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HYPERSTABILITY OF A CLASS OF LINEAR FUNCTIONAL EQUATIONS

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Dedicated to the 60 th birthday of Professor Árpád Varecza

Abstract . The aim of this note is to offer hyperstability results for linear functional equations of the form

$$f(s)+f(t)=\frac{1}{n}\sum_{n}^{i=1} f(s\varphi i(t)) \quad (s,t\in S),$$

where S is a semigroup and where $\varphi 1,...,\varphi n : S\to S$ are pairwise distinct automorphisms of S such that the set $\{\varphi 1,...,\varphi n\}$ is a group equipped with the composition as the group operation . The main results state that if f satisfies a stability inequality related to the above equation then it is also a solution of this equation .

1 . Introduction

In a recent paper of Kocsis and Maksa [KM 98] , the stability problem of a sum form functional equation from information theory led to the investigation of the stability of the equation

$$\varphi(xy)=x^{\alpha}\varphi(y)+y^{\alpha}\varphi(x) \quad (x,y\in]0,1]), \tag{1}$$

where $\alpha\in\mathbb{R}$ is a fixed power and $\varphi :]0,1]\to\mathbb{R}$. It is well - known and easy to see that the general solution of (1) is of the form

$$\varphi(x)=x^{\alpha}\ell(x) \quad (x\in]0,1]),$$

where $\ell :]0,1]\to\mathbb{R}$ satisfies the Cauchy equation

$$\ell(xy)=\ell(x)+\ell(y) \quad (x,y\in]0,1]). \tag{2}$$

The stability problem of (1) can now be formulated as follows :

(P) $\{^{Assumethat}_{for} \quad somedifference \overset{a|^{functi}_{\psi} \quad (xy)}{constantfunction} \varepsilon\psi 0. \quad Doesthereexistasolution\varphi of(1)suchthat^{-\varphi isbounded?}_{\geq -^{on}\psi^{:}_{\alpha x\psi(} {}^{]0,}_{y)}1-]\to^{\mathbb{R}}_{y^{\alpha}\psi(x} {}^{sati}}$

In the case $\alpha=0$ it follows from the Hyers - Ulam stability theorem for the Cauchy functional equation that there exists a solution φ of (1) such that $|\psi-\varphi| \leq \varepsilon$ (see [Hye 4 1]) . The discussion of the case $\alpha=1$ was proposed by Maksa [Mak 97]

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at the 34 th ISFE and an affirmative solution to (P) was found by Jacek Tabor [Tab 97 a] (see also [Bad 0] , [P á l 97] , [Tab 97 b] for related or more general results) . The case $\alpha > 0$ can easily be reduced to the case $\alpha = 1$ by considering the function
$]0,1] \ni x \mapsto \psi(x^{1/\alpha})$ instead of ψ. Thus , it follows from Tabor ' s result that (1) is stable for $\alpha > 0$.

For the sake of completeness now we consider the case $\alpha < 0$, or more generally , we replace the power function $t \mapsto t^\alpha$ in (1) by a function $M :]0,1] \to \mathbb{R}$ satisfying

$$M(xy) = M(x)M(y) \quad (x, y \in]0,1]) \tag{4}$$

and we also suppose that

$$M(x_0) > 1 \quad \text{forsome} \quad x_0 \in]0,1]. \tag{5}$$

Thus , (3) can be rewritten as

$$\mid \psi(xy) - M(x)\psi(y) - M(y)\psi(x) \mid \leq \varepsilon \quad (x, y \in]0,1]). \tag{6}$$

Due to (5), M is positive - valued (see Acz é l and Dhombres [AD 89]) . Therefore , we can introduce the functions

$$\ell(x) = \frac{\psi(x)}{M(x)} \quad (x \in]0,1]) \tag{7}$$

and

$$F(x, y) = \ell(xy) - \ell(x) - \ell(y) \quad (x, y \in]0,1]). \tag{8}$$

With these notations , the stability inequality (6) reduces to

$$\mid F(x, y) \mid \leq \frac{\varepsilon}{M(xy)} \quad (x, y \in]0,1]). \tag{9}$$

It can easily be checked that the function F defined in (8) satisfies the so - called *cocycle functional equation*

$$F(x, y) + F(xy, z) = F(x, yz) + F(y, z) \quad (x, y, z \in]0,1]). \tag{10}$$

With the substitution $z = 0_x^k$, (10) implies that

$$F(x, y) + F(xy, 0_x^k) = F(x, yx_0^k) + F(y, 0_x^k) \quad (x, y \in]0,1], k \in \mathbb{N}). \tag{11}$$

