if we can show that the family is generating .$\mathrm{SL}_2(3)$ has four 3 - Sylow subgroups , giving 8 elements of order 3 . It is not hard to produce with these ones five more elements to show that the family does generate . $line - line$

 Proposition 3 . 10 . *The group* $\mathrm{SL}_2(9)$ *is the active sum of subgroups of order* 3 .

Proof . Consider the family $\mathcal{F}$ consisting of the cyclic group generated by

$$B := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

and all its conjugates . Every element of $\mathcal{F}$ has order 3 . Since $\mathrm{SL}_2(9)$ is perfect , we have that the family is independent . Let i be a square root of -1 in $\mathbb{F}_9$, and consider the following element in $\mathrm{SL}_2(9)$:

$$X := \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}.$$

Since $B^X = B^{-1}$, then $\mathcal{F}$ is regular . And by 3 . 2 . 8 in [37] , the family generates .

Let S be the active sum of the family $\mathcal{F}$, and let $\varphi : S \to \mathrm{SL}_2(9)$ be the projection . To prove that φ is an isomorphism , we will show that $\varphi*$ $: H_2(S) \to H_2(\mathrm{SL}_2(9))$ is an epimorphism . According to [23] , pg .246, $H_2(\mathrm{SL}_2(9)) \simeq \mathbb{Z}_3$. Let $\lambda \in \mathbb{F}_q$ such that

$$\lambda element - slash\{0,1,2\}.\text{Define}$$

$$A := \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}.$$

Observe that A is an element of $\mathrm{SL}_2(9)$ of order 3 , and that $AB = BA$. Thus $P := \langle B, A\rangle$ is a 3 - Sylow subgroup of $\mathrm{SL}_2(9)$. Let $i : P \to \mathrm{SL}_2(9)$ be the inclusion . We have i_* $: H_2(P) \to H_2(\mathrm{SL}_2(9))$ and the map called transfer $t : H_2(\mathrm{SL}_2(9)) \to H_2(P)$. It is well known that the composition $i_* \circ t$ is multiplication by the index of P in $\mathrm{SL}_2(9)$, 80 in this case . Since 80 is relative prime with 3 , we have that $i_* \circ t$ is an isomorphism . Since $H_2(P) \simeq \mathbb{Z}_3$, this means that i_* is an isomorphism . Thus $H_2(\mathrm{SL}_2(9))$ is generated by the element $B \wedge A$(see section 2 . 5) . We need an element in $H_2(S)$ that is mapped to $B \wedge A$. Since $\langle B\rangle$ $\in \mathcal{F}$, we have a generator $\widehat{B} \in S$, corresponding to B. Of course $\varphi(\widehat{B}) = B$. A proof very similar to that of Lemma 1 . 1 . 5 shows that $\varphi^{-1}(C_{SL_2(9)}(B)) = C_S(\widehat{B})$. Choose an element $\widehat{A} \in S$ with $\varphi(\widehat{A}) = A$. Then $\widehat{A}$ and $\widehat{B}$ commute in S, and the element $\widehat{B} \wedge \widehat{A}$ in $H_2(S)$ is mapped to $B \wedge A$. Thus $H_2S \to H_2\mathrm{SL}_2(9)$ is an epimorphism . Therefore $\varphi : S \to \mathrm{SL}_2(9)$ is an isomorphism . $line - line$

3 . 4 . Covering groups

We begin with an extension

$$K \text{\_\_\_\_\_} \widehat{G} \pi \text{\_\_\_\_\_} G, \tag{3}$$

of finite groups . Our first lemma is :

Lemma 3 . 1 1 . *If the extension* (3) *is central , then for e very subgroup* F *of* G, *whose*
order is re latively prime with $|\ K\ |$, *there is a unique homomorphism* $\theta : F \to \widehat{G}$ *such that* $\pi \circ \theta$ *equals the inclusion* $F \to G$. *Proof .* The central extension (3) induces the central extension

$$K \_\_\_\_\_ \pi^{-1}F \_\_\_\_\_ F.$$

Since $(|\ F\ |, |\ K\ |) = 1$, the Schur - Zassenhaus ' theorem [14] , pg . 2 1 1 , tells us that $\pi^{-1}F \simeq K \times F$. There is a unique homomorphism $F \to \pi^{-1}F$ such that the diagram

$$\begin{array}{c} F \_\_\_\_\_ \pi^{-1}F \\ d69 - d69 - d69 - d69 - d69 - d69 - d69_{id} d15 - d15 - line \pi \\ F \end{array}$$

commutes . The homomorphism we are looking for is then

$$F \_\_\_\_\_ \pi^{-1}F \text{———} \widehat{G}. \qquad line - line$$

Suppose now that we have a generating active family $\mathcal{F} = \{F_i\} i \in I$ of subgroups of G, such that $(|\ F_i\ |, |\ K\ |) = 1$ for every $i \in I$. According to the above lemma , there are uniquely determined subgroups $\widehat{F}_i$ of $\widehat{G}$ such that $\pi\,|_{\widehat{F}_i} \colon \widehat{F}_i \to F_i$ is an isomorphism . Since conjugation does not alter order we have that $\widehat{F} := \{\widehat{F}_i\} i \in I$ is an active family , with inclusions as in the family $\mathcal{F}$. We call such a family $\widehat{F}$ a *lifting* of the family $\mathcal{F}$ to $\widehat{G}$. It is clear that the active sum of the family $\mathcal{F}$ is isomorphic to the active sum of the family $\widehat{F}$. Now we have :

