Gegenbauer Matrix Polynomials and Second Order Matrix Differential Equations

Polinomios Matriciales de Gegenbauer y Ecuaciones Diferenciales Matriciales de Segundo Orden

K . A . M . Sayyed , M . S . Metwally , R . S . Batahan

Department of Mathematics , Faculty of Science

Assiut University , 71 5 1 6 , Assiut , Egypt .

Abstract

In this paper , we study the Gegenbauer matrix polynomials . An explicit representation , a three - term matrix recurrence relation and or - thogonality property for the Gegenbauer matrix polynomials are given . These polynomials appear as finite series solutions of second - order ma - trix differential equations .

Key words and phrases : hypergeometric matrix function , orthogo - nal matrix polynomials , Gegenbauer matrix polynomials , three - terms matrix recurrence .

Resumen

En este art í culo se estudian los polinomios matriciales de Gegen - bauer . Se da una representaci ó n expl í cita , una relaci ó n de recurrencia matricial de tres t é rminos y una propiedad de ortogonalidad para los polinomios matriciales de Gegenbauer . Estos polinomios aparecen como soluciones en series finitas de ecuaciones diferenciales matriciales de se - gundo orden .

Palabras y frases c lave : funci ó n hipergeom é trica matricial , polino - mios matriciales ortogonales , polinomios matriciales de Gegenbauer , re - currencia matricial de tres t é rminos .

1   Introduction

It is well known that special matrix functions appear in the study in sta - t ist ics , theoretical physics , groups representation theory and number theory [1 , 2 , 8 , 1 5 , 2 1] .   Orthogonal matrix polynomials have been considered in the book on matrix polynomials by Gohberg , Lancaster and Rodman [7] and in the survey on orthogonal matrix polynomials by Rodman [1 7] and see more papers [4 , 5 , 6 , 1 9 , 20] and the references therein . During the last two decades the classical orthogonal polynomials have been extended to the orthogonal matrix polynomials see for instance [1 2 , 1 3] .   Hermite and Laguerre matrix polynomials was introduced and studied in [9 , 1 0] and an accurate approxi - mation of certain differential systems in terms of Hermite matrix polynomi - als was computed in [3] .   Furthermore , a connection between Laguerre and Hermite matrix polynomials was established in [1 2] .   Recently , the general - ized Hermite matrix polynomials have been introduced and studied in [1 8] . J $\acute{o}$ dar and Cort $\acute{e}$ s introduced and studied the hypergeometric matrix function F (A , B ; C ; z) and the hypergeometric matrix differential equation in [1 1] and the the explicit closed form general solution of it has been given in [14] .

The primary goal of this paper is to consider a new system of matrix poly - nomials , namely the Gegenbauer matrix polynomials .   The structure of this paper is the following .   In section 2 a definition of Gegenbauer matrix poly - nomials is given .   Some differential recurrence relations , in particular Gegen - bauer ' s matrix differential equation are established in section 3 .   Moreover , hypergeometric matrix representations of these polynomials will be given in sec - tion 4 . Finally in section 5 we obtain the orthogonality property of Gegen - bauer matrix polynomials .

Throughout this study , consider the complex space $\mathbb{C}^{N \times N}$ of all square complex matrices of common order N . We say that a matrix A in $\mathbb{C}^{N \times N}$ is a positive stable if Re $(\lambda) > 0$ for all $\lambda \in \sigma(A)$ where $\sigma(A)$ is the set of all eigenvalues of A . If $A_0, A_1, \cdots, A_n$ are elements of $\mathbb{C}^{N \times N}$ and $A_n \neq 0$, then we call

$$P(x) = A_n x^n + A_{n-1} x^{n-1} + ... + A_1 x + A_0,$$

a matrix polynomial of degree n in x .