Using the estimate (9) and equation (4) , we have that

$$\mid F(s, tx_0^k) \mid \leq \frac{\varepsilon}{M(st)[M(x_0)]^k} \quad (s, t \in]0,1]).$$

Hence , by (5) , we obtain

$$\lim_{k \to \infty} F(s, tx_0^k) = 0 \quad (s, t \in]0,1]).$$

Thus , taking the limit $n \to \infty$ in (1 1) , we get that

$$F(x, y) = 0 \quad (x, y \in]0, 1]),$$

that is , ℓ is a solution of (2) . By (7) ,

$$\psi(x) = M(x)\ell(x) \quad (x \in]0, 1])$$

and an easy calculation yields that ψ satisfies the functional equation

$$\psi(xy) = M(x)\psi(y) + M(y)\psi(x) \quad (x, y \in]0, 1]), \tag{12}$$

which is analogous to (1) .

Summarizing our observations , we have proved the following hyperstability result for the functional equation (1 2) .

Theorem 1 . *Let M :$]0, 1] \to \mathbb{R}$ be a s o lution of the functional equation (4) and suppose that (5) als o holds . Assume that the function ψ :$]0, 1] \to \mathbb{R}$ satisfies the s tability inequality (6) for s ome $\varepsilon \geq 0$. Then ψ is a s o lution of (1 2) , that is , (6) is satisfied by $\varepsilon = 0$.*

The above result shows that the solutions of the inequality (6) are j ust the solutions of the corresponding equation (1 2) . Thus , in the particular case $\alpha < 0$, the solutions of (3) and the solutions of (1) are the same . As we have seen , the basic tool for proving the above result is the cocycle equation (1 0) which plays an important role in the theory of group extensions (see [JKT 68] , [Erd 59]) .

We note that if (5) does not hold , that is , $M(x) \leq 1$ for all $x \in]0, 1]$, then , either $M(x) = x^{\alpha}$ $(x \in]0, 1])$ for some $\alpha \geq 0$, or $M(x) = 0(x \in]0, 1])$, or $M(x) = 0$ $(x \in]0, 1[)$ and $M(1) = 1($ see [Acz 66]) . In these cases , the stability problem of the functional equation (1 2) is either solved , or is trivial and uninteresting .

The aim of this paper to extend the above argument to a class of linear functional equations for which a cocycle equation - type identity can be derived .

2 . Main Results

Throughout this section , let $S = (S, \cdot)$ denote a semigroup and let X denote a real normed space . In addition , let $\varphi 1, ..., \varphi n : S \to S$ be pairwise distinct automorphisms of S such that the set $\{\varphi 1, ..., \varphi n\}$ is a group with respect to the composition as group operation .

We consider the following functional equation

$$f(s) + f(t) = \frac{1}{n}\sum_{n}^{i=1} f(s\varphi i(t)) \quad (s, t \in S), \tag{13}$$

There are two important particular cases of the above equation .

- (PC $-1)$: $n = 1$ and $\varphi 1(t) = t(t \in S)$. In this setting , (1 3) reduces to the Cauchy equation (2) .
- (PC $-2)$: $n = 2$ and $\varphi 1(t) = t, \varphi 2(t) = t^{-1}$ $(t \in S)$ and S is an Abelian group . With these assumptions , (1 3) reduces to the so - called *norm - square equation*

$$f(s) + f(t) = \frac{1}{2}(f(st) + f(st^{-1})) \quad (s, t \in S).$$

For further examples and special cases of (1 3) , see [P á l 94] .

The proof of the main results is based on the following lemma ([P á l 94 , Theorem 1]) which derives an identity for the two variable function obtained by taking the difference of the left and right hand sides of (1 3) .