Proposition 3 . 12 . *Assume that the extension* (3) *is s t em , th en , with the above*
notation , $\widehat{F}$ *is a generating active family of subgroups of* $\widehat{G}$
Proof . Let N be the subgroup of $\widehat{G}$ generated by $\widehat{F}$. Since $\widehat{F}$ is active and K is central , $\succ^{\text{thesubgroup}}$ that $\widehat{G}^{N}_{/N=}{}^{\text{is a}} KN^{\text{normal}}/N\text{subgroup}_{\simeq K/(K}\text{of}^{\widehat{G}}_{\cap N)}.\text{It}_{\text{Since}}$ is $\text{nothardthe}_{\text{extension}}$ to see $\text{is}^{\text{that}}\text{central}^{\widehat{G}=} KN_{\cdot}^{\text{It}}{}_{\text{wehave}}$ that $K/(K \cap N)$ is abelian . Therefore $\widehat{G}' \subseteq N$. Since the extension is stem , we have

$$\text{that} K \subseteq \widehat{G}' \subseteq N.\text{Therefore} N = \widehat{G}. \quad line - line$$

An immediate consequence of the previous two propositions is :

Proposition 3 . 13 . *If (3) is a s t em extension and* $\mathcal{F}$ *is an active generating family of subgroups of* G *such that , for e very* $F \in \mathcal{F}$ *the order of* F *is re latively prime with the order of* K, *then there is an epimorphism from the a ctive sum of the family* $\mathcal{F}$

$$to \widehat{G}.$$

Theorem 3 . 14 . *Let G be a finite group . Let p be a prime dividing the order of H_2G. If G is th e active sum of an a ctive generating family $\mathcal{F}$ of subgroups of G, then there exists an $F \in \mathcal{F}$ such that p divides the order of F.*

Proof . Assume $\mathcal{F}$ is an active generating family of subgroups of G. Let S be the active $\sum_{\text{acôve}}$ of $\mathcal{F}$.Assume$_{\text{groupof}}G^{\text{that}}_{,}p$doesthen we nothave $\div_{\text{astem}}$ the$_{\text{extension}}$ order $\mu^{\widehat{G}}_{\text{of}^{\text{any}}_{H\ 2G}_____\in\pi_{\text{of}\mathcal{F}.}_____}Let_{G.}$Since$^{\widehat{G}\text{be}}$
$p \mid\mid H_2G \mid$ and H_2G is abelian , there is a subgroup N of H_2G such that $H_2G/N \simeq \mathbb{Z}_p$.

Since$_{\text{stem}}$extension$^{\text{theabove}}$extension$_{H_2G/N_____}$is$_{\mu}$stem, $\widehat{G}^{\text{theimage}}_{/\ \mu(N\ )\_\_\_\_\_}\pi$ of$^{N}_{G}$,is$_{\text{that}}$normal$_{\text{isto}}$in$^{\widehat{G}}_{\text{say},\cdot\text{a}}We_{\text{stem}}$ob

the form :

$$\mathbb{Z}_{p\_\_\_\_\_}H\_\_\_\_\_G.$$

By what was done in the above propositions , taking into account the fact that p does not divide any element of $\mathcal{F}$, there is a unique lifting $\widehat{F}$ of $\mathcal{F}$ to H, and thus , an epimorphism $S \to H$. Since $\mid H \mid = \mid G \mid p$, we conclude that S is not isomorphic

$$\text{to}G. \quad line - line$$

Theorem 3 . 15 . *Let G be a finite group and $\widehat{G}$ a covering group of G. If $\mathcal{F}$ is a regular and independent generating a ctive family of subgroups of G such that for every $F \in \mathcal{F}$ we have that $\mid F \mid$ and $\mid H_2G \mid$ are re latively prime , then th e active sum of $\mathcal{F}$ is is omorphic to $\widehat{G}_{.}$*

Proof . According to Proposition 3 . 1 3 , the active sum S of the family $\mathcal{F}$ is projected onto $\widehat{G}_{.}$ Let $\varphi : S \to G$ be the usual projection and $K := \ker \varphi$. Consider the following diagram

$$K\_\_\_\_\_S\varphi\_G$$
$$d31^{d31-d15-d15}_{d31} \quad line - d15 - d15 - d15 - d15 \quad vlinevline$$
$$H_2G\_\_\_\_\_\widehat{G}\_\_\_\_\_{}_{\pi}G.$$

Since$^{\text{hand}}_{\text{thediagram}}\square^{\text{commutes}}_{\text{commute}}$.Since$_S$wecaninduceprojects$_{\text{onto}}the_{\widehat{G},}$ dotted$_{\text{wehave}}$arrowthat$K|^{\rightarrow}_{\widehat{G}\ |\leq}H2|^{G,}_{S|.}$

Consider the exact sequence

$$H_2S\varphi^*_{\_}H_2G\_\_\_\_\_K\_\_\_\_\_H_1S\varphi^*_{\_}H_1G\_\_\_\_\_0$$

Since $\mathcal{F}$ is regular and independent , we have that $\varphi*$ is an isomorphism at the H_1 level . This means that the next arrow to the left is zero . Therefore the arrow $H_2G \to K$ is an epimorphism . Thus $\mid K \mid \leq \mid H_2G \mid$. Now , it is j ust a question of cardinality :

$$\mid S \mid = \mid K \mid\mid G \mid \leq \mid H_2G \mid\mid G \mid = |^{\widehat{G}}| \leq \mid S \mid .$$

Therefore S is isomorphic to $\widehat{G}_{.}$ $line - line$

Conjecture . *Every covering group of a finite s imple group is an active sum of cyclic subgroups .*

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