If $P + nI$ is invertible for every integer $n \geq 0$, then from [1 1] it follows that

$$(P)_n = P(P + I)(P + 2I)...(P + (n - 1)I); \quad n \geq 1; \quad (P)0 = I. \qquad (1)$$

From (1) , it is easy to find that

$$(P)_{n-}k = (-1)^k (P)_n [(I - P - nI)k]^{-1}; \quad 0 \leq k \leq n. \qquad (2)$$

Gegenbauer Matrix Polynomials   1 3 From the relation (3) of [1 6 ,   pp . 58] , one obtains

$$((\tfrac{-1)^k}{n-k)!}I = (-n)n_!^k I = (-nI)n_!^k; \quad 0 \le k \le n.$$

(3)

The hypergeometric matrix function F (A , B ; C ; z) has been given in the form [1 1 ,   p . 2 1 0]

$$F\ (\ A\ ,\ B\ ;\ C\ ;\ z\ \ ) = \sum_{n\ge 0} 1_{n!}(A)_n(B)_n[(C)_n]^{-1}z^n,$$

(4)

for matrices A , B and C in $\mathbb{C}^{N\times N}$ such that C $+ nI$ is invertible for all integer

$$n \ge 0 \text{andfor} \mid z \mid < 1.$$

For any matrix P in $\mathbb{C}^{N\times N}$ we will exploit the following relation due to [1 1 ,   p . 2 1 3]

$$(1-x)^{-P} = \sum_{n\ge 0} 1_{n!}(P)_n x^n, \quad \mid x \mid < 1.$$

(5)

It has been seen by Defez and J $\acute{o}$ dar   [3] that ,   for matrices A (k , n) and B (k , n) in $\mathbb{C}^{N\times N}$ where $n \ge 0, k \ge 0$, the following relations are satisfied :

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} A(k,n) = \sum_{n=0}^{\infty}\sum_{k=0}^{[n/2]} A(k,n-2k),$$

(6)

and

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} B(k,n) = \sum_{n=0}^{\infty}\sum_{k=0}^{n} B(k,n-k).$$

(7)

Similarly , we can write

$$\sum_{n=0}^{\infty}\sum_{k=0}^{[n/2]} A\ (\ k\ ,\ n\ \ ) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} A(k,n+2k),$$

(8)

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n} B\ (\ k\ ,\ n\ \ ) = \sum_{n=0}^{\infty}\sum_{k=0}^{[n/2]} B(k,n-k),$$

(9)

$$\sum_{n=0}^{\infty} \sum_{k=0}^{n} B(k, n) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} B(k, n+k), \quad (10)$$

If A is a positive stable matrix in $\mathbb{C}^{N \times N}$, then the nth Hermite matrix polynomials was defined by [1 0 ,   pp . 14]

$$H_n(x, A) = n! \sum_{[n/2]}^{k=0} k! (\tfrac{(-1)^k}{n-2k)!} (x\sqrt{2}A)^{n-2k}; \quad n \geq 0. \tag{11}$$

The expansion of $x^n I$ in a series of Hermite matrix polynomials has been given in [3 ,   pp .   14] in the form

$$\frac{x^n}{n!}I = (\sqrt{2}A)^{-n} \sum_{[n/2]}^{k=0} k!(\tfrac{n!}{n-2k)!}H_{n-2k}(x, A); \quad -\infty < x < \infty. \tag{12}$$

2   Gegenbauer   matrix   polynomials

Let A be a positive stable matrix in $\mathbb{C}^{N \times N}$. We define the Gegenbauer matrix polynomials by means of the relation :

$$F = (1 - 2xt - t^2)^{-A} = \sum_{n=0}^{\infty} C_n^A(x)t^n. \tag{13}$$

By using (5) and (9) , we have

$$(1 - 2xt - t^2)^{-A} = \sum_{\infty}^{n=0} \sum_{[n/2]}^{k=0} (-1)^k \tfrac{(A)_{n-k}}{k!(n-2k)!} (2x)^{n-2k} t^n. \tag{14}$$

By equating the coefficients of t^n in   (1 3)   and   (1 4) ,   we obtain an explicit representation of the Gegenbauer matrix polynomials in the form :

$$C_n^A(x) = \sum_{[n/2]}^{k=0} (-1)^k \tfrac{(A)_{n-k}}{k!(n-2k)!} (2x)^{n-2k}. \tag{15}$$