Lemma . *Let $f : S \to X$ be an arbitrary function . Then th e function $F : S \times S \to X$ defined by*

$$F(s, t) = f(s) + f(t) - \frac{1}{n}\sum_{n}^{i=1} f(s\varphi i(t)) \quad (s, t \in S) \tag{14}$$

satisfies the following functional equation

$$F(x, y) + \frac{1}{n}\sum_{n}^{i=1} F(x\varphi i(y), z) = \frac{1}{n}\sum_{n}^{i=1} F(x, y\varphi i(z)) + F(y, z) \quad (x, y, z \in S). \tag{15}$$

Proof . Let $f : S \to X$ be arbitrary and let F given by (14) . Evaluating the left hand side of (1 5) , we get

$$F(x, y) + \frac{1}{n}\sum_{n}^{i=1} F(x\varphi i(y), z) = f(x) + f(y) + f(z) - \frac{1}{n^2}\sum_{n}^{i=1}\sum_{n}^{j=1} f(x\varphi i(y)\varphi_j(z)).$$

 Similarly, for the right hand side, we deduce

$$F(y,z) \quad + \quad \frac{1}{n}\sum_{n}^{i=1} F(x, y\varphi i(z))$$

$$= \quad f(x)+f(y)+f(z)-\frac{1}{n^2}\sum_{n}^{i=1}\sum_{n}^{j=1} f(x\varphi_j(y\varphi i(z)))$$

$$= \quad f(x)+f(y)+f(z)-\frac{1}{n^2}\sum_{n}^{j=1}\sum_{n}^{i=1} f(x\varphi_j(y)\varphi_j\circ\varphi i(z)))$$

$$= \quad f(x)+f(y)+f(z)-\frac{1}{n^2}\sum_{n}^{j=1}\sum_{n}^{i=1} f(x\varphi_j(y)\varphi i(z)),$$

where, in the last steps, we used that φ_j is a homomorphism and $(\varphi_j\circ\varphi 1,...,\varphi_j\circ\varphi n)$ is a permutation of $(\varphi 1,...,\varphi n)$. Thus (15) turns out to be valid. $\square$

In the particular case (PC-1), the resulting equation (15) is equivalent to the cocycle equation (10). In the second particular case (PC-2), (15) reduces to the equation

$$F(x,y)+\frac{1}{2}(F(xy,z)+F(xy^{-1},z))=\frac{1}{2}(F(x,yz)+F(x,yz^{-1}))+F(y,z)(x,y,z\in S),$$

that was discovered by Székelyhidi [Szék 83] and investigated by Ebanks [Eba 85], [Eba 89] and Székelyhidi [Szék 95].

The following theorem is a *hyperstability* result for (13). It states that if the error bound for the difference of the two sides of (13) satisfies a certain asymptotic property then, in fact, the two sides have to be equal to each other.

Theorem 2. *Let $\varepsilon : S\times S\to\mathbb{R}$ be a function such that there exists a sequence*

$$u_k\in S satisfying$$

$$\lim_{k\to\infty}\varepsilon(u_k s,t)=0 \quad (s,t\in S). \quad (16)$$

Assume that $f : S\to X$ satisfies the stability inequality

$$\|f(s)+f(t)-\frac{1}{n}\sum_{n}^{i=1} f(s\varphi i(t))\| \quad \le\varepsilon(s,t) \quad (s,t\in S). \tag{17}$$

Then f is a solution of (13). *Proof.* Define $F : S\times S\to\mathbb{R}$ by (14). Then (15) is satisfied and (17) yields

$$\| F(s,t)\|\le\varepsilon(s,t) \quad (s,t\in S).$$

Thus, by (16), we have that

$$\lim_{k\to\infty} F(u_k s,t)=0 \quad (s,t\in S). \tag{18}$$

Let $y,z,s_0\in S$ be fixed. Substituting $x=u_k s_0$ into (15), taking the limit as $k\to\infty$ and applying (18), we deduce from (15) that

$$F(y,z)=0 \quad (y,z\in S),$$

that is, f is a solution of (13). $\square$ **Corollary 1.** *Let $\varepsilon : S\times S\to\mathbb{R}$ and suppose that there exist $u\in S$ and $0\le q<1$ such that*

$$\varepsilon(us,t)\le q\varepsilon(s,t) \quad (s,t\in S). \tag{19}$$

Assume that $f : S\to X$ satisfies the stability inequality (17). Then f is a solution of (13).

 Proof . It suffices to show that ε satisfies (1 6) for some sequence u_k. Then , (1 9) yields by induction that

$$\varepsilon(u^k s,t) \le q^k \varepsilon(s,t) \quad (s,t \in S, k \in \mathbb{N}),$$

whence (1 6) follows with the sequence $u_k = u^k$. Thus the statement is the conse - quence of Theorem 2. □ **Theorem 3 .** *Let $\varepsilon : S \times S \to \mathbb{R}$ be a function such that there exists a s equence*

$$u_k \in S satisfying$$

$$\lim_{k\to\infty} \varepsilon(s, t\varphi i(u_k)) = 0 \quad (s,t \in S, i \in \{1,...,n\}). \quad (20)$$

Assume that $f : S \to X$ satisfies the s tability inequality (1 7) . Then f is a s o lution of (1 3) .