Clearly , $C_n^A(x)$ is a matrix polynomial of degree n in x . Replacing x by - x and t by - t in (1 3) , the left side does not exchange . Therefore

$$C_n^A(-x) = (-1)^n C_n^A(x). \tag{16}$$

For $x = 1$ we have

$$(1 - t)^{-2A} = \sum_{n=0}^{\infty} C_n^A(1)t^n.$$

$$C_n^A(1) = (2_n^{A)_n}!. \tag{17}$$

For $x = 0$ it follows

$$(1+t^2)^{-A} = \sum_{n=0}^{\infty} C_n^A(0)t^n.$$

Also , by (5) one gets

$$(1+t^2)^{-A} = \sum_{\infty}^{n=0}(-1)_n^{n(A)_n}!t^{2n}.$$

Therefore , we have

$$C_{2n}^A(0) = (-1)_n^{n(A)_n}!, \quad C_{2n+1}^A(0) = 0. \tag{18}$$

The explicit representation (1 5) gives

$$C_n^A(x) = 2_{n!}^{n(A)_n}x^n + \prod_{n-2,}$$

where $\prod_{n-2}$ is a matrix polynomial of degree $(n-2)$ in x.   Consequently , if $D = dx^d$, then it follows that

$$D^n C_n^A(x) = 2^n(A)_n.$$

3   Differential   recurrence   relations

By differentiating (1 3) with respect to x and t yields respectively

$$\partial_{\partial}^F{}_x = 1 - 2xt^t - t^2 2AF. \tag{19}$$

and

$$\partial_{\partial}^F{}_t = 1(_-^x 2 - xt^{t)}\_t^2 2AF. \tag{20}$$

So that the matrix function F satisfies the partial matrix differential equation :

$$(x - t)\partial_{\partial}^F{}_x - t^{\partial}{}_{\partial t}^F = 0.$$

$$\sum_{n=0}^{\infty} xDC_n^A(x)t^n - \sum_{n=0}^{\infty} nC_n^A(x)t^n = \sum_{n=1}^{\infty} DC_{n-1}^A(x)t^n,$$

where$_{\text{Since}}$$D = \frac{dxDC_0 A_{(x)}}{d} = 0$ and for $n \geq 1$, then we obtain the differential recurrence

relation :
$$xDC_n^A(x) - nC_n^A(x) = DC_{n-1}^A(x). \quad (21)$$

From (1 9) and (20) with the aid of (1 3) we get respectively the following

$$1 - 21_{xt} - t^2 2A(1 - 2xt - t^2)^{-A} = \sum_{\infty}^{n=1} DC_n^A(x)t^{n-1}, \qquad (22)$$

and

$$1x_{-2xt}^{-t} - t^2 2A(1 - 2xt - t^2)^{-A} = \sum_{\infty}^{n=1} nC_n^A(x)t^{n-1}. \qquad (23)$$

Note that $1 - t^2 - 2t(x - t) = 1 - 2xt - t^2$. Thus by multiplying (22) by $(1 - t^2)$ and (23) by $2t$ and subtracting (23) from (22) we obtain

$$2(A + nI)C_n^A(x) = DC_{n+1}^A(x) - DC_{n-1}^A(x). \qquad (24)$$

From (2 1) and (24) , one gets

$$xDC_n^A(x) = DC_{n+1}^A(x) - (2A + nI)C_n^A(x). \qquad (25)$$

Substituting $(n - 1)$ for n in (25) and putting the resulting expression for

$$DC_{n-1}^A(x) \text{into} (21), \text{gives}$$
$$(x^2 - 1)DC_n^A(x) = nxC_n^A(x) - (2A + (n - 1)I)C_{n-1}^A(x). \quad (26)$$

Now , by multiplying (2 1) by $(x^2 - 1)$ and substituting for $(x^2 - 1)DC_n^A(x)$ and $(x^2 - 1)DC_{n-1}^A(x)$ from (26) to obtain the three terms recurrence relation in the form

$$nC_n^A(x) = (2A + 2(n - 1)I)xC_{n-1}^A(x) - (2A + (n - 2)I)C_{n-2}^A(x). \qquad (27)$$

Write (22) in the form :

$$2A(1 - 2xt - t^2)^{-(A+I)} = \sum_{n=0}^{\infty} DC_{n+1}^A(x)t^n. \qquad (28)$$

By applying (1 3) it follows

$$2A(1 - 2xt - t^2)^{-(A+I)} = \sum_{n=0}^{\infty} 2AC_n^{A+I}(x)t^n. \qquad (29)$$

Identification of the coefficients of t^n in (28) and (29) yields

$$DC_{n+1}^A(x) = 2AC_n^{A+I}(x),$$

which gives

$$DC_n^A(x) = 2AC_{n-1}^{A+I}(x). \qquad (30)$$

Iteration (30) yields , for $0 \le r \le n$,

$$D^r C_n^A(x) = 2^r (A)_r C_{n-r}^{A+rI}(x). \qquad (31)$$

The first few Gegenbauer matrix polynomials are listed here ,

$$C_0^A(x) = I,$$
$$C_1^A(x) = 2Ax,$$
$$C_2^A(x) = 2(A)2^{x^2} - A,$$
$$C_3^A(x) = 4_3(A)3^{x^3} - 2(A)2^x,$$
$$\text{and}$$

$$C_4^A(x) = 3^2(A)4^{x^4} - 2(A)3^{x^2} + 2^1(A)2.$$

We conclude this section introducing the Gegenbauer ' s matrix differential equation as follows :

In (25) , replace n by $(n-1)$ and differentiate with respect to x to find

$$xD^2 C_{n-1}^A(x) = D^2 C_n^A(x) - (2A + nI)DC_{n-1}^A(x). \qquad (32)$$

Also , by differentiating (2 1) with respect to x we have

$$xD^2 C_{n-1}^A(x) - (n-1)DC_n^A(x) = D^2 C_{n-1}^A(x). \qquad (33)$$

From (2 1) and (33) by putting $DC_{n-1}^A(x)$ and $D^2 C_{n-1}^A(x)$ into (32) and re - arrangement terms we obtain the Gegenbauer ' s matrix differential equation in the form :

$$(1 - x^2)D^2 C_n^A(x) - (2A + I)xDC_n^A(x) + n(2A + nI)C_n^A(x) = 0. \qquad (34)$$

4   Hypergeometric matrix representations of $C_n^A(x)$

From (1) it is easy to find that

$$[(A)2n]^{-1} = 2^{-2n}[(\tfrac{1}{2}(A+I))_n]^{-1}[(\tfrac{1}{2}A)_n]^{-1},\tag{35}$$

and

$$(A)_n + k = (A)_n(A + nI)k\tag{36}$$

Note that

$$(1 - 2xt - t^2)^{-A} = [1 - 2t(\tbinom{x}{1}_- - t)1_2)]^{-A}(1 - t)^{-2A}.$$

Therefore , by using and (5) and (36) we have

$$\sum_{\infty}^{n=0} C_n^A(x)t^n \;\;=\;\; \sum_{\infty}^{k=0}(A)_k^{2^k t^k \, \tbinom{x}{!(1-t)}2k}1)^k(1-t)^{-2A}$$

$$=\;\; \sum_{\infty}^{k=0}(A)k(1-t)^{-(2A+2kI)}2^k t^k(xk_! - 1)^k$$

$$=\;\; \sum_{\infty}^{n=0}\sum (A)_{\infty}^{k=0}k(2A + 2kI)_n 2^k(xk_{!n!}^{-}1)^k t^{n+k}$$

$$=\;\; \sum_{\infty}^{n=0}\sum (A)_{\infty}^{k=0}k[(A)2k]^{-1}(A)_n + 2k2^k(xk_{!n!}^{-}1)^k t^{n+k},$$

which by inserting (35) and applying (7) with the help of (3) yields

$$\sum_{\infty}^{n=0} C_n^A(x)t^n = \sum_{\infty}^{n=0}(2_n^{A)n}{}_!\sum_{n}^{k=0}(-nI)k(2_k^A{}_! + nI)k[(A + 2^{1I)2k]^{-1}(^x} - 21)^k t^n.$$