Proof . The proof is analogous to that of Theorem 2 . Define F by (1 4) . Instead of (1 8) , we now have that

$$\lim_{k\to\infty} F(s, t\varphi i(u_k)) = 0 \quad (s,t \in S, i \in \{1,...,n\}). \tag{21}$$

Let $x, y, t_0 \in S$ be fixed . Substituting $z = t_0 u_k$ into (1 5) , taking the limit as $k \to \infty$ and applying (2 1) , we obtain that

$$F(x,y) = 0 \quad (x,y \in S).$$

Therefore f is a solution of (13). □ **Corollary 2 .** *Let $\varepsilon : S \times S \to \mathbb{R}$ and suppose that there exist $u \in S$ and $0 \le q < 1$ such that*

$$\varepsilon(s, t\varphi i(u)) \le q\varepsilon(s,t) \quad (s,t \in S, i \in \{1,...,n\}). \tag{22}$$

Assume that $f : S \to X$ satisfies the s tability inequality (1 7) . Then f is a s o lution of (1 3) . *Proof .* In this case , (22) yields by induction that

$$\varepsilon(s, t\varphi i(u^k)) \le q^k \varepsilon(s,t) \quad (s,t \in S, i \in \{1,...,n\}, k \in \mathbb{N}).$$

Therefore (20) is satisfied by $u_k = u^k$ and the statement follows from Theorem 3 .

□

References

[Acz 66] J . Acz é l , *Lectures on functional equations and their applications* , Mathematics in Science and Engineering , vol . 1 9 , Academic Press , New York – London , 1 966 . [AD 89] J . Acz é l and J . Dhombres , *Functional equations in s everal variables (With applications to mathematics , information theory and to the natural and social sciences)* , Cambridge Uni - versity Press , Cambridge , 1 989 . [Bad 0] R . Badora , *Superstability of the Cauchy functional equation* , Publ . Math . Debrecen **57** (2000) , 421 – 424 . [Eba 85] B . Ebanks , *Problems and remarks at the 22 nd ISFE (in Report of Meeting)* , Aequationes Math . **2 9** (1 985) , 105 – 106 . [Eba 89] B . Ebanks , *Differentiable s olutions of a functional equation of Sz é kelyhidi* , Util . Math . **36** (1 989) , 1 97 – 1 99 . [Erd 59] J . Erd hungarumlaut – o s , *A remark on the paper " On som e f – u ncti n – o a* e^{q-u} *ation b S K urep a* comma – quotedblright Glasnik Math . - Fiz . Astr . **14** (1 959) , 3 – 5 . [Hye 41] D . H . Hyers , *On the stability of the linear functional equation* , Pro . Nat . Acad . Sci . U . S . A . **27** (1 941) , 222 – 224 . [JKT 68] B . Jessen , J . Karpf , and A Thorup , *Some functional equations in groups and rings* , Math . Scand . **22** (1 968) , 257 – 265 . [KM 98] I . Kocsis and Gy . Maksa , *The stability of a sum form functional equation arising in information theory* , Acta Math . Hungar . **79** (1 998) , no . 1 - 2 , 39 – 48 . [Mak 97] Gy . Maksa , *1 8 . Problems (in Report of Meeting)* , Aequationes Math . **53** (1 997) , 1 94 .

[Pál94] Zs. Páles, *Bounded solutions and stability of functional equations for two variable functions*, Results Math. **26** (1994), no. 3-4, 360–365.

[Pál97] Zs. Páles, *27. Remark (in Report of Meeting)*, Aequationes Math. **53** (1997), 200–201.

[Szék83] L. Székelyhidi, *Problems and remarks at the 21st ISFE (in Report of meeting)*, Aequationes Math. **26** (1983), no. 2-3, 284.

[Szék95] L. Székelyhidi, *Stability properties of functional equations in several variables*, Publ. Math. Debrecen **47** (1995), no. 1-2, 95–100.

[Tab97a] J. Tabor, *20. Remark, Solution to Problem A in Problem 18. (in Report of Meeting)*, Aequationes Math. **53** (1997), 194–196.

[Tab97b] J. Tabor, *Stability of the Cauchy equation with variable bound*, Publ. Math. Debrecen **51** (1997), no. 1-2, 165–173.

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