Thus , the hypergeometric matrix representation follows by equating the coef - ficients of t^n in the form

$$C_n^A(x) = (2_n^{A)n}{}_!F(-nI, 2A + nI; A + 2^{1I;}1 - 2x_).\tag{37}$$

On applying (1 6) , one gets

$$C_n^A(x) = (-1)^n(2_n^{A)n}{}_!F(-nI, 2A + nI; A + 2^{1I;}1 + 2x_).\tag{38}$$

Gegenbauer Matrix Polynomials   1 9 It is evident that

$$(n1_-2k)!^I = (-nI_{n!})2k; \quad 0 \le 2k \le n.$$

By using (2) and taking into account that

$$(-nI)2k = 2^{-2k}(-n2I)k(-(n-21)I)k,$$

the explicit representation (1 5) becomes

$$C_n^A(x) = (2_n^{x)^n}{}_!(A)_n \sum_{[n/2]}^{k=0} (-\tfrac{n}{2}I)k\binom{-(n-21)I)k}{k}{}_! [(I - A - nI)k]^{-1}\binom{1}{x})^{2k},$$

which gives another hypergeometric matrix representation in the form :

$$C_n^A(x) = (2_{n!}x)^n(A)_n F(-n2I, 1 - 2n_{I;}I - A - nI; 1_{x^2}). \tag{39}$$

Now , we can write

$$(1 - 2xt - t^2)^{-A} = [1 - t^2(\binom{x2}{1}_- - xt1)_2^{)}]^{-A}(1 - xt)^{-2A}.$$

Therefore , by using (5) and (6) we find that

$$\sum_\infty^{k=0} C_n^A(x)t^n = \sum_\infty^{k=0} (A)_k^{k(x^2}{}_!(1_- - 1xt)_{2k}^k t^{2k}(1 - xt)^{-2A}$$

$$= \sum_\infty^{n=0}\sum_\infty^{k=0} (A)k(2A + 2kI)_n(x^2 - {}_k^{1)^k}x^{n-2k}{}_{!n!}t^{n+2k}$$

$$= \sum (A)_n^{n=0}{}_\infty \sum_{[n/2]}^{k=0} [(A + 2^{1}{}_{I)k]^{-1}}\binom{x^2 - 1)^k}{2^2{}_{k_k!(}}{}_n^{x^n -2k}{}_{- 2k)!})t^n.$$

By identification of the coefficients of t^n, another form for the Gegenbauer matrix polynomials follows

$$C_n^A(x) = (A)_n \sum_{[n/2]}^{k=0} [(A + 2^{1I)k]^{-1}}\binom{x^2 - 1)^k}{2^{2k}{}_k !(}{}_n^{x^n -2k}{}_{- 2k)!}. \tag{40}$$

Moreover , by exploiting (40) and using (8) in the sum

$$\sum_{\infty}^{n=0}[(A)_n]^{-1}C_n^A(x)t^n \quad = \quad \sum_{\infty}^{n=0}x_{n!}^{n}t^n\sum_{\infty}^{k=0}k_!^1[(A+2^{1I)k]^{-1}}(4^{1t^2(x^2}-1))^k.$$

By identification of the coefficients of t^n, we obtain a generating relation for the Gegenbauer matrix polynomials in the form :

$$\exp(xt)0^F1(-;A+2^{1I;}4^{1t^2(x^2}-1)) = \sum_{\infty}^{n=0}[(A)_n]^{-1}C_n^A(x)t^n. \qquad (41)$$

5   Orthogonality of Gegenbauer matrix polynomials

Here , we will obtain the most interesting property of the Gegenbauer matrix polynomials , namely the orthogonality of this system of polynomials .   Let A be a positive st able matrix in $\mathbb{C}^{N\times N}$ such that

$A + kI$ is invertible for every integer $k \geq 0$.   (42) By multiplying (34) by $(1 - x^2)^{A-2^{1I}}$ we get

$$D[(1-x^2)^{A+2^{1I}}{}_{DC_n}(^Ax)] + n(1-x^2)^{A-2^{1I}}{}_{(2A}+nI)C_n^A(x) = 0. \qquad (43)$$

Similarly , when $C_m^A(x)$ satisfies (34) it follows

$$D[(1-x^2)^{A+2^1I}{}_{DC_m}^A(x)] + m(1-x^2)^{A-2^{1I}}{}_{(2A}+mI)C_m^A(x) = 0. \qquad (44)$$

By multiplying the equation (43) by $C_m^A(x)$ and the equation (44) by $C_n^A(x)$ and subtracting gives

$$D[(1-x^2)^{A+2^{1I}}{}_{DC_n}(^Ax)]C_m^A(x) - D[(1-x^2)^{A+2^1I}{}_{DC_m}^A(x)]C_n^A(x)+$$
$$n(1-x^2)^{A-2^{1I}}{}_{(2A}+nI)C_n^A(x)C_m^A(x)-$$
$$m(1-x^2)^{A-2^{1I}}{}_{(2A}+mI)C_m^A(x)C_n^A(x) = 0. \qquad (45)$$

Since the multiplication of the matrix in $(A)_n$ is commutative for every integer $n \geq 0$, then $C_n^A(x)C_m^A(x) = C_m^A(x)C_n^A(x)$. We can write

$$D[(1-x^2)^{A+2^{1I}}{}_{\{C_m}(^Ax)DC_n^A(x) - DC_m^A(x)C_n^A(x)\}]$$
$$= (1-x^2)^{A+2^{1I}}{}_{DC_m}(^Ax)DC_n^A(x) + D[(1-x^2)^{A+2^{1I}}{}_{DC_n}(^Ax)]C_m^A(x)-$$

$$(1-x^2)^{A+2^{1I}}_{DC_m}(^A x)DC_n^A(x) + D[(1-x^2)^{A+2^1}I_{DC_m}^A(x)]C_n^A(x).$$

Thus , the equation (45) becomes

$$(n-m)\{A+(n+m)I\}(1-x^2)^{A-2^{1I}}_{C_n}(^A x)C_m^A(x) =$$
$$D[(1-x^2)^{A+2^{1I}}_{\{C_m}{}^A(x)DC_n^A(x) - DC_m^A(x)C_n^A(x)\}].$$

Actually , m and n are non - negative integer and A is a positive stable matrix , hence $A+(m+n)I \neq 0$. Therefore , it follows

$$-^11(1-x^2)^{A-2^{1I}}_{C_n}(^A x)C_m^A(x)dx = 0; \quad m \neq n. \tag{46}$$

That is , for A is a positive stable matrix in $\mathbb{C}^{N \times N}$, the Gegenbauer matrix polynomials form an orthogonal set over the interval (- 1 , 1) with respect to the weight function $(1-x^2)^{A-2^{1I}}$.

One immediate consequence of (46) is

$$-^11(1-x^2)^{A-2^{1I}}_{C_m}{}^A(x)dx = 0; \quad m \neq 0.$$

Now , by multiplying (27) by $(1-x^2)^{A-2^{1I}}_{C_n}{}^A(x)dx$ and integrating between - 1 and 1 and taking into account (46) we get

$$-^11(1-x^2)^{A-2^{1I}}_{[C_n}(^A x)]^2dx = \tag{47}$$
$$2_n(A+(n-1)I) -^1 1x(1-x^2)^{A-2^{1I}}_{C_n}{}^A(x)C_{n-1}^A(x)dx.$$

Again , by replacing n by $n-1$ in (27) and multiply by $(1-x^2)^{A-2^{1I}}_{C_n-1}{}^A(x)dx$ and integrating between - 1 and 1 and taking into account (46) to obtain

$$2(A+nI) -^1 1x(1-x^2)^{A-2^{1I}}_{C_n-1(}{}^A x)C_n^A(x)dx = \tag{48}$$
$$(2A+(n-1)I) -^1 1(1-x^2)^{A-2^{1I}}_{[C_n-1}{}^A(x)]^2dx.$$

Thus , from (47) and (48) we get

$$-^11(1-x^2)^{A-2^{1I}}_{[C_n}{}^A(x)]^2dx = 1_n(A+nI)^{-1}(A+(n-1)I) \tag{49}$$

$$(2A + (n-1)I) -^1 1^{(1} - x^2)^{A-2^{1I}}_{[C_n}\,^A_{-1}(x)]^2 dx.$$

By substituting for n the values $n-1, n-2, \cdots, 1$ in (49) it follows

$$-^1 1^{(1} - x^2)^{A-2^{1I}}_{[C_n}\,^A(x)]^2 dx =$$

$$1_{n!}(A + nI)^{-1} A(2A)_n -^1 1^{(1} - x^2)^{A-2^{1I}}_{[C_0}(\,^A x)]^2 dx.$$

Note that , for A is a positive stable matrix in $\mathbb{C}^{N \times N}$ satisfies (42) it follows

$$-^1 1^{(1} - x^2)^{A-2^{1I}}_{dx} = \sqrt{\pi} \Gamma(A + 2^{1I)\Gamma^{-1}(A} + I).$$

Therefore , we obtain

$$-^1 1^{(1} - x^2)^{A-2^{1I}}_{[C_n}(\,^A x)]^2 dx = 1_{n!}(A + nI)^{-1} A(2A)_n \sqrt{\pi}$$
$$\Gamma(A + 2^{1I)\Gamma^{-1}(A} + I),$$

which can be be written with 46 in the form

$$-^1 1^{(1} - x^2)^{A-2^{1I}}_{C_n}(\,^A x) C^A_m(x) dx = 1_{n!}(A + nI)^{-1} A(2A)_n \sqrt{\pi} \qquad (50)$$
$$\Gamma(A + 2^{1}_{I)\Gamma^{-1}(A} + I)\delta_{mn},$$

where δ_{mn} is Kronecker ' s delta symbol .

Finally , we will expand the Gegenbauer matrix polynomials in series of Hermite matrix polynomials . By employing (1 5) , (8) and (1 2) and taking into account that each matrix commutes with it self , one gets

$$\sum_{\infty}^{n=0} 2^{-n}(\sqrt{2}A_)^n C^A_n(x)t^n = \sum_{\infty}^{n=0} \sum_{\infty}^{k=0} \sum_{[n/2]}^{s=0} (^{-1)^k (A)}_s \,_{!k!(n-}\,^{n+k}_{2s)!} H_{n-2s}(x, A)t^{n+2k},$$

which on applying (8) becomes

$$\sum_{\infty}^{n=0} 2^{-n}(\sqrt{2}A_)^n C^A_n(x)t^n = \sum_{\infty}^{n=0} \sum_{\infty}^{k=0} \sum_{\infty}^{s=0} (-1)^{k}_s\,^{(A)_{n+k+2s}}_{!k!n!} H_n(x, A)t^{n+2k+2s}.$$

By using (7) one gets

$$\sum_{\infty}^{n=0} 2^{-n}(\sqrt{2}A^n_j C^A_n(x)t^n = \sum_{\infty}^{n=0}\sum_{\infty}^{k=0}\sum_{k}^{s=0}(-1)^{k-s}_s {}^{(A)_n+k+s}_{!(k-} {}_{s)!n!}H_n(x,A)t^{n+2k}.$$

Since $(A)_n + k + s = (A + (n + k)I)_s(A)_n + k$, then by using (3) it follows

$$\sum_{\infty}^{n=0} 2^{-n}(\sqrt{2}A^n_j C^A_n(x)t^n = \sum_{\infty}^{n=0}\sum_{\infty}^{k=0}(^{-1)^k}_k {}_{!n!}$$

$$2^F0(-kI, A + (n + k)I; -; 1)(A)_n + k^H n(x,A)t^{n+2k}.$$

By (6) and then equating the coefficients of t^n we obtain an expansion of the Gegenbauer matrix polynomials as series of Hermite matrix polynomials in the form :

$$C^A_n(x) = 2^n(\sqrt{2}A_j^{-n} \sum_{[n/2]}^{k=0} k!(^{(-1)k}_{n-2k)!}2^F0(-kI, A + (n-k)I; -; 1)$$

$$(A)_{n-}k^H n - 2k(x,A). \qquad (51)$$

References

[1]   A . G . Constantine , *Some non - central distribution problem in multivariate*
       analysis , Ann . Math . Statis . , **34** (1 963) , 1 270 − 1 285 .

[2]   A . G . Constantine , R . J . Muirhead , *Parial differential equations for hy -*
       pergeometric functions of two argument matrix , J . Multivariate Anal . **3**
                   (1 972) , 332 − 338 .

[3]   E . Defez , L . J ó dar , *Some applications of the Hermite matrix polynomials*
     s eries expansions , J . Comp . Appl . Math . **99** (1 998) , 1 5 − 1 1 7 .

[4]   A . J . Dur á n , W . Van Assche , *Orthogonal matrix polynomials and higher*
       order recurrence re latio ns , Linear Algebra and its Applications **219** , 26 1
− 280 .

[5]   A . J . Dur á n , *Markov ' s Theorem for o rthogonal matrix polynomials* , Can .
                   J . Math . **48** (1 996) , 1 1 80 − 1 1 95 .

[6]   A .   J . Dur á n and P . Lopez - Rodriguez ,   *Orthogonal matrix po lyn omials :*
zeros and Blumenthal ' Theorem , J . Approx . Theory **84** (1 996) 96 − 1 1 8 .

1 1 4 K . A . M . Sayyed , M . S . Metwally , R . S . Batahan

[7] I . Gohberg , P . Lancaster , L . Rodman , *Matrix Polynomials* , Academic Press , New York , 1 982 .

[8] A . T . James , *Special functions of matrix and s ingle argument in s tatis - tics* , in Theory and applications of Special Functions , Ed . R . A . Askey , Academic Press , 1 975 , 497 – 520 .

[9] L . J *ó* dar , R . Company , E . Navarro , *Laguerre matrix polynomials and sys - t em of s econd - order differential equations* , Appl . Num . Math . **15** (1 994) , 53 – 63 .

[1 0] L . J *ó* dar , R . Company , *Hermite matrix polynomials and s econd order matrix differential equations* , J . Approx . Theory Appl . **12** (2) (1 996) , 20 – 30 .

[1 1] L . J *ó* dar , J . C . Cort *é* s , *On the hypergeometric matrix function* , J . Comp . Appl . Math . **99** (1 998) , 205 – 2 1 7 .

[1 2] L . J *ó* dar , E . Defez , *A connection between Laguerre ' s and Hermite ' s ma - trix polynomials* , Appl . Math . Lett . **1 1** (1) (1 998) , 1 3 – 1 7 .

[1 3] L . J *ó* dar , J . Sastre , *The growth of Laguerre matrix polynomials on bounded intervals* , Appl . Math . Lett . **13** (2000) , 2 1 – 26 .

[1 4] L . J *ó* dar and J . C . Cort *é* s , *Closed form general s o lution of the hyperge - ometric matrix differential equation* , Mathematical and Computer Mod - elling . **32** (2000) , 1 0 1 7 – 1 28 .

[1 5] R . J . Muirhead , *Systems of partial differential equations for hypergeomet - ric functions of matrix argument* , Ann . Math . Statist . **4 1** (1 970) , 99 1 – 1 0 1 .

[1 6] E . D . Rainville , *Special Functions* , The Macmillan Company , New York , 1 960 .

[1 7] L . Rodman , *Orthogonal matrix po lyn omials* , in Orthogonal Polynomials : Theory and Practice , P . Nevai , ed . , vol . 294 of NATO ASI Series C , Kluwer , Dordrecht , 1 990 , 345 – 362 .

[1 8] K . A . M . Sayyed , R . S . Batahan , *On generalized Hermite matrix polyno - mials* , accepted in Electronic Journal of Linear Algebra .

[1 9] A . Sinap , W . Van Assche , *Polynomials interpolation and Gaussian quadrature for matrix - valued functions* , Linear Algebra and its Appli - cations **207** (1 994) , 71 – 1 1 4 .

[20] A . Sinap , W . Van Assche , *Orthogonal matrix po lyn omials and applica* -

tions , J . Comp . Appl . Math . **66** (1 996) , 27 – 52 . [2 1]   A . Terras ,

Special functions for the symmetric space of positive matrices ,

SIAM J . Math . Anal . , **16** (1 985) , 620 – 640 .